Elementary Möbius Geometry II

Circles

by

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Mathematica versions 11.3 or 8.0.1.

In[25]:=
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Date[][[1 ;; 3]]
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I remark, that in some simplifications of symbolic expressions version 8.0.1 works fast, and version 11.3 gives no result in a time of several hours.

Initialization

To work interactively with this notebook: First open the notebook emg0.nb,
read its Section 1 “Initialization” and use Menu/Evaluation/Evaluate Initialization Cells. Then use Menu/Evaluation/Evaluate Initialization Cells in the notebook emg2.nb.

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1. Introduction

This notebook continues the notebook [emg1] about elementary Möbius geometry.   We consider now the circles under the action of the Möbius group.   Again the three-dimensional case is emphasized. The calculations and examples illustrate and complete the presentation of the subject in our book [OS3], Section 2.7. There one finds the foundations of elementary Möbius geometry in terms of linear algebra.

The Möbius group acts transitively on the Möbius space: the sphere S^n, as a point manifold, and on the manifold of the k-dimensional subspheres.   A circle is a 1-dimensional subsphere; thus it is modelled as the set of isotropic 1-dimensional subspaces of a three-dimensional Minkowski space. The manifold of all circles in the n-dimensional Möbius space is equivariantly isomorphic to the manifold of all 3-dimensional pseudo-Euclidean subspaces of the (n+2)-dimensional Minkowski space. Equivalently one may pass to the orthogonal complements being (n-1)-dimensional Euclidean subspaces of the (n+2)-dimensional Minkowski space. Thus. in case n=3, the circle space appears as a 6-dimensional manifold, equivariantly modelled as an open orbit in the Grassmann manifold $G_{5,3}$ of the 3-dimensional sub-spaces of the 5-dimensional pseudo-Euclidean vector space of index 1.

In Chapter 2 the circles are considered as geometric objects in the 3-dimensional Euclidean space. The Chapters 3 -6 may be considered as an extended update of the corresponding chapters in the note-book mcircles.nb of the collection [Spheres]. The contents of each chapter is described in its first paragraph.

1.1. Keywords

Möbius group, isotropic cone, isotropic-orthogonal coordinates, pseudo-orthonormal coordinates, circles, circle pairs, interlaced circles, circle space, pseudo-Euclidean space, pseudo-orthogonal Lie algebra, generalized Coxeter invariant, circle surfaces, tubes, geodesics of the circle manifold.

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2. Circles in the Euclidean 3-space

Mathematica has built-in circles in dimension two only:

In[48]:= **?Circle**

Thus, first we will introduce circles as objects of the three-dimensional Euclidean space. In 2.1 we consider two parameter representations of arbitrary circles; since a circle in the 3-space depends on 6 parameters the circle space is a 6-dimensional manifold. In 2.2 the corresponding plot definitions are presented. In subsection 2.3 the circle (or line, or point) defined by a point tripel of the 3-space is calculated and plotted. As an application we construct tubes of space curves, especially torus knots, in the last subsection.

2.1 Parametrization of the Circle Space. The General Circle

2.1.1. The Unit Sphere with Center o and its Tangential Unit Vectors

2.1.2. Examples

2.1.3. Definition of circle3D

2.1.4. Examples

2.1.5. Another Definition: gencircle

2.1.6. Examples

2.2 The Plot Definitions

2.3 Circles and Lines through Three Points

Definitions: Position Vector, Center, Radius, and Circle

Examples

2.4 An Application: Tubes of a Space Curve

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3. Circles Defined by 2-Dimensional Euclidean Vector Subspaces

In Möbius geometry a circle in the n-sphere, or by stereographic projection, a circle or a line in the Euclidean n-space, is given by a 3-dimensional pseudo-Euclidean vector subspace: the generators of the isotropic cone of this 3-space are the (projective) points of the circle. Equivalently it is defined by the orthogonal complement, that is a (n-1)-dimensional Euclidean subspace of the (n+2)-dimensional pseudo-Euclidean vector space. The sets of these spaces are open orbits of the Möbius group in the corresponding Grassmann manifolds. The Möbius group can be identified with two components of the pseudo-orthogonal group O(1,4). In this section we construct *Mathematica* functions establishing the equivariant bijection between the manifold of Euclidean 2-spaces of the 5-dimensional pseudo-Euclidean vector space of index 1 and the manifold of circles in the 3-dimensional Möbius space explicitly. In the next section we apply this bijection to calculate the Möbius invariants of pairs of circles. Elementary Möbius geometry is presented in [OS3], Section 2.7.

