

Surfaces in the Euclidean 3-Space

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Preface

In this notebook we develop Mathematica tools for applications to Euclidean differential geometry of surfaces. We construct modules for the calculation of all Euclidean invariants like fundamental forms and curvatures for surfaces in the 3-space, and some also for immersions into the n -dimensional Euclidean spaces. This notebook may be considered as a continuation of the notebook `EuCurves.nb` devoted to Euclidean curve theory which may be downloaded from my homepage. It may be used as an interactive introduction into the Euclidean surface geometry, accompanying an introductory course or reading a textbook, e.g. the Introduction [B-R] of W. Blaschke and H. Reichardt which I often use as a reference. It may be useful to compare and to complete this notebook studying the pioneering work of Alfred Gray on applying Mathematica to differential geometry, see the book [G06], or the German translation of the first edition [G94]. The third edition [G06] is accompanied by a series of 27 Mathematica notebooks corresponding to the chapters of the book, which can be downloaded from the website [G06nb]. Alfred Gray presented Euclidean differential geometry with many applications of Mathematica. Writing his book Alfred Gray collected about 200 definitions of concrete curves and as many of concrete surfaces with short descriptions (formulated as usages) and implementations into Mathematica. These catalogs are contained in his Mathematica packages `CURVES.m` and `SURFS.m` available from my homepage. They open many opportunities to apply the functions and miniprograms developed here, to deepen the insight into the differential geometry of curves and surfaces comparing the calculated invariants, e. g. the curvatures, with the shape of the geometric objects using the graphical tools of Mathematica. I am much obliged to Alfred Gray for providing this material and for many discussions introducing me into Mathematica.

Chapter 1 of the notebook is devoted to E. Cartan's method of moving frames using exterior differential forms. It gives the structure forms and the Frenet formulas including the integrability conditions for surfaces in the Euclidean 3-space. Here I followed the presentation in [B-R] using the differential forms ad hoc as elementary as possible. Also the first and the second fundamental forms of surface theory are treated as fields of bilinear forms not using concepts of tensor analysis. Spherical image and Gauss curvature are the subject of Chapter 2 containing a proof of Gauss famous Theorema Egregium. Chapter 3 is devoted to the second fundamental form of the surface, studying the geometry of curves on the surface, in particular their normal curvature. Chapter 4 contains the absolute differentiation on the surface, expressed by the Christoffel symbols as a part of the inner geometry of the surface, invariant under isometric deformations. It is applied to describe the geodesic curvature of a curve on the surface and to define the geodesics as auto-parallel curves. The basic con-

cepts are formulated avoiding the tensor concept, but admitting a straightforward extension to higher dimensional Riemannian and more general geometries, which can be the contents of further Mathematica notebooks.

I hope that the notebook will help students to understand the Euclidean differential geometry of surfaces. By its interactive character it may serve as a starting point for further excursions into this interesting field, which joins algebraic and analytic methods to explore the heart of mathematics: the geometry. Studying older or modern textbooks or monographs of classical differential geometry it may be an exciting challenge to express the objects and concepts in Mathematica terms aiming to explore or illustrate their properties, to verify well known facts, and, with some luck, to gain new insight.

I shall be very thankful for comments and corrections, please send them to sulanke@mathematik.hu-berlin.de.

Important Hint.

Before starting working with the notebook read the Copyright and make the necessary adaption of the Section Initialization to your computer!

Keywords

absolute differential, absolute parallel, arc length, area, asymptotic curves, Christoffel symbols, cone, conical surface, cylindrical surface, developable surface, diffeomorphism, differential, envelope, Euler's formula, evolute, exterior differential, exterior product, flat points, Frenet formulas, fundamental forms, Gauss curvature, Gauss map, geodesic, geodesic curvature, graph, helix, hypersurface, immersion, integrability conditions, integral curvature, involute, isogonal trajectory, Jacobian matrix, line congruences, loxodrome, mean curvature, Meusnier's proposition, minimal surface, moving frame, normal, normal congruence, normal curvature, orientation, osculating sphere, parallel displacement, parameter, Plücker's conoid, principal curvatures, principal curves, principal directions, regular, Riemannian curvature tensor, Riemannian manifold, Rückkehrkante, ruled surface, singular, spherical image, spiral, structure forms, submanifold, surface, surface of revolution, surface with constant Gauss curvature, tangent plane, tangent surface, tangent vector, Theorema egregium, torus, total differential, umbilic point, vector field, volume element.

