Pseudo - Riemannian Geometry and Tensor Analysis

by

Rolf Sulanke

Started February 1, 2015.
Version 1 finished May 20, 2016.
Version 2 started March 30, 2017,
finished July 10, 2017, revised November 8, 2017,
Mathematica v. 9.0.1,, 10.4.1, 11.1.1,11.2.
Version 3. Started December 8, 2017, Mathematica 11.2,
finished January 29, 2018.

Preface

In this notebook I develop and explain Mathematica tools for applications to Riemannian geometry and relativity theory. Together with the notebooks concerned with Euclidean geometry of curves and surfaces in [EDG] it may be used as an interactive introductory textbook of differential geometry. Clearly, these notebooks don't cover the full content of an introductory course on this field. Therein the applications of algebra and calculus to differential geometry, the algorithmic methods are emphasized, and the rich graphical tools of Mathematica are applied to illustrate the concepts in dimensions 2 and 3. In order to learn differential geometry, the notebooks will give best profits when using them accompanying an introductory course or to studying one of the numerous textbooks. Despite of this seemingly restrictive recommendation I tried to compose a coherent textbook containing the fundamental concepts of differential geometry understandable for users with a basic knowledge of calculus, linear and tensor algebra. The Mathematica tools of tensor algebra applied in this notebook can be found in the notebook vectensalg.nb included in the packed file RGv3.zip. Furthermore I hope that the fine structured presentation of the content, the tables of keywords and usages, and the open access to the program sources written as Mathematica packages will be useful for researchers and engineers who need to apply differential geometric methods in their work. This open access may also be useful for those striving to learn programming their own Mathematica functions or to test and, if possible, to improve the proposed programs.

The contents of the notebook can be described shortly as follows. Section 1, based on concepts and functions contained in the notebook euklsfv3.nb of Euclidean surface geometry, treats the Riemannian spaces of constant curvature. They serve as standard examples in the following sections.

Section 2 starts with the definition of differentiable manifolds and its tangent, cotangent, and tensor bundles. I introduce the calculus of differential forms and the corresponding Mathematica functions and operations which form the exterior algebra including the exterior differential operator, a calculus widely used in geometry, physics and topology. This exterior calculus exists on each differentiable manifold independently of any additional structure. Differential forms are antisymmetric covariant tensor fields, and only for such fields the exterior differential is defined. The only differential operator for vector fields existing in such generality is the commutator of two fields, treated in Subsection 2.4.9.

In Section 3 it is shown that the partial derivatives of the coordinates of a vector field can not be considered as the coordinates of a tensor. To get a gradient tensor or a differential of arbitrary tensor fields an additional structure must be given on the manifold, an affine or, more precisely, a linear connection. The theory of linear connections is developed in this Section, gradients and absolute differentials of tensor fields, torsion and curvature tensors of the linear connection are defined. The geometric background of this structure is the question whether there exist absolutely parallel tensor fields, whose gradient is a null tensor. Geodesics are defined as auto-parallel curves.

Section 4 is devoted to pseudo-Riemannian manifolds. The theorem of Levi-Civita is proved: On each such manifold there exists a uniquely defined linear connection with torsion tensor zero for which the absolute differential of the metric tensor is zero. The special properties of this Levi-Civita connection are deduced. Lowering and raising of indices yield canonical isomorphisms between tensor spaces of distinct covariance types of the same degree. Using the duality of vector and tensor spaces canonically defined scalar products in these spaces are introduced. The special properties of the Riemannian curvature tensor, the symmetries of its covariant form, the Bianchi identity are considered. The sectional curvature is introduced and the Gaussian curvature of surfaces is identified as its special case.

The second and higher versions of the notebook are - in difference to the first version - now separated from the elementary Euclidean differential geometry; it can be read and applied independently of these elementary part. All necessary packages and notebooks are collected in the packed file RGv3.zip.

Version 3, RGv3.zip, containes many corrections and some minor completions. The basic package is now eudiffgeov4.m being a tested and corrected union of eudiffgeov3.m with riemannv2.m. The notebook has been adapted to this new composed package and to version 11.2 of Mathematica. Since many definitions in the new version are changed or are new, the Mathematica packages of different versions should not be mixed; use exactly those contained in RGv3.zip. Some functions built-in Mathematica v. 11.2, but not contained in earlier versions, are used in the notebooks and packages of RGv3.zip. Thus RGv3.zip is not compatible with these earlier versions of Mathematica. It is tested with Mathematica v. 11.2 only.

I conclude this preface expressing my sincere thanks to the Humboldt University in Berlin for the continuous support giving me access to Wolfram's Mathematica and the possibility to continue my work also in my retired position.

Keywords

Copyright

Initialization

Imported Functions and their Usages

This Subsection contains tables of the symbols defined in the imported packages and in the Global context.

Hints

Symbol Tables. Usages.

Euclidean Differential Geometry, Linear Connections, and Riemannian Geometry.

Euclidean Linear Algebra

Tensor Algebra

Pseudo-Euclidean Linear Algebra

Alfred Gray's Catalogue of Curves and Surfaces

The Global Context

1. Riemannian Spaces of Constant Curvature

In this Section we introduce n-dimensional Riemannian metrics of constant curvature. The manifolds considered here are open, connected sets of the n-dimensional real vector space ℝ^n provided with Cartesian coordinates. More general manifold concept will be treated in Subsection 2.1.