3.1 From the Subspace to the Circle

In this subsection we start with an arbitrary basis of an Euclidean 2-subspace, construct a pseudo-orthonormal basis of its orthogonal complement and obtain, using this basis and stereographic projection, a parameter representation and plots of the corresponding circle.

3.1.1. The Definition pscomplement

3.1.2. The Plot Command: threepoints, circle2spv, plotcircle2spv

3.1.3. First Examples

3.1.4. The Frame Circles

3.1.5. Random Circle, Generated by a 2-Space Spanned by a Random orthopair

3.2 From the Circle to the Subspace

3.3. Adapting the Frame

3.4. Radius, Center, and Position Vector as Functions of the Subspace

In this section we calculate the basic Euclidean parameters radius, center, and position vector of the circle as functions of the vectors v,w spanning an Euclidean 2-space in the pseudo-Euclidean 5-space. There are two ways to calculate these invariants: one can use the adaption procedure adaptsplframe, or one can use threepoints, and the concepts of section 2.3. For numerical calculations the speed of both methods is nearly the same. We have chosen the first method with the corresponding definitions as standard concepts, and included them into the package mcirc.m, since it seems to be better appropri ate for symbolic calculations.

3.4.1. The Euclidean invariants with Frame Adaption

3.4.2. The Euclidean Invariants with threepoints

3.4.3. Standard Tests

3.4.4. Test with Random Space Vectors

3.5. From Three Points to the Subspace of the Corresponding Circle

3.5.1. Definitions

3.5.2. Examples

4. Möbius Invariants for Pairs of Circles

In this section we apply the general theory, see the paper [S00], or the book [OS3], Section 2.7, to the case of circles in the 3-sphere and the Euclidean 3-space. For the Möbius geometry of spheres see the notebook emg1.nb. Invariants for the other possible pairings (S^k, S^m), 0 ≤ k, m ≤ 3, are treated in the notebook emg3.nb. The expressions of the Möbius invariants of two circles by their Euclidean invariants obtained in subsection 4.8 are analogous to the Coxeter invariant of two spheres, see the notebook emg1.nb. They seems to be new (see also [OS3], subsection 2.7.4).

4.1 Case Discussion

4.2 Invariants for Pairs of Euclidean 2-Subspaces

In Section 3 we established an equivariant bijection between the space of the circles of the Möbius space and the space of 2-dimensional Euclidean subspaces of the corresponding pseudo-Euclidean vector space. Thus pairs of circles correspond bijectively to pairs of such subspaces. In [S00] or [OS3], Section 2.7, is proved that the Eigenvalues of a certain self-adjoint operator of one of the Euclidean 2-spaces are a complete invariant system for the pairs of subspaces and therefore also for the pairs of circles. In Subsection 4.2.5 below we define this operator, named the double projection, using the orthogonal decompositions of the pseudo-Euclidean vector space defined by the subspaces, and construct functions for the calculation of the invariants. First we consider the cross product of orthonormal basis vectors of the subspaces and use it for a case criterion, which is illustrated by examples.

4.2.1. Case Criterion

4.2.2. Example: Random Pair, Case 1.a

4.2.3. Example: Random Pair, Case 1.b

1c

4.2.4. Example: Random Pair, Case 1.c

4.2.5. The Double Projection

4.2.6. The Orthogonal Case

4.2.7. Example: Two Non Orthogonal Circles Intersecting Orthogonally

4.3. Case 1.a: Disjoint, Non-Interlaced Circles

4.4. Case 1.b: Interlaced Circles

Already in section 4.2, the orthogonal case, we have seen: the standard unit circle in the x,y-plane and the z-axis are interlaced "circles". We give now examples of isogonal and non-isogonal interlaced circle pairs.

4.4.1. Example: Isogonal Circles

isogo

4.4.2. Family of Circles, Isogonal to the Unit Circle grc1

4.4.3. Intersection Angle for Spheres Containing Interlaced, But Non Isogonal Circles

4.5 Case 1c: Circles Intersecting in Exactly One Point

4.5.1. First Example

4.5.2. Another Example

4.6 The Cases 2

As remarked in Subsection 4.1 above, in this case the circles lie in a 2-sphere. Thus it would be easy to describe their properties diminishing the dimension, setting dim = 4. We leave this as an exercise to the user and treat here the circles as objects in S^3.