Copyright

Initialization

Before starting the first session insert the address of your working directory, e.g. for Windows 7:

```
SetDirectory["E:\\mymath\\diffgeo\\Euc1SfTheoryv2"]
```

For Linux:

```
SetDirectory["~/mymath/diffgeo/Euc1SfTheoryv2"]
```

Regard eventually deviating notations depending on your operating system! Ensure that the corresponding cell is an initialization cell, and the other cell doesn't have this property (Menu Cell/ Cell Properties).

\$Path

Copy all the needed Mathematica packages into your working directory, or a directory of \$Path.

Obligatory are the packages

tensalg.m, euvec.m, eudiffgeov2.m, Alfred Gray's catalog SURFS.m, and Alfred Gray's catalog CURVES.m.

All these packages together with this notebook are contained in the file euclidsurfacesv2.tgz which can be downloaded from my homepage.

Now save the notebook with your changes.

Start Initialization

The Symbols and their Usages

The next Subsections contain tables of the symbols defined in the package named in the title. Click the symbol (or use ?symbol) to read the usage: a short description of the content and the syntax of the function called with the symbol, e.g.

?vec

The last method may be applied to all built-in symbols of Mathematica too, e.g.

?D

Click Button[StyleBox[">>", "SR"], Inherited, Active -> True, BaseStyle -> "Link", ButtonData -> "paclet:ref/D"] to read the corresponding help page. To see the explicit definition use

?? d

I recommend to look into the package and see the definition without the prefixed context, here eudiffgeov2`Private`.

tensalg.m

euvec.m

eudiffgeov2

SURFS.m

CURVES.m

Global

I. The Local Structure Forms of a Surface

I.1. Parameter Representation and Differential

I.2. An Orthonormal Moving 3-Frame of the Surface

I.2.1. The Definition

I.2.2. Example: Plane

I.2.3. Metric Invariants. First Fundamental Form

I.2.4. Isometries. Inner Geometry

I.2.5. The Adapted Moving Frame

I.3. Differential Forms

In the following we will use Cartan's method of moving frames. We apply the calculus of differential forms on the parameter manifold U in an elementary form in dimension 2 avoiding tensor algebra, as it is presented in the Introduction of W. Blaschke and H. Reichardt [B-R].

I.3.1. The Differential of a 0-Form (= Function)

