Elementary Möbius Geometry I

Points and Spheres

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Starting date December 28, 2010,
finished February 7, 2011.
Revised December 15, 2011.
2. Revision started February 2, 2018,
finished July17, 2018.
Mathematica v. 11.3.

In[25]:=
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Date[][[1 ;; 3]]
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Initialization

To work interactively with this notebook: First open the notebook emg0.nb, read its Sectio 1 “Initialization” and use Menu/Evaluation/Evaluate Initialization Cells. Then use Menu/Evaluation/Evaluate Initialization Cells in the notebook emg1.nb.

The notebook emg1.nb is the first of a series of notebooks about elementary Möbius geometry. The notebook emg2.nb contains the geometry of circles, and in the notebook emg3.nb the Möbius geometry of subspheres S^k, S^l with 0≤k≤l<3 is treated as far as the cases are not considered in the earlier notebooks of the series.

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1. Introduction

In this notebook we apply *Mathematica* to create tools for working in n-dimensional Möbius geometry. The three-dimensional case is emphasized. The calculations and examples illustrate and complete the presentation of the subject in our book [OS3], Section 2.7. There one finds the foundations of elementary Möbius geometry in terms of linear algebra; also hyperbolic and elliptic geometry as used in this notebook are contained therein. We intend to apply the functions and modules developed here to Möbius differential geometry in notebooks of the series mdg-i.nb, i 0 1, 2,...

In the elementary context the Möbius group is defined to be the group of projective transformations leaving an elliptic hyperquadric S^n of the (n+1)-dimensional real projective space invariant. For dimension n≥2 the Möbius group coincides with the group of conformal transformations of the n-sphere S^n. This follows from Liouville's theorem proved in differential geometry. Thus the Möbius geometry coincides with the conformal geometry of the n-sphere (with respect to its standard metric). In homogeneous coordinates {x(i)} the equation of the n-sphere as a projective hyper-quadric has the normal form (for n = 3)

$$-x(1)^2 + x(2)^2 + x(3)^2 + x(4)^2 + x(5)^2 = 0$$

In [OS3] it is proved that the Möbius group is isomorphic to the pseudo-orthogonal group $O(1, n+1))^+$, that is the group leaving the pseudo-orthogonal scalar product of index 1 in the (n+2)-dimensional real vector space

$$\text{pssp}(\text{vec}(x), \text{vec}(y)) := -x(1)\, y(1) + x(2)\, y(2) + x(3)\, y(3) + x(4)\, y(4) + x(5)\, y(5)$$

and a time orientation invariant; these linear transformations generate the projective maps forming the Möbius group. Therefore our considerations are based on the linear algebra of n-dimensional pseudo-Euclidean vector spaces of index 1, sometimes named Minkowski spaces.

Section 2. Here some functions and Modules of pseudo-Euclidean linear algebra contained in the package neuvecv2.m are shown and tested. A notebook “pseuklid.nb” about pseudo-Euclidean geometry is contained in the packed files sphs4.tgz or sphs4.zip on my homepage, see [Spheres].

The Möbius group acts transitively on the Möbius space: the sphere S^n as a point manifold, and on the manifold of the k-dimensional subspheres. Here the case of the hyperspheres, k = n-1, is the simplest and most important. In this notebook we consider mainly the Möbius geometry as the geometry of the point manifold and the geometry of the manifold of hyperspheres, the spheres in case n=3. A pseudo-orthonormal frame as the standard basis is very appropriate if one treats subspheres; for the geometry of the point manifold the isotropic-orthonormal bases are often useful, because they may be better adapted to point configurations. Both methods are described and applied inSection 2. In the last subsection the projective model of the Möbius space is introduced.

Section 3 contains the description of the Möbius space and the conformal models of the simply connected space forms. Besides of Euclidean geometry the hyperbolic geometry is presented in greater detail. For the hyperbolic plane geometry we created a special notebook, see [hyp2D]. There one finds the classification of quadrics in the hyperbolic plane.

