Preliminaries

(1) Throughout the book we assume ZFC. We use "virtual classes", writing $\{x|\varphi(x)\}$ for the class of x such that $\varphi(x)$. We also write:

$$\begin{array}{l}\{t(x_1,\ldots,x_n)|\varphi(x_1,\ldots,x_n)\},\ \text{(where e.g.}\\ t(x_1,\ldots,x_n)=\{y|\psi(y,x_1,\ldots,x_n)\})\end{array}$$

for:

$$\{y|\bigvee x_1,\ldots,x_n(y=t(x_1,\ldots,x_n)\wedge\varphi(x_1,\ldots,x_n))\}$$

We also write

$$\begin{array}{l}\mathbb{P}(A)=\{z|z\subset A\},A\cup B=\{z|z\in A\vee z\in B\}\\ A\cap B=\{z|z\in A\wedge z\in B\},\neg A=\{z|\notin A\}\end{array}$$

(2) Our notation for ordered n–tuples is $\langle x_1,\ldots,x_n\rangle$. This can be defined in many ways and we don't specify a definition.

(3) An *n–ary relation* is a class of n–tuples. The following operations are defined for all classes, but are mainly relevant for binary relations:

$$\begin{array}{l}\mathrm{dom}(R)=:\{x|\bigvee y\langle y,x\rangle\in R\}\\ \mathrm{rng}(R)=:\{y|\bigvee x\langle y,x\rangle\in R\}\\ R\circ P=\{\langle y,x\rangle|\bigvee z|\langle y,z\rangle\in R\wedge\langle z,x\rangle\in P\}\\ R\restriction A=\{\langle y,x\rangle|\langle y,x\rangle\in R\wedge x\in A\}\\ R^{-1}=\{\langle y,x\rangle|\langle x,y\rangle\in R\}\end{array}$$

We write $R(x_1,\ldots,x_n)$ for $\langle x_1,\ldots,x_n\rangle\in R$.

(4) A *function* is identified with its *extension* or *field* — i.e. an n–ary function is an $n+1$–ary relation F such that

$$\begin{array}{l}\bigwedge x_1\ldots x_n\bigwedge z\bigwedge w((F(z,x_1,\ldots,x_n)\wedge F(w,x_1,\ldots,x_n))\to\\ \qquad\to z=w)\end{array}$$

$F(x_1,\ldots,x_n)$ then denotes the value of F at $x_1,\ldots,x_n$.

(5) "*Functional abstraction*" $\langle t_{x_1,\ldots,x_n} | \varphi(x_1,\ldots,x_n)\rangle$ denotes the function which is defined and takes value $t_{x_1,\ldots,x_n}$ whenever $\varphi(x_1,\ldots,x_n)$ and $t_{x_1,\ldots,x_n}$ is a set:

$$\langle t_{x_1,\ldots,x_n} | \varphi(x_1,\ldots,x_n)\rangle =: \\ \{\langle y, x_1,\ldots,x_n\rangle | y = t_{x_1,\ldots,x_n} \wedge \varphi(x_1,\ldots,x_n)\},$$

where e.g. $t_{x_1,\ldots,x_n} = \{z|\psi(z,x_1,\ldots,x_n)\}$.

(6) *Ordinal numbers* are defined in the usual way, each ordinal being identified with the set of its predecessors: $\alpha = \{\nu|\nu < \alpha\}$. The *natural numbers* are then the finite ordinals: $0 = \emptyset, 1 = \{0\},\ldots,n = \{0,\ldots,n-1\}$.

(7) *A note on ordered n–tuples.* A frequently used definition of ordered pairs is:

$$\langle x, y\rangle =: \{\{x\},\{x,y\}\}.$$

One can then define n–tuples by:

$$\langle x\rangle =: x,\ \langle x_1, x_2,\ldots,x_n\rangle =: \langle x_1, \langle x_1,\ldots,x_n\rangle\rangle.$$

However, this has the disadvantage that every $n+1$–tuple is also an n–tuple. If we want each tuple to have a fixed length, we could instead identify the n–tuples with *vecton of length n* — i.e. functions with domain n. This would be circular, of course, since we must have a notion of ordered pair in order to define the notion of "function". Thus, if we take this course, we must first make a "preliminary definition" of ordered pairs — for instance:

$$(x,y) =: \{\{x\},\{x,y\}\}$$

and then define:

$$\langle x_0,\ldots,x_{n-1}\rangle = \{(x_0,0),\ldots,(x_{n-1},n-1)\}.$$

If we wanted to form n–tuples of proper classes, we could instead identify $\langle A_0,\ldots,A_{n-1}\rangle$ with:

$$\{\langle x,i\rangle | (i = 0 \wedge x \in A_0) \vee \ldots \vee (i = n-1 \wedge x \in A_{n-1})\}.$$

(8) *Overhead arrow notation.* The symbol $\vec{x}$ is often used to donate a vector $\langle x_1,\ldots,x_n\rangle$. It is not surprising that this usage shades into what I shall call the *informal mode* of overhead arrow notation. In this mode $\vec{x}$ simply stands for a string of symbols $x_1,\ldots,x_n$. Thus we write $f(\vec{x})$ for $f(x_1,\ldots,x_n)$, which is different from $f(\langle x_1,\ldots,x_n\rangle)$. (In informal mode we would write the latter as $f(\langle\vec{x}\rangle)$.) Similarly, $\vec{x} \in A$ means that each of $x_1,\ldots,x_n$ is an element of A, which is different from $\langle\vec{x}\rangle \in A$.

We can, of course, combine several arrows in the same expression. For instance we can write $f(\vec{g}(\vec{x}))$ for $f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$. Similarly we can write $f(\overrightarrow{g(\vec{x})})$ or $f(\vec{g}(\vec{x}))$ for

$$f(g_1(x_{1,1}, \ldots, x_{1,p_1}), \ldots, g_m(x_{m,1}, \ldots, x_{m,p_m})).$$

The precise meaning must be taken from the context. We shall often have recourse to such abbreviations. To avoid confusion, therefore, we shall use overhead arrow notation *only* in the informal mode.

(9) A *model* or *structure* will for us normally mean an $n{+}1$–tuple $\langle D, A_1, \ldots, A_n\rangle$ consisting of a domain D of individuals, followed by relations on that domain. If φ is a first oder formula, we call a sequence $v_1, \ldots, v_n$ of distinct variables *good for* φ iff every free variable of φ occurs in the sequence. If M is a model, φ a formula, $v_1, \ldots, v_n$ a good sequence for φ and $x_1, \ldots, x_n \in M$, we write: $M \models \varphi(v_1, \ldots, v_n)[x_1, \ldots, x_n]$ to mean that φ becomes true in M if v_i is interpreted by x_i for $i = 1, \ldots, n$. This is the *satisfaction relation.* We assume that the reader knows how to define it. As usual, we often suppress the list of variables, writing only $M \models \varphi[x_1, \ldots, x_n]$. We may sometimes indicate the variables being used by writing e.g. $\varphi = \varphi(v_1, \ldots, v_n)$.