I.3.2. Sum and Product Rules

I.3.3. Exterior Product

I.3.4. The Exterior Differential

I.4. The Derivation Equations

I.4.1. The Orthonormal Cobasis $\{\sigma[1], \sigma[2]\}$

I.4.2. Derivation of the Orthonormal Frame

I.5. The Integrability Conditions

I.5.1. Summary

I.5.2. Proof of the First Three Integrability Conditions

1.5.3. Proof of the Last Three Integrability Conditions

1.6. The Local Uniqueness Theorem

2. Spherical Image and Gaussian Curvature

2.1. The Gauss Map

2.1.1. Orientation and Spherical Image

2.1.2. Ruled Surfaces

2.1.3. Circular Cones

2.1.4. Conical Surfaces

2.1.5. Cylindrical Surfaces

2.1.6. Tangent Surfaces

2.1.7. Developable Surfaces I

2.1.8. One-Sheeted Hyperboloids

2.2. The Gaussian Curvature

2.2.1. Integral Curvature and Curvature Integral. Orientation

2.2.2. Theorema Egregium

2.2.3. Developable Surfaces 2

3. The Second Fundamental Form

3.1. The Definition

3.2. Gaussian Curvature and Second Fundamental Form

3.3. An Equivalent Definition of the Second Fundamental Form

3.4. The Curvature Vector

3.4.1. Curves on Surfaces

3.4.2. Angles

3.4.3. Curvature

3.5. Normal and Principal Curvatures

3.6. Graphs of Functions

3.7. Ruled Surfaces: Criterion of Developability

3.8. The Normal Congruences

3.8.1. Normal Congruence of a Regular Curve

3.8.2. Normal Congruence of the Helix

3.8.3. Normal Surfaces of a Surface

3.9. Surfaces of Revolution

Any body produced on a lathe or a potter's wheel is bordered by a surface of revolution. Such surfaces are not only easy to produce; they may serve also as easy to handle examples in surface geometry. The reason is that the geometric properties of such a surface are defined by the properties of its generating curve, the profile curve or meridian of the surface. Thus two-dimensional problems for surfaces can be reduced to one-dimensional problems for curves, e.g. partial differential equations for surfaces to ordinary differential equations for the meridians, which are much easier to solve. In this section we consider some of these applications.

3.9.1. Definition and Fundamental Invariants

3.9.2. Minimal Surfaces of Revolution

3.9.3. Surfaces of Revolution of Constant Gaussian Curvature

3.9.4. Surfaces of Revolution of Constant Positive Gaussian Curvature

3.9.5. Surfaces of Revolution of Constant Negative Gaussian Curvature

3.9.6. Alfred Gray's Exposition

3.9.6.a. Surfaces of Revolution of Constant Positive Gaussian Curvature

3.9.6.b. Surfaces of Revolution of Constant Negative Gaussian Curvature

4. Absolute Differential and Christoffel Symbols

In general, the differential and the derivatives of a tangent vector field on a surface are not tangent to the surface. The tangential component of the differential of the vector field is called its absolute differential; we will calculate it in this section. The calculation leads to the Christoffel symbols which depend on the metric of the surface only and so belong to the inner geometry. These concepts play a fundamental role in the n-dimensional differential geometry of Riemannian manifolds and in Einstein's general relativity theory. Therefore we begin this section considering m-dimensional submanifolds as immersions of m-dimensional parameter manifolds into the n-dimensional Euclidean space.

4.1. The First Fundamental Form of an Immersion. Riemannian Metric

4.2. Absolute Differential of Tangent Vector Fields

We consider a general immersion $smf[f[u\dots]]$ into the n-dimensional Euclidean space, $n = \dim$, e.g.

? smf

Example.

$\dim = 4; smf[f][u, v, w]$

We fix the necessary assumptions and notations: The immersion is $smf[f][x[1], \dots, x[m]]$.

The dimension of the parameter manifold is $mf = \text{Length}[\{x\}]$. For the symbolic calculation we fix

$\text{Clear}[\dim, f]; \dim = 4; mf = 2; smf[f][x[1], x[2]]$

4.2.1. The Total Differential of a Tangent Vector Field

4.2.2. The Absolute Differential of a Tangent Vector Field

4.2.3. Absolute Parallel Vector Fields

4.2.4. Parallel Displacement Along Curves

4.2.5. Geodesics as Auto-Parallel Curves

4.3. Geodesic Curvature

Already in Section 3.4 we introduced the geodesic curvature of a curve on the surface $sf[u;v]$:

$cx[s_] := sf[hu[s], hv[s]]; \gamma[s_] := \{hu[s], hv[s]\}$

as the norm of the tangential projection of its curvature vector. We name it the "outer" definition of the geodesic curvature, since the embedding of the curve into the Euclidean space is explicitly

used. In this subsection we will give the "inner" definition of the geodesic curvature which is the norm of the absolute derivative of its tangent unit vector with respect to its arc length. This definition uses the Riemannian metric of the surface only and can be easily generalized to curves in Riemannian spaces. We show that both definitions yield the same geodesic curvature In the case of curves on surfaces in the Euclidean 3-space.

4.3.1. The Inner Definition of the Geodesic Curvature

4.3.2. Circles on the Sphere

4.3.3. General Curves on the Unit Sphere

4.3.4. The Loxodromes

4.3.5. Another Curve on the Sphere

4.3.6. Further Examples

References

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