Semi-quasi-homogeneous Singularities with Triangle Weights in Positive Characteristics

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Dedicated to Herbert Kurke on the occasion of his sixtieth birthday

Introduction

k denotes an algebraically closed field of any characteristic $p \geq 0$. Let $f \in k[[X]]$, $X = (X_0, \ldots, X_n)$ be a (semi-) quasi-homogeneous[1] power series of weight $w = (w_0, \ldots, w_n)$ defining an isolated singularity. We associate w with the rational number $s(w)$ introduced by K. Saito (see [12]), which is also said to be the singularity index (see [1]). For $k = \mathbb{C}$, properties of sqh singularities are intensively studied, and sqh singularities of small modality are known.

In arbitrary characteristics, a complete list is available in the case of $s \leq 1$ (cf. [2], [6], [5] for $s < 1$ and [9], [10] for $s = 1$).

This paper is a proposal to continue the classification of the above mentioned singularities following increasing values of s. As we will see, we obtain (as well as some other weights) in the interval $(1, \frac{7}{6}]$ the weights of all triangle singularities from the complex analytic case.

The values $s \in \mathbb{Q}$ coming from such an isolated sqh singularity have as their smallest accumulation point $s = 1$, with the next accumulation point being $\frac{4}{3}$.

Thus, in the interval above (which is the first half of $(1, \frac{4}{3}]$ and gives the smallest upper bound for all "triangle weights") there is at most a finite number of such values of s. Those are the ones we calculate to give a list of corresponding sqh hypersurface singularities in any characteristic.

The computational approach to the classification of singularities has made some progress last year with the edition of Singular 1.2, where algorithms of [1] have been implemented. Still, there seems to be nothing comparable for the case of $p > 0$. Thus, a list of sqh singularities following increasing values of s, instead of the integer invariant of modality, may give an approach towards using computer algebra for classification in any characteristic.

Several of the assertions here are obtained using the computer algebra systems Singular 1.2 ([3]) and Reduce 3.6 ([8]).

1. The index set S

Let S be the set of all numbers $s \in \mathbb{R}$ such that over some algebraically closed field there exists a quasi-homogeneous polynomial of weight $w = (w_0, \ldots, w_n)$ having in its set of zeros an isolated singularity at the origin, and $s = s(w) = n + 1 - 2(w_0 + \ldots + w_n)$.

[1] We say sqh and qh, respectively.

1.1 Proposition:

S is contained in $\mathbb{Q}$; its smallest accumulation points are 1 and $\frac{4}{3}$.

Proof: Since in any characteristic, a quasi-homogeneous polynomial $f \in k[X_0, \ldots, X_n]$ defining an isolated singularity at the origin must have in its support terms of the type X_i^a or $X_i^a X_j$ (for some $j \neq i$), $i = 0, \ldots, n$ we have as a necessary condition for $w = (w_0, \ldots, w_n)$ the system

$a_i w_i = 1$, or $a_i w_i + w_j = 1$ respectively

of linear equations, which is uniquely solvable in $\mathbb{Q}$. Now consider the equations for w:

We may suppose $w_i \leq \frac{1}{2}$ for all i with equality for at most one i (cf. [7]). Further, we exploit the above equations to obtain conditions for weights w with $s = s(w) < \frac{4}{3}$. Now 1.1. is obtained as indicated by the following two remarks.

1.2 Remark:

Let m be the number of weights $< \frac{1}{2}$ for a polynomial f as above, then for $s = s(w) < \frac{4}{3}$ we have $m \leq 3$.

(This can be seen as in the proof of [10], p. 4, Lemma.)