(10) $\in$*–models.* $M = \langle D, E, A_1, \ldots, A_n\rangle$ is an $\in$*–model* iff E is the restriction of the $\in$–relation to D^2. Most of the models we consider will be $\in$–models. We then write $\langle D, \in, A_1, \ldots, A_n\rangle$ or even $\langle D, A_1, \ldots, A_n\rangle$ for $\langle D, \in \cap D^2, A_1, \ldots, A_n\rangle$. M is *transitive* iff it is an $\in$–model and D is transitive.

(11) *The Levy hierarchy.* We often write $\bigwedge x \in y\varphi$ for $\bigwedge x(x \in y \to \varphi)$, and $\bigvee x \in y\varphi$ for $\bigvee x(x \in y \wedge \varphi)$. Azriel Levy defined a hierarchy of formulae as follows:

A formula is Σ_0 (or Π_0) iff it is in the smallest class Σ of formulae such that every primitive formula is in Σ and $\bigwedge v \in u\varphi$, $\bigvee v \in u\varphi$ are in Σ whenever φ is in Σ and v, u are distinct variables.

(Alternatively we could introduce $\bigwedge v \in u$, $\bigvee v \in u$ as part of the primitive notation. We could then define a formula as being Σ_0 iff it contains no unbounded quantifiers.)

The Σ_{n+1} formulae are then the formulae of the form $\bigvee v\varphi$, where φ is Π_n. The Π_{n+1} formulae are the formulae of the form $\bigwedge v\varphi$ when φ is Σ_n.

If M is a transitive model, we let $\Sigma_n(M)$ denote the set of realations on M which are definable by a Σ_n formula. Similarly for $\Pi_n(M)$. We say that a relation R is $\Sigma_n(M)(\Pi_n(M))$ *in parameters* $p_1, \ldots, p_m$ iff

$$R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n, p_1, \ldots, p_m)$$

and R' is $\Sigma_n(M)(\Pi_n(M))$. $\underline{\Sigma}_1(M)$ then denotes the set of relations which are $\Sigma_1(M)$ in some parameters. Similarly for $\underline{\Pi}_1(M)$.

(12) *Kleene's equation sign.* An equation '$L \simeq R$' means: 'The left side is defined if and only if everything on the right side is defined, in which case the sides are equal'. This is of course not a strict definition and must be interpreted from case to case.

$F(\vec{x}) \simeq G(H_1(\vec{x}), \ldots, H_n(\vec{x}))$ obviously means that the function F is defined at $\langle x_1, \ldots, x_n \rangle$ iff each of the H_i is defined at $\langle \vec{x} \rangle$ and G is defined at $\langle H_1(\vec{x}), \ldots, H_n(\vec{x}) \rangle$, in which case equality holds.

The recursion schema of set theory says that, given a function G, ther is a function F with:

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y \rangle).$$

This says that F is defined at $\langle y, \vec{x} \rangle$ iff $F$is defined at $\langle z, \vec{x} \rangle$ for all $z \in y$ and G is defined at $\langle y, \vec{x}, \langle F(z, \vec{x}) | z \in y \rangle \rangle$, in which case equality holds.

(13) By the recursion theorem we can define:

$$TC(x) = x \cup \bigcup_{z \in x} TC(z)$$

(the transitive closure of x)

$$\mathrm{rn}(x) = \mathrm{lub}\{\mathrm{rn}(z) | z \in x\}$$

(the rank of x).

Chapter 1

Transfinite Recursion Theory

1.1 Admissibility

Some fifty years ago Kripke and Platek brought out about a wide ranging generalization of recursion theory — which dealt with "effective" functions and relations on ω — to transfinite domains. This, in turn, gave the impetus for the development of fine structure theory, which became a basic tool of inner model theory. We therefore begin with a discussion of Kripke and Platek's work, in which ω is replaced by an arbitrary "admissible" structure.

1.1.1 Introduction

Ordinary recursion theory on ω can be developed in three different ways. We can take the notion of *algorithm* on basic, defining a recursive function on ω to be one given by an algorithm. Since, however, we have no definition for the general notion of algorithm, this approach involves defining a special class of algorithms and then convincing ourselves that "Church's thesis" holds — i.e. that every function generated by an algorithm is, in fact, generated by one which lies in our class. Alternatively we can take the notion of *calculus* on basic, defining an n–ary relation R on ω to be recursively enumerable (r.e.) if for some calculus involving statements of the form "$R(i_1, \ldots, i_n)$" $(i_1, \ldots, i_n < \omega)$, R is the set of tuples $\langle i_1, \ldots, i_n \rangle$ such that "$R(i_1, \ldots, i_n)$" is provable. R is then recursive if both it and its complement are r.e. A function defined on ω is recursive if it is recursive as a relation. But again, since we have no general definition of calculus, this involves specifying a special class of calculi and appealing to the appropriate form of Church's thesis.

A third alternative is to base the theory on *definability*, taking the r.e. relation as those which are definable in elementary number theory by one of a certain class of formulae. This approach has often been applied, but characterizing the class of defining formula tends to be a bit unnatural. The situation changes radically, however, if we replace ω by the set $H = H_\omega$ of heredetarily finite sets. We consider definability over the structure $\langle H, \in \rangle$, employing the familiar Levy hierarchy of set theoretic formulae:

$\Pi_0 = \Sigma_0 =:$ formulae in which all quantifiers are bounded

$\Sigma_{n+1} =:$ formulae $\bigvee x\varphi$ where φ in Π_n

$\Pi_{n+1} =:$ formulae $\bigwedge x\varphi$ where φ in Σ_n.

We then call a relation on H r.e. (or H–r.e.) iff it is definable by a Σ_1 formula. Recalling that $\omega \subset H$ it then turns out that a relation on ω is H–r.e. iff it is r.e. in the classical sense. Moreover, there is an H–recursive map $\pi : H \leftrightarrow \omega$ such that $A \subset H$ is H–r.e. iff $\pi''A$ is r.e. in the classical sense.

This suggests a very natural way of relativizing recursion theory to transfinite domains. Let $N = \langle |N|, \in, A_1, \ldots, A_n \rangle$ be any transitive structure. We first define:

Definition 1.1.1. A relation on N is $\Sigma_n(N)$ (in the parameters $p_1, \ldots, p_n \in N$) iff it is N–definable (in $\vec{p}$) by a Σ_n formula. It is $\Delta_n(N)$ (in $\vec{p}$) if both it and its completement are $\Sigma_n(N)$ (in $\vec{p}$). It is $\underline{\Sigma}_n(N)$ iff it is $\Sigma_n(N)$ in some parameters. Similarly for $\underline{\Delta}_n(N)$.

Following our above example of $N = \langle H, \in \rangle$, it is natural to define a relation on N as being *N–r.e.* iff it is $\underline{\Sigma}_1(N)$, and *N–recursive* iff it is $\underline{\Delta}_1(N)$. A partial function F on N is *N–r.e.* iff it is N–r.e. as a relation. F is *N–recursive* as a function iff it is N–r.e. and dom(F) in $\underline{\Delta}_1(N)$.

(Note that $\underline{\Sigma}_1(\langle H, \in \rangle) = \Sigma_1(\langle H, \in \rangle)$, which will not hold for arbitrary N.)

