Curves of Constant Curvatures in Möbius Geometry

Dedicated to the memory of Alfred Gray

by Rolf Sulanke

Starting date April 13, 2010
Finished December 7, 2010

Introduction

In this notebook we apply *Mathematica* to describe the curves of constant curvatures in the three-dimensional Möbius geometry. The calculations and examples illustrate and complete the presentation of the subject in my not yet finished book [MDG]. In our book [OS3] one finds the foundations of elementary Möbius geometry in terms of linear algebra; also hyperbolic and elliptic geometry as used in this notebook are contained there.

The classical work treating Möbius differential geometry is W. Blaschke & G. Thomsens book [BT]. A contemporary presentation of this field is given by Udo Hertrich-Jeromin in his book [H- J]. There one can find more literature and some aspects of the history of this field. This interesting book accumulates a lot of material of the subject, but it contains nothing about the Möbius differential geometry of curves. In my paper [MGII] I treated this matter by E. Cartan's method of moving frames. The curves of constant curvatures are classified there solving a differential equation. In the presentation [MDG] and in chapter 2 below I applied the classification of 1-parameter subgroups of the Möbius group to describe all the curves of constant curvatures in the 3-dimensional Möbius space. In the present notebook the theory as described in [MDG] is applied and illustrated. Some calculations not contained in [MDG] are performed only here. I recommend to look at chapter 3 of [MDG]; this notebook can be considered as a part of that chapter. A special role play the circles and lines in Möbius geometry. These curves are locally Möbius equivalent; they form a Cartan class of immersions for which local invariants like curvatures do not exist. Of course they are also orbits of 1-parametric subgroups. Under a curve of constant curvatures in the Möbius space we always understand a generally curved curve with constant curvatures.

Any 1-parametric subgroup whose orbit is a curve of constant curvatures can be completed to a uniquely defined connected 2-dimensional maximal abelian subgroup of the Möbius group, see chapter 3 below. The orbits of these subgroups are the Dupin cyclides. The Dupin cyclides can be characterised as surfaces being envelopes of 1-parametric sphere families in a twofold way: There exist two distinct such families having the same envelope. They appear as the main part of homogeneous surfaces in the 3-dimensional Möbius space, see [MGV]. The characteristics of the generating sphere families form two families of circles (or lines) each generating the cyclide. These families are transformed into themselves by the 1-parametric subgroup generating the curve of constant curvature defining the cyclide; since the transformations are conformal the curves of constant curvatures are isogonal trajectories of the circle families. A well known example are the helices on the circle cylinder in Euclidean geometry. In dimension 3 the classification in chapter two shows that any curve of constant Möbius curvatures is Möbius equivalent to a curve of constant Riemannian curvatures in an Euclidean, hyperbolic or elliptic space, whose Riemannian geometries are subgeometries of the Möbius space: their isometry groups are subgroups of the Möbius group.

For the application of *Mathematica* to Euclidean differential geometry we mention the pioneering work of Alfred Gray, who introduced me into *Mathematica,* see [G06]. A short presentation of n-dimensional Euclidean curve theory is given in my paper [ECG] which together with a corresponding notebook EuCurves.nb may be downloaded from my homepage. From there one may download also other notebooks treating Möbius elementary geometry, pseudo-Euclidean geometry and Lie algebras.

I hope that the present notebook may serve as a good example of applying *Mathematica* to differential geometry. Of special interest are the many symbolical calculations carried out here. This and the graphical and numerical tools of *Mathematica* have been very useful in exploring the subject.

■ **Keywords**

Möbius group, curves of constant curvatures, space forms (conformal models), Dupin cyclides, cone, isotropic-orthogonal coordinates, pseudo-orthonormal coordinates, spherical reflections, 2D-spirals, spiral cylinder, stereographic projection, pseudo-Euclidean space

■ **Copyright**

Initialization

Before starting to work interactively with this notebook first time, read this section carefully and make the necessary preparation. Later it suffices to call the menu item "Evaluation. Evaluate Initialization Cells".

For working with the notebook you need the packages
euvec.m, vectorcalc.m, neuvec.m, eudiffgeo.m, mdg.m.
You may download these and other packages from my Homepage . From there you find all the packages mentioned above in the file moebpack.tgz.
Before initializing the notebook ensure that these packages are laying in a directory of your

■ **The Initialization**

1. List of Symbols and their Usages

In this section one finds tables of the symbols introduced in the imported packages and in the Global Context.
To get the usages click on the name! If this does not work,enable Dynamic Updating in the Evaluation Menu.

■ **1.1. Symbols in the Package vectorcalc.m**

■ **1.2. Symbols in the Package euvec.m**

■ **1.3. Symbols in the Package eudiffgeo.m**

■ **1.4. Symbols in the Package neuvec.m**

■ **1.5. Symbols in the Package mdg.m**

■ **1.6. Symbols in the Global Context**

2. Space Curves of Constant Curvatures

In this chapter we describe the curves of constant curvatures as orbits of 1-parameter subgroups of the Möbius group, see section 3.2 of [MDG]. There is shown that any curve of constant Möbius curvatures k, h is Möbius equivalent to an orbit of the 1-parametric subgroup g(t) = exp[cc[k,h]t] of the Möbius group. In section 2.1 we calculate these subgroups, what leads to a very large complex formula, hardly to work with manually. Nevertheless, by the help of *Mathematica* we may classify and visualize the curves of constant curvatures as shown in the subsequent sections of this chapter. Since equivalent curves (under the action of the Möbius group) correspond to conjugated subgroups we obtain at the same time the classification of the 1-parametric subgroups of the Möbius group of S^3 in conjugacy classes.

■ **2.1. The 1-Parametric Subgroups of the Möbius Group Generate the Curves of Constant Curvatures**

■ **2.2. Classification and Properties of the 1-Parameter Subgroups**

■ **2.3. The Eigensystem of cc[k,h]**

2.4. Case 3, the Euclidean Case chB=0

- **2.5. Case 2, the Hyperbolic Case chB>0**
- **2.6. Complete Systems of Representatives in the Hyperbolic Case**
- **2.7. The Elliptic Case chB<0**
- **2.8. Transformation between the Euclidean and the Möbius Space**
- **2.9. Lines. The Conformal Representation of the Translation Group**
- **2.10. Circles. The Conformal Representation of the Rotation Group.**

---

3. Dupin Cyclides

The Dupin cyclides are the orbits of the two-parametric Abelian subgroups of the Möbius space, see [MGV]. In section 3.1 we start with the element cc[k,h] of the Lie algebra and complete it to a maximal Abelian Lie subalgebra of the Lie algebra of the Möbius group. In section 3.2 we consider the corresponding two-dimensional Abelian subgroups and apply the classification obtained in chapter 2. In each case we plot a corresponding two-dimensional orbit and a curve of constant curvatures k, h lying in it.

- **3.1. Two-Parametric Abelian Subgroups of the Möbius Group**
- **3.2 The Dupin Cyclides as Orbits of 2-Dimensional Commutative Subgroups**
- **3.3. The Euclidean Realization of the Hyperbolic Curves of Constant Curvatures**

References

Homepage

Home

http://www- irm.mathematik.hu- berlin.de/~sulanke