4.6.1. Definition of vspace 2d

4.6.2. a) Two Disjoint Circles:

4.6.3. b) Two Intersecting Circles:

4.6.4. c) Two Tangent Circles

4.7 Further Examples and Applications

4.7.1. Standard Unit Circle Compared with a Circle Family with Increasing Radius

4.7.2. Osculating Circles of a Plane Curve

4.7.3. Osculating Circles and Osculating Spheres of a Space Curve

4.7.4. The same for an Elliptically Distorted Helix

4.8 The Möbius Invariants Expressed by Euclidean Invariants

A complete system of Euclidean invariants of a pair (C0,C1) of circles is obtained by the following entities:
The radii r of C0, r1 of C1,
the distance d1 of the centers,
the angle a between the position vectors nc0, nc1,
the angle b between the position vector nc0, and the line connecting the centers,
the angle c between the position vector nc1, and the line connecting the centers.

4.8.1. The Calculation of the Expressions

4.8.2. Test

5. Normal Forms of Pairs of Circles

In the general classification of pairs {k-sphere, l-sphere} contained in an n-sphere (see [OS3] or [S00]) we obtained normal forms for the bases of the generating vector spaces, where the coefficients of one basis with respect to the other are defined by the invariant eigenvalues of the matrix pp. In this section we describe these normal forms for each of the cases mentioned in the subsection 4.1 Case discussion,

above. The advantage of the normal form representation is that the relations between the invariants and the geometry of the circle pair can be studied directly. For the first circle we always take the unit circle grc1 = plotcircle2spv [stb[4],stb[5]] with center o in the x,y-plane, defined with the initialization; the other circle is given by the corresponding normal form of the base. We remark that the case numbering and the notations differ from those used in [S00].
ATTENTION: For getting the normal forms the definitions in subsection 5.1 are needed. Evaluate 5.1 first!

In[77]:= **Show [grc1]**

v5.1

5.1 Some Vector Functions Needed for the Definition of the Normal Forms

5.2 Case 1a: Disjoint, Non-Interlaced Circles

5.3 Case 1b: Interlaced Circles

5.3.1. Introduction

5.3.2. Radius, Center, Position Vector

5.3.3. The Möbius Invariant

5.3.4. A Circle Family and a Circle Surface

5.3.5. The Circle Family and the Circle Surface with b = 1.2 (case 1b)

5.3.6. The Circle Family and the Circle Surface with a = 1 (case 2b)

5.3.7. The Isogonal Case

5.4 Case 2a: Non Intersecting Circles on a Sphere

5.5 Case 2b: 2-Point Intersecting Circles in the x,y-Plane

5.6 Cases 1c and 2c: Circles Intersecting in One Point

6. Circle Surfaces, Circle Orbits, Geodesics in the Circle Space

A circle surface is a surface in the 3-space through any point of which exists a circle contained in the surface. With other words: A circle surface can be described as a curve in the circle space. The spheres (and planes) are the simplest circle surfaces. The tubes considered in section 2.4 are circle surfaces.

More general, envelopes of one-parameter sphere families: the channel surfaces, are circle surfaces. A special case are the Dupin cyclides, which are channel surfaces in a twofold way. That means, that there exist two different sphere curves whose envelopes are the same channel surface: tori, circular cones, circular cylinders, and their images under Möbius transformations. Other special circle surfaces are the orbits of 1-parameter subgroups of the Möbius group in the circle space.

The circle space is pseudo-Riemannian symmetric of rank two. The geodesics of the circle space are orbits of certain 1-parameter subgroups of the Möbius group.   As in all reductive spaces, the geodesics through an element C are the orbits of one-parameter subgroups exp(At) C, $t \in \mathbb{R}$, where in our case C is the standard unit circle grc1 and A an element of the orthogonal complement of the isotropy algebra of C in the Lie algebra o(5,1); here orthogonality is defined by the Killing form of the Lie algebra o(5,1), see my notebook liealgeb.nb, in *MathSource* [MS]. It would be interesting to investigate the relation between the pseudo-Riemannian geometry of the circle space and the circle manifolds more basically. But this may be the subject of another notebook. Some results about the Möbius geometry of circle surfaces are contained in the diploma paper [UM] of Uwe May; especially the geodesics defined in the next subsections can be found there. We emphasize that we do not give here a classification of all the geodesics with respect to the Möbius group. We only treat three simple cases of geodesics lying in a 2-sphere.

6.1 Spacelike Geodesics

6.2 Timelike Geodesics

6.3 Isotropic Geodesics

6.4 Character of the Geodesics

References

Homepage

Home

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