However, this will only work for an N satisfying rather strict conditions since, when we move to transfinite structures N, we must relativize not only the concepts "recursive" and "r.e.", but also the concept "finite". In the theory of H the finite sets were simply the elements of H.

Correspondingly we define:

$$u \text{ is } N\text{–}\textit{finite} \text{ iff } u \in N.$$

But there are certain basic properties which we expect any recursion theory to have. In particular:

- If A is recursive and u is finite, then $A \cap u$ is finite.
- If u is finite and $F : u \to N$ is recursive, then $F''u$ is finite.

Those transitive structures $N = \langle |N|, \in A_1, \ldots, A_n \rangle$ which yield a satisfactory recursion theory are called *admissible*. An ordinal α is then called *admissible* iff L_α is admissible. The admissible structures were characterized by Kripke and Platek as those transfinite structures which satisfy the following axioms:

(1) $\emptyset, \{x, y\}, \cup x$ are sets

(2) *The Σ_0 axiom of subsets:*

$$x \cap \{z | \varphi(u)\} \text{ is a set}$$

(where φ is any Σ_0–formula)

(3) *The Σ_0 axiom of collection:*

$$\bigwedge x \in u \bigvee y \ \varphi(x, y) \to \bigvee v \bigwedge x \in u \bigvee y \in v \ \varphi(x, y),$$

(where φ is any Σ_0–formula).

Note *Kripke–Platek set theory (KP)* consists of the above axioms together with the axoim of extensionality and the full axiom of foundation (i.e. for all formulae, not just the Σ_0 ones).

Note Although the definability approach is the one most often employed in transfinite recursion theory, the approaches via algorithms and calculi have also been used to define the class of admissible ordinals.

1.1.2 Properties of admissible structures

We now show that admissible structures satisfy the two criteria stated above. In the following let $M = \langle |M|, \in A_a, \ldots, A_n \rangle$ be admissible.

Lemma 1.1.1. *Let $u \in M$. Let A be $\underline{\Delta}_1(M)$. Then $A \cap u \in M$.*

Proof: Let $Ax \leftrightarrow \bigvee y A_0 yx; \neg Ax \leftrightarrow \bigvee y A_1 yx$, where A_0, A_1 are $\underline{\Sigma}_0(M)$. Then $\bigwedge x \in u \bigvee y (A_0 yx \vee A_1 yx)$. Hence there is $v \in M$ such that $\bigwedge x \in u \bigvee y \in v (A_0 yx \vee A_1 yx)$. QED

Before verifying the second criterion we prove:

Lemma 1.1.2. *M satisfies:*

$$\bigwedge x \in u \bigvee y_1 \ldots y_n \varphi(x, \vec{y}) \to \bigvee u \bigwedge x \in u \bigvee y_1 \ldots y_n \in u \varphi(x, \vec{y})$$

for Σ_0–formulae φ.

Proof. Assume $\bigwedge x \in u \bigvee y_1 \ldots y_n \varphi(x, \vec{y})$. Then

$$\bigwedge x \in u \bigvee w \underbrace{\bigvee y_1 \ldots y_n \in w \varphi(x, \vec{y})}_{\Sigma_0}.$$

Hence there is $v' \in M$ such that $\bigwedge x \in u \bigvee w \in v' \bigvee y_1 \ldots y_n \in w \varphi(x, \vec{y})$. Take $v = \bigcup v'$. QED (Lemma 1.1.2)

We now verify the second criterion:

Lemma 1.1.3. *Let $u \in M, u \subset \mathrm{dom}(F)$, where F is a $\underline{\Sigma}_1(M)$ funcion. Then $F''u \in M$.*

Proof. Let $y = F(x) \leftrightarrow \bigvee z F'zyx$, where F' is a $\underline{\Sigma}_0(M)$ relation. Then $\bigwedge x \in u \bigvee z, y F'zyx$. Hence there is $v \in M$ such that $\bigwedge x \in u \bigvee z, y \in v F'zyx$. Hence $F''u = v \cap \{y | \bigvee x \in u \bigvee z \in v F'zxy\}$.
QED (Lemma 1.1.3)

Assuming the admissibility of M, we immediately get from Lemma 1.1.2:

Lemma 1.1.4. *Let $\varphi(y, \vec{x})$ be a Σ_1–formula. Then $\bigvee y \varphi(y, \vec{x})$ is uniformly Σ_1 in M.*

Note "Uniformly" is a word which recursion theorists like to use. Here it means that $M \models \bigvee y \varphi(y, \vec{x}) \leftrightarrow \Psi(\vec{x})$ for a Σ_1 formula Ψ which depends only on φ and not on the choice of M.

Lemma 1.1.5. *Let $\varphi(y, \vec{x})$ be Σ_1. Then $\bigwedge y \in u \varphi(y, \vec{x})$ is uniformly Σ_1 in M.*

Proof. Let $\varphi(y, \vec{x}) = \bigvee z \varphi'(z, y, x)$, where φ' is Σ_0. Then

$$\bigwedge y \in u \varphi(y, \vec{x}) \leftrightarrow \bigvee v \underbrace{\bigwedge y \in u \bigvee z \in u \varphi'(z, y, x)}_{\Sigma_0}$$

in M. QED (Lemma 1.1.5)

Lemma 1.1.6. *Let $\varphi_0(\vec{x}), \varphi_1(\vec{x})$ be Σ_1. Then $(\varphi_0(\vec{x}) \wedge \varphi_1(\vec{x})), (\varphi_0(\vec{x}) \vee \varphi_1(\vec{x}))$ are uniformly Σ_1 in M.*

Proof. Let $\varphi_i(\vec{x}) = \bigvee y_i \varphi'_i(y_i, \vec{x})$ where without loss of generality $y_0 \neq y_1$. Then

$$(\varphi_0(\vec{x}) \wedge \varphi_1(\vec{x})) \leftrightarrow \bigvee y_0 \bigvee y_1(\varphi'_0(y_0, x) \wedge \varphi'_1(y_1, x)).$$

Similarly for $\vee$. QED (Lemma 1.1.6)

Putting this together:

Lemma 1.1.7. *Let $\varphi_1, \ldots, \varphi_n$ be Σ_1–formulae. Let Ψ be formed from $\varphi_1, \ldots, \varphi_n$ using only conjunction, disjunction, existence quantification and bounded universal quantification. Then $\Psi(x_1, \ldots, x_n)$ in uniformly $\Sigma_1(M)$*

An immediate consequence of Lemma 1.1.7 is:

Lemma 1.1.8. *$R \subset M^n$ in $\Sigma_1(M)$ in the parameter $\emptyset$ iff it is $\Sigma_1(M)$ in no parameter.*

Proof. Let $R(\vec{x}) \leftrightarrow R'(\emptyset, \vec{x})$. Then

$$R(\vec{x}) \leftrightarrow \bigvee z(R'(z, \vec{x}) \wedge \bigwedge y \in zy \neq y).$$

QED (Lemma 1.1.8)

Note R is in fact *uniformly* $\Sigma_1(M)$ in the sense that its Σ_1 definition depends only on the original Σ_1 definition of R from $\emptyset$, and not on M.