Se ction 4. The Sections 4, 5, 6 may be considered as an extended update of the notebook mspheres.nb in the collection [Spheres]. Section 4 describes the Möbius Geometry of Spheres in pseudo-Euclidean termes. The manifold of all oriented spheres is identified with the set of all spacelike unit vectors. This is the one-sheeted hyper-hyperboloid defined in the 5-dimensional pseudo-Euclidean vector space of index 1 by the equation

```
pssp[vec[x], vec[x]] == -x[1]^2 + x[2]^2 + x[3]^2 + x[4]^2 + x[5]^2 == 1
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We remark that in Möbius geometry planes are special spheres, more exactly, they are not defined. Considering the model of the Euclidean space as the complement of a distinguished “infinite” point in S^n, the planes are the Möbius geometric spheres containing this point. This is exactly as they appear under stereographic projection with the infinite point as North Pole . The stereographic projection is a conformal map of the not directly visible Möbius space S^3 onto the usual Euclidean space E^3 .

In Section 5 we study the mutual position of two hyperspheres up to Möbius equivalence. It is characterized by a single invariant, the "inversive distance", a generalized angle, or the Coxeter invariant, which is well known as

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coxeterinv[r1, r2, d] == (-d^2 + r1^2 + r2^2)/(2 r1 r2)
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$$\text{coxeterinv}[r1, r2, d] == \frac{-d^2 + r1^2 + r2^2}{2\, r1\, r2}$$

Here r1, r2 denote the radii of the hyperspheres and d the distance of their centers. The Coxeter invariant is an expression in Euclidean terms which are not invariants in Möbius geometry. We obtain a new derivation of the Coxeter invariant showing that it is a conformal invariant.

In Section 6 the geodesics of the sphere space are classified from an elementary viewpoint. There are three classes of geodesics corresponding to the types of the central hyperplanes whose intersections with the hyper-hyperboloid they are. In each of these classes the Möbius group acts transitively.

1.1. Keywords

angles, conformal models of space forms , Coxeter invariant, geodesics of the sphere manifold, hyperbolic space, isotropic cone, isotropic-orthogonal coordinates, Möbius group, planes, pseudo-Euclidean space, pseudo-orthogonal Lie algebra, pseudo-orthonormal coordinates, spheres, spherical reflections, stereographic projection

2. Pseudo-Euclidean Linear Algebra

The package neuvecv2.m contains interesting tools enhancing *Mathematica* with concepts of pseudo-Euclidean linear algebra. In this chapter we describe some of these tools and introduce some new functions which we need in Möbius differential geometry.

2.1. Test of some Functions of neuvecv2.m

2.2. Isotropic-Orthonormal Bases

2.3. The Projective Model of the Möbius Space

2.3.1. The Model

2.3.2. The Map sphmap

3. The Möbius Space and the Space Forms

In this Section we describe the Möbius space and the Möbius group. Furthermore we consider the Euclidean, spherical and hyperbolic geometries as subgeometries of Möbius geometry.

3.1. The Pseudo-Euclidean Model of the Möbius Space. The Möbius Group

3.2. Spherical Riemann Geometry as Subgeometry of the Möbius Geometry

3.3. An isometric Model of the Euclidean Space in the isotropic hypercone

3.4. Isotropic-Orthonormal Coordinates

3.5. The Möbius Model of the Euclidean Space in Isotropic-Orthonormal Coordinates

3.6. Hyperbolic Geometry as a Subgeometry of Möbius Geometry

In this section we describe some elementary features of the hyperbolic geometry embedded into Möbius geometry.

3.6.1. The Pseudo-Euclidean Model of the Hyperbolic Space

3.6.2. Möbius Geometric Embedding and F. Klein's Conformal Disk Model of the Hyperbolic Space

3.6.3. The Poincaré Model of the Hyperbolic Plane

3.6.4. Hyperbolic Lines in the Hyperbolic Plane

3.6.5. Example. Random Central Linesdetail.

3.6.6. The Poincaré Model of the Hyperbolic Space

3.6.7. Hyperbolic Lines in the Hyperbolic Space

3.6.8. Hyperbolic Planes in the Hyperbolic Space

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4. Spheres