Consider the case of two essential variables:

1.3 Lemma:

Let $w = (w_0, w_1)$ with $1 < s(w) < \frac{4}{3}$. Then $\frac{1}{3} < w_0 + w_1 < \frac{1}{2}$, and w is (up to permutation of components) one of the following[2]:

(i) $w = (\dfrac{1}{3}, \dfrac{1}{a})$, $a \geq 7$, $s(w) \to \dfrac{4}{3}$ for $a \to \infty$,

further $w = (\dfrac{1}{4}, \dfrac{1}{a})$, $a = 5, \ldots, 11$

and $w = (\frac{1}{5}, \frac{1}{5}), (\frac{1}{5}, \frac{1}{6}), (\frac{1}{5}, \frac{1}{7})$

(ii) $w = (\dfrac{1}{a}, \dfrac{a-1}{3a})$, $a \geq 5$, $s(w) \to \dfrac{4}{3}$ for $a \to \infty$,

further $w = (\dfrac{1}{3}, \dfrac{2}{3a})$, $a \geq 9$, $s(w) \to \dfrac{4}{3}$ for $a \to \infty$,

and $w = (\frac{1}{4}, \frac{3}{16}), (\frac{1}{5}, \frac{1}{5}), (\frac{1}{6}, \frac{5}{24}), (\frac{1}{7}, \frac{3}{14}), (\frac{1}{8}, \frac{7}{32}), (\frac{1}{3}, \frac{2}{15}), (\frac{1}{4}, \frac{3}{20}), (\frac{1}{5}, \frac{4}{25}), (\frac{1}{3}, \frac{1}{9}),$
$(\frac{1}{4}, \frac{1}{8}), (\frac{1}{3}, \frac{2}{21}), (\frac{1}{4}, \frac{3}{28}), (\frac{1}{3}, \frac{1}{12}), (\frac{1}{4}, \frac{3}{32})$

[2]cases depending if for some corresponding polynomial f there are two, one or no powers of variables in the support; some weights appear several times

(iii) $w = (\frac{a-1}{3a-1}, \frac{2}{3a-1})$, $a \geq 4$, $s(w) \to \frac{4}{3}$ for $a \to \infty$,
and $w = (\frac{1}{5}, \frac{1}{5}), (\frac{4}{19}, \frac{3}{19}), (\frac{5}{23}, \frac{3}{23})$

To complete the proof of 1.1, write down the equations for the weights with 3 essential variables (seven cases) and check the condition $1 < s(w) < \frac{4}{3}$; this gives inequalities excluding accumulation points for s in that interval.

Note that the set S may be of more interest if we understand better the case of positive characteristics. Considering only the values coming from a quasi-homogeneous isolated hypersurface singularity over a field of characteristic 0, a Thom-Sebastiani like construction shows us that such values of s form a subgroup of $\mathbb{Q}$. It is not clear whether this assertion holds generally.

Nevertheless, increasing numbers s are presenting remarkable combinatorial difficulties.

2. The interval of the 14 triangle weights

As we have seen, the interval $(1, \frac{7}{6}]$ contains, at most, a finite number of points of S. This "first half" of the interval $(1, \frac{4}{3})$ is given by the smallest upper bound for s such that all "triangle weights" are contained as is shown in the following table with the names for the weights taken from the corresponding (semi-)quasi-homogeneous polynomials in the list of singularities (cf. [1], [14]). Thus, a "triangle weight" is one of the weights of the 14 so called exceptional right-unimodal hypersurface singularities, known in characteristic 0 ("triangle singularities"), and is denoted by one of the corresponding symbols:

$E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, Q_{10}, Q_{11}, Q_{12}, W_{12}, W_{13}, S_{11}, S_{12}, U_{12}$

There are four pairs among those which have the same index s: (E_{13}, Z_{11}), (E_{14}, Q_{10}),(Z_{13}, Q_{11}) and (W_{13}, S_{11}).[3]

As we see in the table below, there are further weights which do not correspond to any of the triangle singularities. In characteristic 0, they are realized by several hypersurface singularities of (right-) modality 2 and one of (right-) modality 3. As before, notations for the weights are preserved for arbitrary characteristics, but note that indices of the symbols do not necessarily coincide with the Tjurina-number of a corresponding isolated singularity if $p \neq 0$. In the table, the right hand column contains the modality of a nondegenerate sqh polynomial of the given weight in characteristic 0.