Lemma 1.1.9. *Let $R(y_1, \ldots, y_n)$ be a relation which is $\Sigma_1(M)$ in the the parameter p. For $i = 1, \ldots, n$ let $f_i(x_1, \ldots, x_m)$ be a partial function on M which (as a relation) is $\Sigma_1(M)$ in p. Then the following relation is uniformly $\Sigma_1(M)$ in p:*

$$R(f_1(\vec{x}), \ldots, f_n(\vec{x})) \leftrightarrow: \bigvee y_1 \ldots y_n(\bigwedge_{i=1}^{n} y_i = f_i(\vec{x}) \wedge R(\vec{y})).$$

This follows by Lemma 1.1.7. ("Uniformly" again mean that the Σ_1 definition depends only on the Σ_1 definition of $R, f_1, \ldots, f_n$.)

Similarly:

Lemma 1.1.10. *Let $f(y_1, \ldots, y_n), g_i(x_1, \ldots, x_n)(i = 1, \ldots, n)$ be partial functions which are $\Sigma_1(M)$ in p, then the function $h(\vec{x}) \simeq f(g(\vec{x}))$ is uniformly $\Sigma_1(M)$ in p.*

Proof.

$$z = h(\vec{x}) \leftrightarrow \bigvee y_1 \ldots y_n(\bigwedge_{i=1}^{n} y_i = g_i(\vec{x}) \wedge z = f(\vec{y})).$$

QED (Lemma 1.1.10)

Lemma 1.1.11. *Let $f_i(\vec{x})$ be a function which is $\Sigma_1(M)$ in $p (i = 1, \dots, n)$. Let $R_i(\vec{x}) (i = 1, \dots, n)$ be mutually exclusive relations which are $\Sigma_1(M)$ in p. Then the function*

$$f(\vec{x}) \simeq f_i(\vec{x}) \text{ if } R_i(\vec{x})$$

is uniformly $\Sigma_1(M)$ in p.

Proof.

$$y = f(\vec{x}) \leftrightarrow \bigvee_{i=1}^{n} (y = f_i(\vec{x}) \wedge R_i(\vec{x})).$$

QED (Lemma 1.1.11)

Using these facts, we see that the restrictions of many standard set theoretic functions to M are $\Sigma_1(M)$.

Lemma 1.1.12. *The following functions are uniformly $\Sigma_1(M)$:*

(a) $f(x) = x, f(x) = \cup x, f(x, y) = x \cup y, f(x, y) = x \cap y, f(x, y) = x \setminus y$ (set difference)

(b) $f(x) = C_n(x)$, where $C_0(x) = x, C_{n+1}(x) = C_n(x) \cup \bigcup C_n(x)$

(c) $f(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$

(d) $f(x) = i$ (where $i < \omega$)

(e) $f(x_1, \dots, x_n) = \langle x_1, \dots, x_n \rangle$

(f) $f(x) = \mathrm{dom}(x)$, $f(x) = \mathrm{rng}(x)$, $f(x, y) = x''y, f(x, y) = x \upharpoonright y$, $f(x) = x^{-1}$

(g) $f(x_1, \dots, x_n) = x_1 \times x_2 \times \dots \times x_n$

(h) $f(x) = (x)_i^n$ where $(\langle z_0, \dots, z_{n-1} \rangle)_i^n = z_i$ and $(u)_i^n = \emptyset$ in all other cases

(i) $f(x, z) = x[z] = \begin{cases} x(z) \text{ if } x \text{ is a function} \\ \text{and } z \in \mathrm{dom}(x) \\ \emptyset \text{ otherwise.} \end{cases}$

Proof. We display sample proofs. (a) is straightforward. (b) follows by induction on n. To see (c), $y = \{x_1, \dots, x_n\}$ can be expressed by the Σ_0–statement

$$x_1, \dots, x_n \in y \wedge \bigwedge z \in y (z = x_1 \vee \dots \vee z = x_n).$$

(d) follows by induction on i, since

$$0 = \emptyset, i+1 = i \cup \{i\}.$$

The proof of (e) depends on the precise definition of $\langle x_1, \ldots x_n\rangle$. If we want each tuple to have a unique length, then the following definition recommends itself: First define a notion of ordered pair by: $(x, y) =: \{\{x\}, \{x, y\}\}$ Then (x, y) is a Σ_1 function. Then iff: $\langle x_1, \ldots, x_n\rangle =: \{(x_1, 0), \ldots, (x_n, n-1)\}$, the conclusion is immediate.

For (f) we display the proof that $\mathrm{dom}(x)$ is a Σ_1 function. Note that $x, y \in C_n(\langle x, y\rangle)$ for a sufficient n. But since every element of $\mathrm{dom}(x)$ is a component of a pair lying in x, it follows that $\mathrm{dom}(x) \subset C_n(x)$ for a sufficient n. Hence $y = \mathrm{dom}(x)$ can be expressed as:

$$\bigwedge z \in y \bigvee w \langle w, z\rangle \in x \wedge \bigwedge z, w \in C_n(x)(\langle w, z\rangle \in x \to z \in y).$$

To see (g), note that $y = x_1 \times \ldots \times x_n$ can be expressed by:

$$\begin{array}{l} \bigwedge z_1 \in x_1 \ldots \bigwedge z_n \in x_n \langle z_1, \ldots, z_n\rangle \in y \\ \wedge \bigwedge w \in y \bigvee z_1 \in x_1 \ldots \bigvee z_n \in x_n w = \langle z_1, \ldots, z_n\rangle. \end{array}$$

To see (h) note that, for sufficiently large $m, y = (x)^n_i$ can be expressed by:

$$\begin{array}{l} \bigvee z_0 \ldots z_{n-1}(x = \langle z_0, \ldots, z_{n-1}\rangle \wedge y = z_i) \\ \vee(y = \emptyset \wedge \bigwedge z_0 \ldots z_{n-1} \in C_m(x) x \neq \langle z_0, \ldots, z_{n-1}\rangle) \end{array}$$

(i) is similarly straightforward. QED (Lemma 1.1.12)

The *recursion theorem* of classical recursion theory says that if $g(n, m)$ is recursive on ω and $f : \omega \to \omega$ is defined by:

$$f(0) = k, f(n+1) = g(n, f(n)),$$

then f is recursive. The point is that the value of f at any n is determined by its values at smaller numbers. Working with H instead of ω we can express this in the elegant form:

> Let $g : \omega \times H \to \omega$ be Σ_1.
> Then $f : \omega \to \omega$ is Σ_1, where $f(n) = g(n, f \upharpoonright n)$.

If we take $g : H^2 \to H$, then f will be Σ_1 where $f(x) = g(x, f \upharpoonright x)$ for $x \in H$. We can even take g as being a partial function on H^2. Then f is Σ_1 where:

$$f(x) \simeq g(x, \langle f(z) | z \in x\rangle).$$

(This means that $f(x)$ is defined if and only if $f(z)$ is defined for $z \in x$ and g is defined at $\langle x, f \restriction x \rangle$, in which case the above equality holds.)

