Curves of Constant Curvatures in Möbius Geometry

Dedicated to the memory of Alfred Gray
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Introduction

In this notebook we apply Mathematica to describe the curves of constant curvatures in the three-dimensional Möbius geometry. The calculations and examples illustrate and complete the presentation of the subject in my not yet finished book [MDG]. In our book [OS3] one finds the foundations of elementary Möbius geometry in terms of linear algebra; also hyperbolic and elliptic geometry as used in this notebook are contained there.

The classical work treating Möbius differential geometry is W. Blaschke & G. Thomsens book [BT]. A contemporary presentation of this field is given by Udo Hertrich-Jeromin in his book [H-J]. There one can find more literature and some aspects of the history of this field. This interesting book accumulates a lot of material on the subject, but it contains nothing about the Möbius differential geometry of curves. In my paper [MGII] I treated this matter by E. Cartan's method of moving frames. The curves of constant curvatures are classified there solving a differential equation. In the presentation [MDG] and in chapter 2 below I applied the classification of 1-parameter subgroups of the Möbius group to describe all the curves of constant curvatures in the 3-dimensional Möbius space. In the present notebook the theory as described in [MDG] is applied and illustrated. Some calculations not contained in [MDG] are performed only here. I recommend to look at chapter 3 of [MDG]; this notebook can be considered as a part of that chapter. A special role play the circles and lines in Möbius geometry. These curves are locally Möbius equivalent; they form a Cartan class of immersions for which local invariants like curvatures do not exist. Of course they are also orbits of 1-parametric subgroups. Under a curve of constant curvatures in the Möbius space we always understand a generally curved curve with constant curvatures.

Any 1-parametric subgroup whose orbit is a curve of constant curvatures can be completed to a uniquely defined connected 2-dimensional maximal abelian subgroup of the Möbius group, see chapter 3 below. The orbits of these subgroups are the Dupin cyclides. The Dupin cyclides can be characterised as surfaces being envelopes of 1-parametric sphere families in a twofold way: There exist two distinct such families having the same envelope. They appear as the main part of homogeneous surfaces in the 3-dimensional Möbius space, see [MGV]. The characteristics of the generating sphere families form two families of circles (or lines) each generating the cyclide. These families are transformed into themselves by the 1-parametric subgroup generating the curve of constant curvature defining the cyclide; since the transformations are conformal the curves of constant curvatures are isogonal trajectories of the circle families. A well known example are the helices on the circle cylinder in Euclidean geometry. In dimension 3 the classification in chapter two shows that any curve of constant Möbius curvatures is Möbius equivalent to a curve of constant Riemannian curvatures in an Euclidean, hyperbolic or elliptic space, whose Riemannian geometries are subgeometries of the Möbius space; their isometry groups are subgroups of the Möbius group.
For the application of Mathematica to Euclidean differential geometry we mention the pioneering work of Alfred Gray, who introduced me into Mathematica, see [G06]. A short presentation of n-dimensional Euclidean curve theory is given in my paper [ECG] which together with a corresponding notebook EuCurves.nb may be downloaded from my homepage. From there one may download also other notebooks treating Möbius elementary geometry, pseudo-Euclidean geometry and Lie algebras.

I hope that the present notebook may serve as a good example of applying Mathematica to differential geometry. Of special interest are the many symbolical calculations carried out here. This and the graphical and numerical tools of Mathematica have been very useful in exploring the subject.

- **Keywords**
  - Möbius group, curves of constant curvatures, space forms (conformal models), Dupin cyclides, cone, isotropic-orthogonal coordinates, pseudo-orthonormal coordinates, spherical reflections, 2D-spirals, spiral cylinder, stereographic projection, pseudo-Euclidean space

- **Copyright**

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**Initialization**

Before starting to work interactively with this notebook first time, read this section carefully and make the necessary preparation. Later it suffices to call the menu item "Evaluation. Evaluate Initialization Cells".

- **The needed packages**

  For working with the notebook you need the packages euvec.m, vectorcalc.m, nevec.m, eudiffgeo.m, mdg.m.

  You may download these and other packages from my Homepage. From there you find all the packages mentioned above in the file mdgpack.zip. download.

  Before initializing the notebook ensure that these packages are laying in a directory of your choice, e.g. ~/mathpack, in your

```
In[1]:= $Path
```

If necessary, insert the addresses of your package- and your working directory into the next cells which correspond to your operating system:

**For Windows:**

```
In[2]:= PrependTo[$Path, "E:\mymath\mathpack"];
```

```
In[3]:= SetDirectory["E:\mymath\diffgeo\mdg"];
```

**For Linux:**

```
PrependTo[$Path, "/mymath/mathpack"];  
```

```
SetDirectory["~/mymath/diffgeo/mdg"];  
```

```
Directory[]
```

Now give the cells corresponding to your operating system the properties "Cell Evaluatable" and "Initialization cell" (Menu Cell/Cell properties), and inactivate these properties for the cells corresponding to the other operating system. If this is done, save the notebook. Next time you may start the notebook directly with the evaluation of the initialization, as follows:

- **The Initialization**

  Before starting to work interactively with the notebook
**Activité Evaluation/Evaluate Initialization from the menu.**