We assume all weights to be $\leq \frac{1}{2}$. Since $w_i = \frac{1}{2}$ does not change s (counting the appropriate number of variables), we only list the remaining ones ("essential weights"). In characteristic 2, we have a different classification of quadratic forms

[3]These are the dual pairs in the sense of Pinkham's "strange duality".

which gives rise to splitting a corresponding polynomial f into a quadratic form and its "essential part" requiring one variable of weight $\frac{1}{2}$ in several cases.

Distribution of quasi-homogeneous polynomials defining isolated singularities for $1 < s \leq 7/6$ (ordered by increasing values of s)				
s	essential weights	support in	name of weight	right-modality for $p = 0$
$\frac{22}{21}$	$(\frac{1}{3}, \frac{1}{7})$	x^3, y^7	E_{12}	1
$\frac{16}{15}$	$(\frac{1}{5}, \frac{4}{15})$	x^5, xy^3	Z_{11}	1
$\frac{16}{15}$	$(\frac{1}{3}, \frac{2}{15})$	x^3, xy^5	E_{13}	1
$\frac{13}{12}$	$(\frac{1}{3}, \frac{1}{8})$	x^3, y^8	E_{14}	1
$\frac{13}{12}$	$(\frac{1}{3}, \frac{1}{4}, \frac{3}{8})$	x^3, y^4, yz^2	Q_{10}	1
$\frac{12}{11}$	$(\frac{3}{11}, \frac{2}{11})$	x^3y, xy^4	Z_{12}	1
$\frac{11}{10}$	$(\frac{1}{4}, \frac{1}{5})$	x^4, y^5	W_{12}	1
$\frac{10}{9}$	$(\frac{1}{6}, \frac{5}{18})$	x^6, xy^3	Z_{13}	1
$\frac{10}{9}$	$(\frac{1}{3}, \frac{7}{18}, \frac{2}{9})$	x^3, y^2z, xz^3	Q_{11}	1
$\frac{10}{9}$	$(\frac{1}{3}, \frac{1}{9})$	x^3, y^9, xy^6, x^2y^3	J_{16}	2
$\frac{9}{8}$	$(\frac{1}{4}, \frac{3}{16})$	x^4, xy^4	W_{13}	1
$\frac{9}{8}$	$(\frac{1}{4}, \frac{5}{16}, \frac{3}{8})$	x^4, y^2z, xz^2	S_{11}	1
$\frac{17}{15}$	$(\frac{1}{3}, \frac{1}{10})$	x^3, y^{10}	E_{18}	2
$\frac{17}{15}$	$(\frac{1}{3}, \frac{1}{5}, \frac{2}{5})$	x^3, y^5, yz^2, y^3z	Q_{12}	1
$\frac{8}{7}$	$(\frac{2}{7}, \frac{1}{7})$	x^3y, xy^5, y^7, x^2y^3	Z_{15}	2
$\frac{8}{7}$	$(\frac{1}{3}, \frac{2}{21})$	x^3, xy^7	E_{19}	2
$\frac{38}{33}$	$(\frac{1}{3}, \frac{1}{11})$	x^3, y^{11}	E_{20}	2
$\frac{15}{13}$	$(\frac{4}{13}, \frac{5}{13}, \frac{3}{13})$	x^2y, y^2z, xz^3	S_{12}	1
$\frac{7}{6}$	$(\frac{1}{3}, \frac{1}{12})$	$x^3, y^{12}, xy^8, x^2y^4$	J_{22}	3
$\frac{7}{6}$	$(\frac{1}{4}, \frac{1}{6})$	x^4, y^6, x^2y^3	W_{15}	2
$\frac{7}{6}$	$(\frac{1}{8}, \frac{7}{24})$	x^8, xy^3	Z_{17}	2
$\frac{7}{6}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{4})$	$x^3, y^3, z^4, x^2y, xy^2$	U_{12}	1
$\frac{7}{6}$	$(\frac{1}{3}, \frac{1}{6}, \frac{5}{12})$	$x^3, y^6, yz^2, x^2y^2, xy^4$	Q_{14}	2

The entries in the tables below give principal parts of sqh singularities. We obtain any sqh singularity with the same principal part by adding a linear combination of the listed super-diagonal monomials (with constant coefficients). This property is based on the following generalization of theorem 12.6 from [1].

