

Elementary Möbius Geometry III

Pairs of Subspheres in S^3

by
Rolf Sulanke

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Introduction

This notebook continues the notebooks [emg1] and [emg2] about elementary Möbius geometry. We consider here those pairs of subspheres of the Möbius space being not treated in these notebooks under the action of the Möbius group; the titles of sections 2 - 5 show the pairings considered here. Pairs of spheres are treated in the notebook [emg1], and pairs of circles in [emg2]. The last section treats some examples of geodesics in the space of 0-spheres. In this section some concepts of Lie algebras are needed. They are contained in the package `liealgsh.m`, added to `moebpack.tar.gz`. Again the three-dimensional case is emphasized. The calculations and examples illustrate and complete the presentation of the subject in our book [O-S], Section 2.7, or the paper [S4]. There one finds the foundations of elementary Möbius geometry in terms of linear algebra. See also the Introductions to the notebooks mentioned above.

Our consideration are based on the linear algebra of n -dimensional pseudo-Euclidean vector spaces of index 1, sometimes named Minkowski spaces. The tools enhancing *Mathematica* for applications in this field are contained in the package `neuvec.m`, which needs the packages `vectorcalc.m` and `euvec.m` containing useful modules for working in elementary geometry and linear algebra. A notebook “`pseuklid.nb`” about pseudo-Euclidean geometry is contained in the packed files `sphs4.tgz` or `sphs4.zip` on my homepage, see [S1]. In Chapter 1 below the usages of the new Modules and Functions defined in the loaded packages and in the Global Context can be seen after evaluating the initialization cells of the notebook. Loading the package `mpairs.m` containing functions applied to the Möbius geometry of pairs the other packages already used in Elementary Möbius Geometry I, II are loaded automatically. All the packages needed to work with this notebook are contained in the packed file `moebpack.tar.gz`; download it and unzip the files to your working directory.

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Keywords

Möbius group, isotropic cone, pseudo-orthonormal coordinates, spheres, circles, point pairs, pairs of subspheres, pseudo-Euclidean space, pseudo-orthogonal Lie algebra, Killingform, Möbius invariants of pairs of subspheres, Möbius invariant expressed by Euclidean invariants, geodesics of the 0-spheres manifold.

Initialization

1. List of Symbols and their Usages

In this Chapter one finds tables of the symbols introduced in the imported packages and in the Global Context.

To get the usages click on the name! If this does not work correctly, enable Dynamic Updating in the Evaluation Menu.

- 1.1. Symbols in the Package `vectorcalc.m`
 - 1.2. Symbols in the Package `euvec.m`
 - 1.3. Symbols in the Package `neuvec.m`
 - 1.4. Symbols in the Package `mspher.m`
 - 1.5. Symbols in the Package `mcirc.m`
 - 1.6. Symbols in the Package `moeb.m`
 - 1.7. Symbols in the Package `mpairs.m`
 - 1.8. Symbols in the Global Context
-

2. Spheres and circles

A pair (S, C) , consisting of a sphere S and a circle C of the 3-dimensional Euclidean space or the 3-dimensional sphere, is defined up to a Möbius transform by exactly a single invariant, if C is not contained in S ("general position"). It is easy to show that all pairs (S, C) with $C \subset S$ are Möbius equivalent; indeed, this follows from the transitivity of the action of the Möbius group of the sphere S^2 on the manifold of all circles contained in S^2 . In the first subsection we calculate this invariant `moebsc`, then we give some examples, and finally we derive an expression of `moebsc` in terms of Euclidean invariants of the pair (S, C) .

- 2.1 Definition of the Invariant `moebsc`
 - 2.2 Examples
 - 2.3 Euclidean Interpretation of `moebsc`
-

3. Spheres and Point Pairs

- 3.1. Point Pairs as 0-Spheres
 - 3.2. The Invariant `moeb spp` and the Mutual Position
 - 3.3. An Expression of the Invariant `moeb spp` in Euclidean Terms
 - 3.4. Orthogonality
-

4. Circles and Point Pairs

In this section we consider pairs (C, pp) consisting of a Circle $C = S_1^1$ and a point pair $pp = (a_1, a_2) = S_2^0$ contained in the Euclidean 3-space or the 3-sphere. Such a pair is said to be in general position, if the union $C \cup pp$ is not

contained in a 2-sphere. Then its mutual position is defined up to Möbius equivalence by two invariants, which we are going to calculate. We study the relation between the values of these invariants and the geometry of the pair. In the last subsection we find formulas expressing the invariants in terms of the Euclidean invariants of the pairs.

- **4.1. The invariants**
- **4.2. Examples**
- **4.3. The Separation Property**
- **4.4. Two Associated Spheres Belonging to a Pair (S_1^1, S_2^0)**
- **4.5. Möbius Invariants Expressed by Euclidean Invariants**

5. Point Quadrupels

We consider pairs of 0-spheres, i.e. point quadrupels $\{\{a_1, a_2\}, \{b_1, b_2\}\}$. Since each point quadrupel belongs to a sphere, we could consider spherical Möbius geometry: $\dim = 4$, but for later applications we will continue with $\dim = 5$ here. Since the Möbius group of the sphere coincides with the broken linear transformations in one complex variable of the Riemann sphere, and this is the projective complex line, an invariant of the quadrupel coincides with the complex cross ratio, the basic invariant of complex projective geometry. Historically, the Möbius group has been found by Möbius in this important context. We considered the original complex Möbius group, and its relation to the 2-dimensional real case, in the notebook [RS]. Here we are interested more in the real situation, for later application in differential geometry. Therefore we will apply the general method of Möbius invariants for pairs of subspheres also in this lowest dimensional case. Since the cross ratio is a complex number we may expect to obtain two real invariants describing the mutual position of two point pairs in the Möbius space.

- **5.1. A complete invariant system**
- **5.2. Throws**
- **5.3. Four points on the standard unit circle**

6. Geodesics in the Space of 0-Spheres

- **6.1. Introduction**
- **6.2. Spacelike geodesics**
- **6.3. Timelike geodesics**
- **6.4. Isotropic geodesics**

References

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