We now prove the same thing for an arbitrary admissible M. If f is a partial $\underline{\Sigma}_1$ function and $x \subset \mathrm{dom}(f)$, we know by Lemma 2.2.3 that $f''x \in M$. But then $f \restriction x \in M$, since $f^*(z) \simeq \langle f(z), z \rangle$ is a $\underline{\Sigma}_1$ function with $x \subset \mathrm{dom}(f^*)$, and $f^{*\prime\prime}x = f \restriction x$. The *recursion theorem for admissibles* $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ then reads:

Lemma 1.1.13. *Let $G(y, \vec{x}, u)$ be a $\Sigma_1(M)$ function in the parameter p. Then there is exactly one function $F(y, \vec{x})$ such that*

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y \rangle).$$

Moreover, F is uniformly $\Sigma_1(M)$ in p (i.e. the Σ_1 definition depends only on the Σ_1 definition of G.)

Proof. We first show existence. Set:

$$\Gamma_{\vec{x}} =: \{f \in M | f \text{ is a function } \wedge \mathrm{dom}(f) \text{ is transitive } \wedge \bigwedge y \in \mathrm{dom}(f) f(y) = G(y, \vec{x}, f \restriction y)\}$$

Set $F_{\vec{x}} = \bigcup \Gamma_{\vec{x}}; F = \{\langle y, \vec{x} \rangle | y \in F_{\vec{x}}$. Then F is in $\Sigma_1(M)$ in p uniformly.

(1) F is a function.

Proof. Suppose not. Then for some $\vec{x}$ there are $f, f' \in \Gamma_{\vec{x}}$, $y \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$ such that $f(y) \neq f'(y)$. Let y be $\in$–minimal with this property. Then $f \restriction y = f' \restriction y$. But then $f(y) = G(y, \vec{x}, f \restriction y) = G(y, \vec{x}, f', \restriction y) = f'(y)$. Contradiction! QED (1)

Hence $F(y) = f(y)$ if $y \in \mathrm{dom}(f)$ and $f \in \Gamma_{\vec{x}}$.

(2) Let $\langle y, \vec{x} \rangle \in \mathrm{dom}(F)$. Then $y \subset \mathrm{dom}(F_{\vec{x}}), \langle y, \vec{x}, \langle F(z, \vec{x}) | z \in y \rangle \rangle \in \mathrm{dom}(G)$ and

$$F(y, \vec{x}) = G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y \rangle).$$

Proof. Let $y \in \mathrm{dom}(f), f \in \Gamma_{\vec{x}}$. Then

$$\begin{aligned} F(y, \vec{x}) = f(y) &= G(y, \vec{x}, f \restriction x) \\ &= G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y \rangle). \end{aligned}$$

QED (2)

(3) Let $y \subset \mathrm{dom}(F_{\vec{x}}), \langle y, \vec{x}, F_{\vec{x}} \restriction y \rangle \in \mathrm{dom}(G)$. Then $y \in \mathrm{dom}(F_{\vec{x}})$.

Proof. By our assumption: $\bigwedge z \in y \bigvee f (f \in \Gamma_{\vec{x}} \wedge z \in \mathrm{dom}(f))$. Hence there is $u \in M$ such that

$$\bigwedge z \in y \bigvee f \in u (f \in \Gamma_{\vec{x}} \wedge z \in \mathrm{dom}(f)).$$

Set: $f' = \bigcup(u \cap \Gamma_{\vec{x}})$. Then $f' \in \Gamma_{\vec{x}}$ and $y \subset \mathrm{dom}(f')$. Moreover $f' \upharpoonright y = F_{\vec{x}} \upharpoonright y$. Set $f'' = f' \cup \{\langle G(y, \vec{x}, f' \upharpoonright y), y\rangle\}$. Then $f'' \in \Gamma_{\vec{x}}$ and $y \in \mathrm{dom}(f'')$, where $f'' \subset F_{\vec{x}}$. QED (3)

This proves existence. To show uniqueness, we virtually repeat the proof of (1): Let F^* satisfy the same condition. Set $F^*_{\vec{x}}(y) \simeq F^*(y, \vec{x})$. Suppose $F^* \neq F$. Then $F^*_{\vec{x}}(y) \not\simeq F_{\vec{x}}(y)$ for some $\vec{x}, y$. Let y be $\in$–minimal ect. $F^*_{\vec{x}}(y) \not\simeq F_{\vec{x}}(y)$. Then $F^*_{\vec{x}} \upharpoonright y = F_{\vec{x}} \upharpoonright y$. Hence

$$\begin{aligned} F^*_{\vec{x}}(y) &\simeq G(y, \vec{x}, \langle F^*_{\vec{x}}(z) | z \in y\rangle) \\ &\simeq G(y, \vec{x}, \langle F_{\vec{x}}(z) | z \in y\rangle) \\ &\simeq F_{\vec{x}}(y). \end{aligned}$$

Contradiction! QED (Lemma 1.1.13)

We recall that the transitive closure $TC(x)$ of a set x is recursively definable by: $TC(x) = x \cup \bigcup_{z \in x} TC(z)$. Similarly, the rank $rn(x)$ of a set is definable by $rn(x) = \mathrm{lub}\{rn(z) | z \in x\}$. Hence:

Corollary 1.1.14. *TC, rn are uniformly $\Sigma_1(M)$.*

The successor function $s\alpha = \alpha + 1$ on the ordinals is defined by:

$$sx = \begin{cases} x \cup \{x\} \text{ if } x \in On \\ \text{undefined if not} \end{cases}$$

which is Σ_1. The function $\alpha + \beta$ is defined by:

$$\begin{aligned} &\alpha + 0 = \alpha \\ &\alpha + s\nu = s(\alpha + \nu) \\ &\alpha + \lambda = \textstyle\bigcup_{\nu < \lambda} \alpha + \nu \text{ for limit } \lambda. \end{aligned}$$

This has the form:

$$x + y \simeq G(y, x, \langle x + z | z \in y\rangle).$$

Similarly for the function $x \cdot y, x^y, \ldots$ etc. Hence:

Corollary 1.1.15. *The ordinal functions $\alpha + 1, \alpha + \beta, \alpha^\beta, \ldots$ etc. are uniformly $\Sigma_1(M)$.*

We note that there is an even more useful form of Lemma 1.1.13:

Lemma 1.1.16. *Let G be as in Lemma 1.1.13. Let $h : M \to M$ be $\Sigma_1(M)$ in p r.t. $\{\langle x, y\rangle | x \in h(y)\}$ is well founded. There is a unique f such that*

$$F(y) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | x \in h(y)\rangle).$$

Moreover, F is uniformly[1] *$\Sigma_1(M)$ in p.*

[1](“uniformly” meaning, of course, that the Σ_1 definition of F depends only on the Σ_1 definition of G, h.)

The proof is exactly like that of Lemma 1.1.13, using minimality in the relation $\{\langle x, y\rangle | x \in h(y)\}$ in place of $\in$–minimality. We now consider the structure of “really finite” sets in an admissible M.