1.1. Pseudo-Riemannian Metrics

1.1.1. A General pseudo-Riemannian Metric

1.1.2. Inner Geometry of an Immersion

1.1.3. The Inverse Metric Matrix

1.1.4. The Euclidean Metric

1.2. Riemannian Spaces of Constant Positive Curvature

1.3. Riemannian Spaces of Constant Negative Curvature

1.3.1. Spheres of imaginary Radius (n-Pseudo-Spheres).

1.3.2. Conformal Model of the hyperbolic n-spaces

---

2. Tensor Fields

In this and the following Sections we often use the functions and modules of tensor algebra contained in the package tensalgv3.m. To learn this matter I recommend first to study the notebook vectensalgv3.nb.

2.1. Manifolds

2.1.1. The Definition

2.1.2. The R^n and the n-Spheres S^n

2.1.3. The Category of Differentiable Manifolds

2.2. Tangent Bundle

2.2.1 Tangent Spaces

2.2.2. Cotangent Bundle

2.3. Vector and Tensor Fields on a Manifold

2.3.1. Definition of Tensor Fields

2.3.2. Functions as Tensor Fields of Type {0, 0}

2.3.3. Vector Fields

2.3.4. Directional Derivatives

2.3.5. Local Bases. Commuting Properties

2.4. Exterior Differential in Tensor Description

Mathematica functions modelling the exterior differential calculus for p-forms and antisymmetric covariant tensors are contained in the package eudiffgeov4.m. Their applications to p-forms described in the notebook euklsfv3.nb in [Button[$CellContext`EDG, Inherited, ButtonData -> "EDG"]]. The exterior differential is denoted by dd.
It remains to consider the exterior differential in the realm of tensors.

? dd

2.4.1. The Exterior Algebra of Antisymmetric Covariant Tensor Fields

2.4.2. The Exterior Differential of Antisymmetric Arrays in n Dimensions

2.4.3. The Exterior Differential of Antisymmetric Covariant Tensors

2.4.4. Linearity and Product Rules

2.4.5. Exterior Differential of 1-Forms and Commutator of Vector Fields

---

3. Absolute Differentials and Gradients of Vector and Tensor Fields

3.1. The Differentiation of Vector Fields

3.2. Linear Connections

3.2.1. Introduction

3.2.2. The Gradient of a Vector Field

3.2.3. The Absolute Differential of Functions, Vector, and Covector Fields

3.2.4. The Absolute Differential of a Tensor Field of Type (p, q)

3.2.5. The Absolute Differential, Modules absD, tensorD

3.2.6. Properties of the Absolute Differential

4.2.2. Curvature of Immersions. Examples

4.2.3. The Covariant Curvature Tensor

4.2.4. Duality and Tensor Metric

4.2.5. Sectional Curvature

Epilog

In this Epilog I mention some important subjects not treated in the notebooks; the user may continue the study of differential geometry in one or several of the mentioned directions and elaborate Mathematica functions for the corresponding calculations. The references given in the epilog as well as in the Notebooks are very subjective and in no way complete. Please use the mathematical databases Mathematical Reviews or Zentralblatt to find the rich actual literature. Furthermore I recommend to look into the Wolfram Library Archive to find information on Mathematica projects and packages for the interested item. In particular, search there for "Tensor" tom find information and sources on work based on Mathematica, e. g. MathTensor, H. Soleng, José M. Martín-García, and others.

1. Relativity Theory

The space-time world of special relativity theory is the Lorentz space: the four-dimensional pseudo-Euclidean vector space of index 1, and that of the general relativity theory is a 4-dimensional pseudo-Riemannian manifold of index 1. Thus the Mathematica concepts contained in this notebook and the accompanying packages may serve as a starting point to go ahead into this field. There exists a lot of applications of Mathematica to this field. Many concrete calculations cam be found in the work of Sortirios Bonanos who unfortunately already died in 2013. Further see the commercial package Cartan of Harald. H. Soleng, the package ccgrg of Andrzej Woszczyna and others.

2. Lie Groups and Lie Algebras

The theory of Lie groups and algebras is needed for a wide field of developments in differential geometry, geometric analysis, and algebra with important applications to physics. There are some special programs written in Mathematica, e. g. the work LieArt of Robert Feger and Thomas w. Kephart, or my old package, but a systematic treatment of this theory in Mathematica concepts seems to be missing yet. An interactive textbook developed along the lines of S. Helgason's book would be an important step in using Mathematica methods in transformation groups, fiber bundles and holonomy theory, homogeneous, in particular symmetric spaces.

3. Submanifolds of Homogeneous Spaces with Cartan Methods

Here I have in mind less the general theory but the investigation of special problems in projective, affine, symplectic or Möbius spaces. For these fields there exists an overwhelming large number of publications, but not so much is done, outside of Euclidean geometry, with Mathematica. Just appeared the book of G. R. Jensen, E. Musso, and L. Nicolodi which could stimulate interesting Mathematica projects in this field. Clearly there are many further subjects which could be mentioned, like complex geometry, geometric theory of differential equations, Hamiltonian equations and calculus of variations, but to discuss them here would break the limits of this notebook.

I will be very grateful for all comments, critical remarks or hints to errors. Reports on experiences of using the notebooks in teaching differential geometry are welcome. Please, write me an e-mail to sulanke@-mathematik.hu-berlin.de.

Berlin. January 22, 2018.
Rolf Sulanke

References

Homepage

The newest versions and all my public Mathematica notebooks and packages can be downloaded from my homepage

http://www-irm.mathematik.hu-berlin.de/~sulanke

I will be grateful for all feedback, in particular for hints to errors, or reports about problems, failures, or success at work with my packages or notebooks. Please write to

e-mail: sulanke@mathematik.hu-berlin.de