The first command loads the packages mentioned above:

In[4]:= Needs["mdg`"]

In this notebook we shall consider the 3-dimensional Möbius space as the elliptical quadrik, the 3-dimensional sphere, in the 4-dimensional real projective space. Therefore we set the dimension dim and the index ind of the vector space as follows:

In[5]:= dim::usage = "dim is the dimension of the vector space under consideration; it must be set within the Global Context.";

? dim

In[6]:= dim = 5;

In[7]:= ind::usage = "ind is the index of the pseudo-Euclidean vector space under consideration; it must be set within the Global Context.";

In[8]:= ind = 1;

- Examples:

- $\$Assumptions

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1. **List of Symbols and their Usages**

In this section one finds tables of the symbols introduced in the imported packages and in the Global Context. To get the usages click on the name! If this does not work, enable Dynamic Updating in the Evaluation Menu.

- **1.1. Symbols in the Package vectorcalc.m**

In[10]:= ? vectorcalc`*

```
vectorcalc`*

- basis matrix outzero randomv stb
  - b1f noprops projpt rank trp
  - dotnorme null projpt3D renorm vec
  - dv nullmatrix randommatrix smoothing wedge
```
1.2. Symbols in the Package euvec.m

In[11]:= ? euvec`*

```
    euvec``
    center3pts	 normed	 sphericalreflection
    circle3D	 orthocomplement	 spherpt3D
    circle3pts	 plotcircle3D	 sterproj
    cross	 plotcircle3pts	 subsphere
    esorthonorm	 plotgencircle	 subspheremf
    gencircle	 posvec3pts	 tg4frame
    hyperplane	 print	 tube
    innerprod	 radius3pts	 unit
    invsterproj	 sph3D	 unitvec
    neglect	 sphere
    norm	 sphereplot3D
```

1.3. Symbols in the Package eudiffgeo.m

In[12]:= ? eudiffgeo`*

```
eudiffgeo``
```
    arc	 circ2D	 curvatures	 spiral	 torsion
    arclength	 circle2D	 frenet	 spirgr
    assu	 curvature	 graph	 tangent
```

1.4. Symbols in the Package neuvec.m

In[13]:= ? neuvec`*

```
    neuvec``
    ch	 ide	 orthonorm	 pscross	 psgram
    chsort	 indexorder	 orthopair	 psCross	 pssp
    dual	 normalize	 pr	 pssfilter
```

Examples

1.5. Symbols in the Package mdg.m

In[14]:= ? mdg`*

```
    mdg``
    dimm	 io	 mfre3D	 transio
    indd	 iob	 sphmap	 transoi
```

```
mdg`dimm
```

```
mdg`dimm = dim
```

Examples

1.6. Symbols in the Global Context

Evaluate the next cell to get the actual list of the symbols in the Global context.
2. Space Curves of Constant Curvatures

In this chapter we describe the curves of constant curvatures as orbits of 1-parameter subgroups of the Möbius group, see section 3.2 of [MDG]. There is shown that any curve of constant Möbius curvatures \(k, h\) is Möbius equivalent to an orbit of the 1-parametric subgroup \(g(t) = \exp[cc[k,h]t]\) of the Möbius group. In section 2.1 we calculate these subgroups, what leads to a very large complex formula, hardly to work with manually. Nevertheless, by the help of Mathematica we may classify and visualize the curves of constant curvatures as shown in the subsequent sections of this chapter. Since equivalent curves (under the action of the Möbius group) correspond to conjugated subgroups we obtain at the same time the classification of the 1-parameter subgroups of the Möbius group, see section 3.2 of [MDG].
parametric subgroups of the Möbius group of $S^3$ in conjugacy classes.

- 2.1. The 1-Parametric Subgroups of the Möbius Group Generate the Curves of Constant Curvatures
- 2.2. Classification and Properties of the 1-Parameter Subgroups
- 2.3. The Eigensystem of $cc[k,h]$
- 2.4. Case 3, the Euclidean Case $chB=0$
- 2.5. Case 2, the Hyperbolic Case $chB>0$
- 2.6. Complete Systems of Representatives in the Hyperbolic Case
- 2.7. The Elliptic Case $chB<0$
- 2.8. Transformation between the Euclidean and the Möbius Space
- 2.9. Lines. The Conformal Representation of the Translation Group

3. Dupin Cyclides

The Dupin cyclides are the orbits of the two-parametric Abelian subgroups of the Möbius space, see [MGV]. In section 3.1 we start with the element $cc[k,h]$ of the Lie algebra and complete it to a maximal Abelian Lie subalgebra of the Lie algebra of the Möbius group. In section 3.2 we consider the corresponding two-dimensional Abelian subgroups and apply the classification obtained in chapter 2. In each case we plot a corresponding two-dimensional orbit and a curve of constant curvatures $k, h$ lying in it.

- 3.1. Two-Parametric Abelian Subgroups of the Möbius Group
- 3.2 The Dupin Cyclides as Orbits of 2-Dimensional Commutative Subgroups
- 3.3. The Euclidean Realization of the Hyperbolic Curves of Constant Curvatures

References

Homepage

http://www-irm.mathematik.hu-berlin.de/~sulanke