Lemma 1.1.17. *Let $u \in H_\omega$. The class u and the constant function $f(x) = u$ are uniformly $\Sigma_1(M)$.*

Proof. By $\in$–induction on u: Let $u = \{z_1, \ldots, z_n\}$.

$$x \in u \leftrightarrow \bigvee_{i=1}^{n} x = z_i$$
$$x = u \leftrightarrow \bigwedge y \in x\, y \in u \wedge \bigwedge_{i=1}^{n} z_i \in x.$$

QED

$x \in \omega$ is clearly a Σ_0 condition. But then:

Lemma 1.1.18. *Let $\omega \in M$. Then the constant function $f(x) = \omega$ is uniformly $\Sigma_1(M)$.*

Proof.

$$x = \omega \leftrightarrow (\bigwedge z \in x z \in \omega \wedge \emptyset \in x \wedge \bigwedge z \in x z \cup \{z\} \in x)$$

(where '$z \in \omega$' is Σ_0) QED

Lemma 1.1.19. *The class* Fin *and the function $f(x) = \mathbb{P}_\omega(x)$ are uniformly $\Sigma_1(M)$, where* $\text{Fin} = \{x \in M | \overline{\overline{x}} < \omega\}, \mathbb{P}_\omega(x) = \mathbb{P}(x) \cap \text{Fin}$.

Proof.

$$\begin{aligned} x \in \text{Fin} &\leftrightarrow \bigvee n \in \omega \bigvee f\, f : n \leftrightarrow x \\ y = \mathbb{P}_\omega(x) &\leftrightarrow \bigwedge u \in y (u \subset x \wedge u \in \text{Fin}) \wedge \emptyset \in y \wedge \\ &\wedge \bigwedge z \in x \{z\} \in y \wedge \bigwedge u, v \in y u \cup v \in y. \end{aligned}$$

We must show that $\mathbb{P}_\omega(x) \in M$. If $\omega \notin M$, then $rn(x) < \omega$ for all $x \in M$, Hence $M = H_\omega$ is closed under $\mathbb{P}_\omega$. If $\omega \in M$, there is $\underline{\Sigma}_1(M)$ f defined by

$$f(0) = \{\{z\} | z \in x\}, f(n+1) = \{u \cup v | \langle u, v\rangle \in f(n)^2\}.$$

Then $\mathbb{P}_\omega(x) = \bigcup f''\omega \in M$. QED (Lemma 1.1.19)

But then:

Lemma 1.1.20. *If $\omega \in M$, then $H_\omega \in M$ and the constant function $f(x) = H_\omega$ is uniformly $\Sigma_1(M)$.*

Proof. $H_\omega \in M$, since there is a $\Sigma_1(M)$ function g defined by $g(0) = \emptyset, g(n+1) = \mathbb{P}_\omega(g(n))$. Then $H_\omega = \bigcup g''\omega \in M$ and $f(x) = H_\omega$ is $\Sigma_1(M)$ since g and the constant function ω are $\Sigma_1(M)$. QED (Lemma 1.1.20)

1.1.3 The constructible hierarchy

We recall Gödel's definition of the *constructible hierarchy* $\langle L_r | r \in \mathrm{On}\rangle$:

$$\begin{aligned} &L_0 = \emptyset \\ &L_{\nu+1} = \mathrm{Def}(L_\nu) \\ &L_\lambda = \bigcup_{\nu<\lambda} L_\nu \text{ for limit } \lambda, \end{aligned}$$

where $\mathrm{Def}(u)$ is the set of all $z \subset u$ which are $\langle u, \in\rangle$–definable in parameters from u (taking $\mathrm{Def}(\emptyset) = \{\emptyset\}$). (Note that if u is transitive, then $u \subset \mathrm{Def}(u)$ and $\mathrm{Def}(u)$ is transitive.) Gödel's *constructible universe* is then $L =: \bigcup_{\nu\in\mathrm{On}} L_\nu$.

By fairly standard methods one can show:

Lemma 1.1.21. *Let $\omega \in M$. Then the function $f(u) = \mathrm{Def}(u)$ is uniformly $\Sigma_1(M)$.*

We omit the proof, which is quite lengthy. It involves "arithmetizing" the language of first order set theory by identifying formulae with elements of ω or H_ω, and then showing that the relevant syntactic and semantic concepts are M–recursive.

By the recursion theorem we can of course conclude:

Corollary 1.1.22. *Let $\omega \in M$. The function $f(\alpha) = L_\alpha$ is uniformly $\Sigma_1(M)$.*

The constructible hierarchy *over* a set u is defined by:

$$\begin{aligned} &L_0(u) = TC(\{u\}) \\ &L_{\nu+1}(u) = \mathrm{Def}(L_\nu(u)) \\ &L_\lambda(u) = \bigcup_{\nu<\lambda} L_\nu(u) \text{ for limit } \lambda. \end{aligned}$$

Oviously:

Corollary 1.1.23. *Let $\omega \in M$. The function $f(u, \alpha) = L_\alpha(u)$ is uniformly $\Sigma_1(M)$.*

The constructible hierarchy *relative to* classes $A_1, \ldots, A_n$ is defined by:

$$\begin{aligned} &L_0[\vec{A}] = \emptyset \\ &L_{\nu+1}[\vec{A}] = \mathrm{Def}(L_\nu[\vec{A}], \vec{A}) \\ &L_\lambda[\vec{A}] = \bigcup_{\nu<\lambda} L_\nu[\vec{A}] \text{ for limit } \lambda, \end{aligned}$$

where $\mathrm{Def}(U, A_1, \ldots, A_n)$ is the set of all $z \subset u$ which are $\langle u, \in, A_1 \cap u, \ldots, A_n \cap u\rangle$–definable in parameters from u.

Much as before we have:

Lemma 1.1.24. *Let $\omega \in M$. Let $A_1, \ldots, A_n$ be $\Delta_1(M)$ in the parameter p. Then the function $f(u) = \mathrm{Def}(u, A_1, \ldots, A_n)$ is uniformly $\Sigma_1(M)$ in p.*

Corollary 1.1.25. *Let $\omega \in M$. Let $A_1, \ldots, A_n$ be as above. Then the function $f(\alpha) = L_\alpha[\vec{A}]$ is uniformly $\Sigma_1(M)$ in p.*

(In particular, if $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$. Then $f(\alpha) = L_\alpha[\vec{A}]$ is uniformly $\Sigma_1(M)$.)

(One could, of course, also define $L_\alpha(u)[\vec{A}]$ and prove the corresponding results.)

Any well ordering r of a set u induces a well ordering of $\mathrm{Def}(u)$, since each element of $\mathrm{Def}(u)$ is defined over $\langle u, \in\rangle$ by a tuple $\langle \varphi, x_1, \ldots, x_n\rangle$, where φ is a formula and $x_1, \ldots, x_n$ are elements of u which interpret free variables of φ. If u is transitive (hence $u \subset \mathrm{Def}(u)$), we can also arrange that the well ordering, which we shall call $< (u, r)$, is an end extension of r. The function $< (u, r)$ is uniformly Σ_1. If we then set:

$$<_0 = \emptyset, <_{\nu+1} = < (L_\nu, <_\nu)$$
$$<_\lambda = \bigcup_{\nu<\lambda} <_\nu \text{ for limit } \lambda,$$

it follows that $<_\nu$ is a well ordering of L_ν for all ν. Moreover $<_\alpha$ is an end extension of $<_\nu$ for $\nu < \alpha$.