In this Section we establish the basic bijection between spheres (including planes as spheres of infinite radius) of the Euclidean 3-space E^3 and one-dimensional Euclidean subspaces of the 5-dimensional pseudo-Euclidean vector space V^5 of index 1. Any spacelike vector defines the orthogonal complement of its span, being a 4-dimensional pseudo-Euclidean subspace, the isotropic vectors of which correspond to the points of a sphere S^2 contained in the Möbius space S^3. In the converse direction, the span of the isotropic vectors representing the points of such a sphere is a pseudo-Euclidean 4-space, defining its normal, an Euclidean 1-space.
Remark: Center and radius are metric concepts. From the viewpoint of Möbius geometry they serve as parameters only.

4.1. Spheres through Four Points and Corresponding Spacelike Vectors: vradius, vcenter, sphereplot3D

4.1.1. The Definitions of the Spacelike Vector Function sph4ptsvec

4.1.2. The Spacelike Vector as a Function of Center and Radius of the Sphere: spherevec, hspherevector

4.1.3. Some Tests. A Graphic Application.

4.1.4. Inversion of hspherevector: vradius, vcenter

4.1.5. Oriented Spheres

4.1.6. Parameter Representation of the Sphere Corresponding to a Spacelike Vector

4.2. Planes

We shortly consider some special aspects of the correspondence between planes and their space-like vectors, see the Corollary in section 3.1.

4.2.1. The Definition planevec

4.2.2. Random Planes

4.2.3. The Plane through Three Points

4.2.4. Planes through Three Random Points

4.3. The Sphere or Plane Corresponding to a Spacelike Vector

We used the Module euklidsphereplot3D already very often. In this subsection we explain its definition.

4.3.1. The Definition of the Parameter Representation euklidsphere

4.3.2. Examples

4.3.3. The Module euklidsphereplot3D

4.4. Examples, Comparision of Sphere Plot Methods

4.4.1. Example: Equidistant Unit Spheres

4.4.2. Example: The Spacelike Frame Vectors

4.4.3. Example : A Plane Bundle

4.4.4. Evaluation Problems: Floating Point Numbers

4.5. The Manifold of All Spheres, Random Spheres

4.5.1. The Sphere Manifold

4.5.2. A Random Spacelike Unit Vector

5. The Generalized Angle

In analogy to Euclidean geometry the generalized angle is defined applying the scalar product. It describes the basic invariant of two spheres. More generally there exist stationary angles between a k-sphere and an l-sphere forming a complete system of Möbius invariants for such pairs, see [OS3], Section 2.7. The case of two circles in the 3-sphere are considered in the *Mathematica* notebook Elementary Möbius Geometry II, and the remaining parings of subspheres of the three-dimensional Möbius space are treated in the notebook Elementary Möbius Geometry III.

5.1. Definition. Cases. Coxeter Invariant

5.3. Non-intersecting Spheres

5.4. Tangential Spheres

5.5. Intersection of a Sphere and a Plane

6. Geodesics of the Sphere Manifold

6.1. Introduction

In subsection 4.5.1 we identified the space of oriented spheres with the hypersurface H: pssp[x,x] = 1 of spacelike unit vectors in the pseudo-Euclidean vector space of index 1: a "pseudo-hypersphere". Since the pseudo-orthogonal group O(1,4) acts transitively on the pseudosphere, it is a pseudo-Riemannian symmetric space of constant scalar curvature. Since the origin as the center of H is fixed under the action of O(1,4) we define as in the sphere geometry: The geodesics of H are the intersections of the pseudo-hypersphere H with 2-planes through the center o. But now we have three kinds of such planes: pseudo-Euclidean, Euclidean, and isotropic, and the corresponding geodesics. We take the standard unit sphere stb[5] as starting point of the geodesics. To find the types of geodesic we may restrict to the linear hull[stb[1],stb[2]], as a subspace of the tangent space of this hyper-hyperboloid at stb[5]: look at the isotropy action, the action of the subgroup of the Möbius group preserving stb[5] on the tangential space of H at stb[5]. Obviously, this is the usual standard action of the pseudo-orthogonal group O(1,3) on the 4-dimensional pseudo-

Euclidean vector space of index 1.
In the next subsection we interpret the three types of geodesics geometrically.

6.2. Timelike Geodesics

6.2.1. Definitions and general considerations

6.2.2. Another Example

6.3. Spacelike Geodesics

6.4. Isotropic Geodesics

References

OS3

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Homepage

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