2.1 Lemma (cf. [11]):

Let f_1 be quasi-homogeneous of some weight w and choose monomials $m_1, \ldots, m_s$ of weights $q_1, \ldots, q_s$ respectively, $q_i > 1$, such that their classes form a base of the k-subspace S of the Tjurina-algebra

$$T(f_1) := k[[X]]/(f_1, \frac{\partial f_1}{\partial X_0}, \ldots, \frac{\partial f_1}{\partial X_n}),$$

where S is generated by the classes of all monomials of weight > 1.

Now take any power series f which is sqh with respect to w and assume $f = f_1 + \sum_{q>1} f_q$. Choose a rational number $q' > 1$ and suppose

$$f_{q'} \equiv \sum_{i, q_i = q'} \alpha_i m_i \; mod \; (f_1, \frac{\partial f_1}{\partial X_0}, \ldots, \frac{\partial f_1}{\partial X_n})$$

for some $\alpha_i \in k$ (the sum over an empty set is defined to be 0). Then f is contact equivalent with

$$f_1 + \sum_{q, 1<q<q'} f_q + \sum_{i, q_i = q'} \alpha_i m_i + g,$$

where $w(g) > q'$.

The following tables give more detailed information on singularities with triangle weights. In particular, all sqh forms can be obtained. The information is obtained in two steps:

(i) Find a representative for all quasi-homogeneous polynomials with a given weight in the contact class ("geometric part").

(ii) Find the super-diagonal monomials in the Tjurina algebra ("standard base calculation").

2.2 Example: For $p \neq 2$ and the weight of U_{12} (which is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{4})$), consider the monomials of weight 1. We are essentially done, if we write down a nondegenerate cubic form in 2 variables, e.g. the product of the 3 linear forms $x,\ y,\ x + y$.

In the case of characteristic 2, we obtain a further form with an additional variable of weight 1/2.

Semi-quasi-homogeneous singularities with triangle weights, $p \neq 2$		
name	principal part	super-diagonal basis monomials
E_{12}	$x^3 + y^7$	xy^5 (for $p \neq 3, 7$) xy^5, xy^6 (for $p = 7$) $xy^5, x^2y^3, x^2y^4, x^2y^5$ (for $p = 3$)
Z_{11}	$x^5 + xy^3$	x^4y (for $p \neq 3$) x^4y, x^3y^2, x^4y^2 (for $p = 3$)
E_{13}	$x^3 + xy^5$	y^8 (for $p \neq 3, 5$) y^8, y^9 (for $p = 5$) x^2y^3 (for $p = 3$)
E_{14}	$x^3 + y^8$	xy^6 (for $p \neq 3$) $x^2y^3, xy^6, x^2y^4, x^2y^5, x^2y^6$ (for $p = 3$)
Q_{10}	$x^3 + y^4 + yz^2$	xy^3 (for $p \neq 3$) $x^2z,\ xy^3,\ x^2y^2,\ x^2y^3$ (for $p = 3$)
Z_{12}	$x^3y + xy^4$	y^6 (for $p \neq 3$) x^2y^3 (for $p = 3$)
W_{12}	$x^4 + y^5$	x^2y^3 (for $p \neq 5$) $xy^4,\ x^2y^3,\ x^2y^4$ (for $p = 5$)
Z_{13}	$x^6 + xy^3$	x^5y (for $p \neq 3$) x^3y^2, x^5y, x^5y^2 (for $p = 3$)
Q_{11}	$x^3 + y^2z + xz^3$	z^5 (for $p \neq 3$) x^2y, x^2z^2 (for $p = 3$)
W_{13}	$x^4 + xy^4$	y^6
S_{11}	$x^4 + y^2z + xz^2$	x^3z
Q_{12}	$x^3 + yz^2 + y^3z$	xy^4 (for $p \neq 3$) $x^2y^2, x^2z, xy^4, x^2y^3, x^2y^4$ (for $p = 3$)
S_{12}	$x^2y + y^2z + xz^3$	z^5
U_{12}	$x^2y + xy^2 + z^4$	y^2z^2

