Elementary Möbius Geometry III

Pairs of Subspheres in S^3

by
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In[•]:=
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Date[][[1 ;; 3]]
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Out[•]= {2018, 7, 18}

Initialization

$Assumptions

1. Introduction

This notebook continues the notebooksemg1.nb, emg2.nb.   We consider here those pairs of subspheres of the Möbius space S^3 not treated in these notebooks; the titles of sections 2 - 5 show the pairings considered here. Pairs of spheres are treated in the notebook emg1.nb, and pairs of circles in emg2.nb. The last section treats some examples of geodesics in the space of 0-spheres. In this section some concepts of Lie algebras are needed. They are contained in the package liealgsh.m, added to elmoeb.zip.   The calculations and examples illustrate and complete the presentation of the subject in our book [O-S], Section 2.7, or the paper [S4]. There one finds the foundations of elementary Möbius geometry in terms of linear algebra. See also the Introductions to the notebooks mentioned above.

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Keywords

Möbius group, isotropic cone, pseudo-orthonormal coordinates, spheres, circles, point pairs, pairs of subspheres, pseudo-Euclidean space, pseudo-orthogonal Lie algebra, Killing form, Möbius invariants

of pairs of subspheres, Möbius invariants expressed by Euclidean invariants, geodesics of the 0-spheres manifold.

2. Spheres and circles

A pair (S, C), consisting of a sphere S and a circle C of the 3-dimensional Euclidean space or the 3-dimensional sphere, is defined up to a Möbius transform by exactly a single invariant, if C is not contained in S ("general position"). It is easy to show that all pairs (S, C) with C⊂S are Möbius equivalent; indeed, this follows from the transitivity of the action of the Möbius group of the sphere S^2 .on the manifold of all circles contained in S^2. In the first subsection we calculate this invariant moebsc, then we give some examples, and finally we derive an expression of moebsc in terms of Euclidean invariants of the pair (S,C).

2.1 Definition of the Invariant moebsc

2.2 Examples

2.2.1 Inclusion: rank 2

2.2..2 Example 1. Tangent Pairs

2.2.3 Example 2. Bundle of Tangent Circles

2.2.4 Example 3. Circle and Sphere Rosette

2.2.5 Test of the Conformity

2.2..6 Intersecting Pairs

2.2.7 Orthogonal Intersections

2.2.8 Disjoint Pairs

2.3 Euclidean Interpretation of moebsc

2.3.1 A General Formula

2.3.2 Evaluation of moebsc

2.3.3 The Invariant moebsc and the Mutual Position

2.3.4 Plane and Circle

3. Spheres and Point Pairs

In this Section we consider pairs { S^0, S^2}, S^0 ={pt[a],pt[b]}, a≠b, a 0-sphere (point pair) and a sphere S^2 and find the corresponding Möbius invariants.

3.1. Point Pairs as 0-Spheres

3.1.1 An Adapted Frame

3.1.2 Test of vspacepp

3.1.3 Examples

3.1.4 Random Test

3.2. The Invariant moebspp and the Mutual Position

3.2.1. The Definition

3.2.2. The Mutual Position

3.3. An Expression of the Invariant moebspp in Euclidean Terms

3.3.1. The result

3.3.2. Test

3.3.3. Sphere

3.3.4. Random Test in Dimension 5, i.e. dim = 7, ind = 1

3.4. Orthogonality

4. Circles and Point Pairs

In this section we consider pairs (C, pp) consisting of a Circle C = S_1^1 and a point pair pp = (a1,a2) = S_2^0 contained in the Euclidean 3-space or the 3-sphere. Such a pair is said to be in general position, if the union C∪pp is not contained in a 2-sphere. Then its mutual position is defined up to Möbius equivalence by two invariants, which we are going to calculate. We study the relation between the values of these invariants and the geometry of the pair. In the last subsection we find formulas expressing the invariants in terms of the Euclidean invariants of the pairs.

4.1. The Invariants

4.2. Examples

4.2.1. Pairs not in General Position

4.2.2. Pairs in General Position

4.2.3. The Circle may be a Line

4.2.4. One Eigenvalue 0

4.2.5. Standard Unit Circle and Random Point Pairs

4.2.6. The Eigenspheres of the Random Pair

4.3. The Separation Property

4.4. Two Associated Spheres Belonging to a Pair (S_1^1, S_2^0)

4.5. Möbius Invariants Expressed by Euclidean Invariants

In this subsection we will express the determinant and the trace of ppcpp[(C,pp)] by Euclidean invariants of the configuration obtaining formulas similar, but more complicated, to the Coxeter distance of two spheres. The calculations show very well the excellent performance of *Mathematica* in doing symbolical simplifications.

4.5.1. Adaption of the Euclidean Coordinates to the Point Pair

4.5.2. Adaption of the Euclidean Coordinates to the Circle

4.5.3. Calculation of the invarints

4.5.4. Invariant Angles

4.5.5. Control, OK!

5. Point Quadruples

We consider pairs of 0-spheres, i.e. point quadruple {{a1,a2},{b1,b2}}. Since each point quadruple belongs to a sphere, we could consider spherical Möbius geometry: dim = 4, but for later applications we will continue with dim =5 here. Since the Möbius group of the sphere coincides with the broken linear transformations in one complex variable of the Riemann sphere, and this is the projective complex line, an invariant of the quadruple coincides with the complex cross ratio, the basic invariant of complex projective geometry. Historically, the Möbius group has been found by Möbius in this important context. We considered the original complex Möbius group, and its relation to the 2-dimensional real case, in the notebook [RS]. Here we are interested more in the real situation, for later application in differential geometry. Therefore we will apply the general method of Möbius invariants for pairs of subspheres also in the lowest dimension. Since the cross ratio is a complex number we may expect to obtain two real invariants describing the mutual position of two point pairs in the Möbius space.

5.1. A complete invariant system

5.2. Throws

5.2.1. Definition of a throw and the matrix pppts

5.2.2. The eigenvalue 1

5.2.3. The other eigenvalues

5.2.4. The eigenspheres

5.3. Four points on the standard unit circle

5.3.1. Two Point Pairs

5.3.2.The orthogonal case

5.3.3. Two Point Pairs on the Line. Another Approach

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6. Geodesics in the Space of 0-Spheres

6.1. Introduction

6.2. Spacelike geodesics

6.3. Timelike geodesics

6.4. Isotropic geodesics

References

Homepage

Home

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