Similarly, if A is $\Sigma_1(M)$ in p, there is a hierarchy $<_\nu^A$ ($\nu \in \mathrm{On} \cap M$) such that $<_\nu^A$ well orders $L_\nu[A]$ and the function $f(\nu) = <_\nu^A$ is $\Sigma_1(M)$ in p (uniformly relative to the Σ_1 definition of A).

By corollary 1.1.25 we easily get:

Lemma 1.1.26. *Let $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$ be admissible. Let $\alpha = \mathrm{On} \cap M$. Then $\langle L_\alpha[\vec{A}], \in \vec{A}\rangle$ is admissible.*

Proof: Set: $L_\nu^{\vec{A}} = \langle L_\nu[\vec{A}], \in, \vec{A}\rangle$. Axiom (1) holds trivially in $L_\nu^{\vec{A}}$.

To verify the Σ_0–axiom of subsets, let B be $\underline{\Sigma}_0(L_\alpha^{\vec{A}})$. Let $u \in L_\alpha^{\vec{A}}$.

Claim $u \cap B \in L_\alpha^{\vec{A}}$.

Proof: Pick $\nu < \alpha$ such that $u \in L_\nu^{\vec{A}}$ and B is $\underline{\Sigma}_0$ in parameters from $L_\nu^{\vec{A}}$. By $\underline{\Sigma}_0$–absoluteness we have:

$$u \cap B \in \mathrm{Def}(L_\nu^{\vec{A}}) = L_{\nu+1}^{\vec{A}} \subset L_\alpha^{\vec{A}}.$$

QED (Claim)

We now prove Σ_0–collection. Let Rxy be a $\underline{\Sigma}_0$–relation. Let $u \in L_\alpha^{\vec{A}}$ such that $\bigwedge x \in u \bigvee yRxy$.

Claim $\bigvee v \in L_\alpha^{\vec{A}} \bigwedge x \in u \bigvee y \in vRxy$.

For each $x \in u$ let $g(x)$ be the least $\nu < \alpha$ such that $x \in L_\nu^{\vec{A}}$. Then g is in $\underline{\Sigma}_1(M)$ and $u \subset \mathrm{dom}(g)$. Hence $\delta = \sup g''u < \alpha$ and

$$\bigwedge x \in u \bigvee y \in L_\delta^{\vec{A}} Rxy.$$

QED (Lemma 1.1.26)

Definition 1.1.2. Let α be an ordinal.

- α is *admissible* iff L_α is admissible
- α is *admissible in* $A_1, \ldots, A_n \subset$ iff $L_\alpha^{\vec{A}} =: \langle L_\alpha[\vec{A}], \in \vec{A}\rangle$ is admissible
- $f : \alpha^n \to \alpha$ is *α–recursive* (in $\vec{A}$) iff f is $\underline{\Sigma}_1(L_\alpha)(\underline{\Sigma}_1(L_\alpha^{\vec{A}}))$
- $R \subset \alpha^n$ is *r.e.* (in $\vec{A}$) iff R is $\underline{\Sigma}_1(L_\alpha(\Sigma_1(L_\alpha^{\vec{A}}))$.

(**Note** The theory of α–recursive functions and relations on an admissible α has been built up without references to L_α, using a formalized notion of α–bounded calculus (Kripke) or α–bounded algorithm (Platek).

Similarly for α–recursiveness in $A_1, \ldots, A_n$, taking the A_i as "oracles")

A transitive structure $M = \langle |M|, \in \vec{A}\rangle$ is called *strongly admissible* iff, in addition to the Kripke–Platek axioms, it satisfies the Σ_1 *axiom of subsets*:

$$x \cap \{z | \varphi(z)\} \text{ is a set (for } \Sigma_1 \text{ formulae } \varphi).$$

Kripke defines the *projectum* δ_α of an admissible ordinal α to be the least δ such that $A \cap \delta \notin L_\alpha$ for some $\underline{\Sigma}_1(M)$ set A. He shows that $\delta_\alpha = \alpha$ iff α is strongly admissible. He calls α *projectible* iff $\delta_\alpha < \alpha$. There are many projectible admissibles — e.g. $\delta_\alpha = \omega$ if α is the least admissible greater than ω. He shows that for every admissible α there is a $\underline{\Sigma}_1(L_\alpha)$ injection f_α of L_α into δ_α.

The definition of projectum of course makes sense for *any* $\alpha \geq \omega$. By refinements of Kripke's methods it can be shown that f_α exists for every $\alpha \geq \omega$ and that $\delta_\alpha < \alpha$ whenever $\alpha \geq \omega$ is not strongly admissible. We shall — essentially — prove these facts in chapter 2 (except that, for technical reasons, we shall employ a modified version of the constructible hierarchy).

1.2 Primitive Recursive Set Functions

1.2.1 PR Functions

The *primitive recursive set functions* comprise a collection of functions

$$f : V^n \to V$$

which form a natural analogue of the primitive recursive number functions in ordinary recursion theory. As with admissibility theory, their discovery arose from the attempt to generalize ordinary recursion theory. These functions are ubiquitous in set theory and have very attractive absoluteness properties. In this section we give an account of these functions and their connection with admissibility theory, though — just as in §1 — we shall suppress some proofs.

Definition 1.2.1. $f : V^n \to V$ is a *primitive recursive (pr) function* iff it is generated by successive application of the following schemata:

(i) $f(\vec{x}) = x_i$ (here $\vec{x}$ is $x_1, \ldots, x_n$)

(ii) $f(\vec{x}) = \{x_i, x_j\}$

(iii) $f(\vec{x}) = x_i \setminus x_j$

(iv) $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$

(v) $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$

(vi) $f(y, \vec{x}) = g(y, \vec{x}, \langle f(z, \vec{x}) | z \in y \rangle)$

We also define:

Definition 1.2.2. $R \subset V^n$ is a *primitive recursive relation* iff there is a primitive recursive function r such that $R = \{\langle \vec{x} \rangle | r(\vec{x}) \neq \emptyset\}$.

(**Note** It is possible for a function on V to be primitive recursive as a relation but not as a function!)

We begin by developing some elementary consequences of these definitions:

Lemma 1.2.1. *If $f : V^n \to V$ is primitive recursive and $k : n \to m$, then g is primitive recursive, where*

$$g(x_1, \ldots, x_m) = f(x_{k(1)}, \ldots, x_{k(n)}).$$

proof by (i), (iv).

Lemma 1.2.2. *The following functions are primitive recursive*

(a) $f(\vec{x}) = \bigcup x_j$

(b) $f(\vec{x}) = x_i \cup x_j$

(c) $f(\vec{x}) = \{\vec{x}\}$

(d) $f(\vec{x}) = n$, *where* $n < \omega$

(e) $f(\vec{x}) = \langle \vec{x} \rangle$

Proof.