Semi-quasi-homogeneous singularities with triangle weights, $p=2$		
name	principal part	super-diagonal basis monomials
E_{12}	x^3+y^7	xy^5
	$x^3+y^7+z^2$	$xy^5, y^4z, xy^2z, y^5z, xy^3z, xy^4z, xy^5z$
Z_{11}	x^5+xy^3	x^4y
	$x^5+xy^3+z^2$	$y^2z, xy^4, x^3z, x^2yz, x^4z, x^3yz, x^4yz$
E_{13}	x^3+xy^5	y^8
	$x^3+xy^5+z^2$	$y^4z, y^8, xy^2z, y^5z, xy^3z, y^6z, y^7z, y^8z$
E_{14}	x^3+y^8	xy^6, xy^7
	$x^3+y^4z+z^2$	xy^2z, xy^3z
Q_{10}	$x^3+y^4+yz^2$	$xy^3,\ y^3z,\ xy^2z,\ xy^3z$
	$x^3+yz^2+y^2u+u^2$	$xyu,\ yzu,\ xzu,\ xyzu$
Z_{12}	x^3y+xy^4	y^6
	$x^3y+xy^4+z^2$	$y^3z,\ x^2z,\ y^6,\ xy^2z,\ y^4z,\ xy^3z,\ y^5z,\ y^6z$
W_{12}	x^4+y^5	$x^2y^3,\ x^3y^2,\ x^3y^3$
	$y^5+x^2z+z^2$	$y^3z,\ xy^2z,\ xy^3z$
Z_{13}	x^6+xy^3	x^5y
	$xy^3+x^3z+z^2$	y^2z, x^2yz
Q_{11}	$x^3+y^2z+xz^3$	$yz^3, z^5, xyz^2, yz^4, yz^5$
W_{13}	x^4+xy^4	x^2y^3, x^3y^2, x^3y^3
	$xy^4+x^2z+z^2$	y^3z, xy^2z, xy^3z
S_{11}	$x^4+y^2z+xz^2$	x^3y, x^3z, x^2yz, x^3yz
	$y^2z+xz^2+x^2u+u^2$	$xyu, xzu, yzu, xyzu$
Q_{12}	$x^3+yz^2+y^3z$	xy^2z
	$x^3+yz^2+y^3z+u^2$	$xyu, yzu, xy^2z, xy^2u, xzu, y^2zu, xyzu, xy^2zu$
S_{12}	$x^2y+y^2z+xz^3$	xyz^2
	$x^2y+y^2z+xz^3+u^2$	$xzu, yzu, xyz^2, xyu, yz^2u, xyzu, xz^2u, xyz^2u$
U_{12}	$x^2y+xy^2+z^4$	xz^3, yz^3, xyz^2, xyz^3
	$x^2y+xy^2+z^2u+u^2$	$xzu, yzu, xyu, xyzu$

2.3 Remark:

We obtain from the above classification:

(i) Let p be none of the characteristics 2, 3, 5 or 7, then for all of the 14 triangle weights there is only one non-quasi-homogeneous form up to contact-equivalence which is obtained by adding the only super-diagonal monomial from the table to the corresponding quasi-homogeneous part.

(ii) "Moduli" appear e.g. in the first equation of U_{12} in characteristic 2, if we consider the negative weight deformation $f_{s,t} = x^2y + xy^2 + z^4 + sxz^3 + tyz^3$ of $x^2y+xy^2+z^4$. We may choose $(s,t) \in \mathbb{P}^1$ and obtain contact equivalence only in finite classes.

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