(a) By (i), (v), Lemma 1.2.1, since $\bigcup x_j = \bigcup_{z \in x_j} z$

(b) $x_i \cup x_j = \bigcup \{x_i, x_j\}$

(c) $\{\vec{x}\} = \{x_1\} \cup \ldots \cup \{x_m\}$

(d) By in induction on n, since $0 = x \setminus x, n + 1 = n \cup \{n\}$

(e) The proof depends on the precise definition of n–tuple. We could for instance define $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$ and $\langle x_1, \ldots, x_n \rangle = \langle x_1, \langle x_2, \ldots, x_n \rangle \rangle$ for $n > 2$.

If, on the other hand, we wanted each tuple to have a unique length, we could call the above defined ordered pair (x, y) and define:

$$\langle x_1, \ldots, x_n \rangle = \{(x_1, 0), \ldots, (x_n, n - 1)\}.$$

QED (Lemma 1.2.2)

Lemma 1.2.3. *(a)* $\notin$ *is pr*

(b) If $f : V^n \to V, R \subset V^n$ *are primitive recursive, then so is*

$$g(\vec{x}) = \begin{cases} f(\vec{x}) \text{ if } R\vec{x} \\ \emptyset \text{ if not} \end{cases}$$

(c) $R \subset V^n$ *is primitive recursive iff its characteristic functions* X_R *is a primitive recursive function*

(d) If $R \subset V^n$ *is primitive recursive so is* $\neg R =: V^n \setminus R$

(e) Let $f_i : V^n \to V, R_i \subset V^n$ *be* $pr(i = 1, \ldots, m)$ *where* $R_1, \ldots, R_n$ *are mutually disjoint and* $\bigcup_{i=1}^{n} R_i = V^n$*. Then* f *is pr where:*

$$f(\vec{x}) = f_i(x) \text{ when } R_i\vec{x}.$$

(f) If $Rz\vec{x}$ *is primitive recursive, so is the function*

$$f(y, \vec{x}) = y \cap \{z | Rz\vec{x}\}$$

(g) If $Rz\vec{x}$ *is primitive recursive so is* $\bigvee z \in yRz\vec{x}$

(h) If $R_i\vec{x}$ *is primitive recursive* $i = 1, \ldots, m)$*, then so is* $\bigvee_{i=1}^{m} R_i\vec{x}$

(i) If $R_1, \ldots, R_n$ *are primitive recursive relations and* φ *in a* Σ_0 *formula, then* $\{\langle\vec{x}\rangle | \langle V, R_1, \ldots, R_n\rangle \models \varphi[\vec{x}]\}$ *is primitive recursive.*

(j) If $f(z, \vec{x}$ *is primitive recursive, then so are:*

$$g(y, \vec{x}) = \{f(z, \vec{x} | z \in y\}$$
$$g'(y, \vec{x}) = \langle f(z, \vec{x}) | z \in y\rangle$$

(k) If $R(z, \vec{x})$ *is primitive recursive, then so is*

$$f(y, \vec{x}) = \begin{cases} \text{That } z \in y \text{ such that } Rz\vec{x} \text{ if exactly} \\ \text{one such } z \in y \text{ exists;} \\ \emptyset \text{ if not.} \end{cases}$$

Proof.

(a) $x \notin y \leftrightarrow \{x\} \setminus y \neq \emptyset$

(b) Let $R\vec{x} \leftrightarrow r(\vec{x}) \neq \emptyset$. Then $g(\vec{x}) = \bigcup_{z \in r(\vec{x})} f(\vec{x})$.

(c) $X_r(\vec{x}) = \begin{cases} 1 \text{ if } R\vec{x} \\ 0 \text{ if not} \end{cases}$

(d) $X_{\neg R}(\vec{x}) = 1 \setminus X_R(\vec{x})$

(e) Let $f'_i(\vec{x}) = \begin{cases} f_i(\vec{x}) \text{ if } R_i\vec{x} \\ \emptyset \text{ if not} \end{cases}$
Then $f(\vec{x}) = f'_i(\vec{x}) \cup \ldots \cup f'_m(\vec{x})$.

(f) $f(y, \vec{x}) = \bigcup_{z \in y} h(z, \vec{x})$, where:

$$h(z, \vec{x}) = \begin{cases} \{z\} \text{ if } Rz\vec{x} \\ \emptyset \text{ if not} \end{cases}$$

(g) Let $Py\vec{x} \leftrightarrow: \bigvee z \in yRz\vec{x}$. Then $X_P(\vec{x}) = \bigcup_{z \in y} X_R(z, \vec{x})$.

(h) Let $P\vec{x} \leftrightarrow \bigvee_{i=1}^{m} R_i\vec{x}$. Then

$$X_P(\vec{x}) = X_{R_1} \cup \ldots \cup X_{R_n}(\vec{x}).$$

(i) is immediate by (d), (g), (h)

(j) $g(y, \vec{x}) = \bigcup_{z \in y} \{f(z, \vec{x})\}, g'(y, \vec{x}) = \bigcup_{z \in y} \{\langle f(z, \vec{x}), z \rangle\}$

(k) $R'zu\vec{x} \leftrightarrow: (z \in u \wedge Rz\vec{x} \wedge \bigwedge z' \in u(z \neq z' \to \neg Rz'\vec{x}))$ is primitive recursive by (i). But then:

$$f(y, \vec{x}) = \bigcup (y \cap \{z | R'zy\vec{x}\})$$

QED (Lemma 1.2.3)

Lemma 1.2.4. *Each of the functions listed in §1 Lemma 1.1.12 is primitive recursive.*

The proof is left to the reader.

Note Up until now we have only made use of the schemata (i) – (v). This will be important later. The functions and relations obtainable from (i) – (v) alone are called *rudimentary* and will play a significant role in fine structure theory. We shall use the fact that Lemmas 1.2.1 – 1.2.3 hold with "rudimentary" in place of "primitive recursive".

Using the recursion schema (vi) we then get:

Lemma 1.2.5. *The functions $TC(x), rn(x)$ are primitive recursive.*

The proof is the same as before (§1 Corollary 1.1.14).

Definition 1.2.3. $f : \mathrm{On}^n \times V^m \to V$ is primitive recursive iff f' is primitive recursive, where

$$f'(\vec{y},\vec{x}) = \begin{cases} f(\vec{y},\vec{x}) \text{ if } y_1,\dots,y_n \in \mathrm{On} \\ \emptyset \text{ if not} \end{cases}$$

As before:

Lemma 1.2.6. *The ordinal function $\alpha+1, \alpha+\beta, \alpha\cdot\beta, \alpha^\beta, \dots$ are primitive recursive.*

Definition 1.2.4. Let $f : V^{n+1} \to V$.

$f^\alpha (\alpha \in \mathrm{On})$ is defined by:

$$\begin{aligned} &f^0(y,\vec{x}) = y \\ &f^{\alpha+1}(y,\vec{x}) = f(f^\alpha(y,\vec{x}),\vec{x}) \\ &f^\lambda(y,\vec{x}) = \bigcup_{r<\lambda} f^r(y,\vec{x}) \text{ for limit } \lambda. \end{aligned}$$

Then:

Lemma 1.2.7. *If f is primitive recursive, so is $g(\alpha,y,\vec{x}) = f^\alpha(y,\vec{x})$.*

There is a strengthening of the reursion schema (vi) which is analogue to §1 Lemma 1.1.16. We first define:

Definition 1.2.5. Let $h : V \to V$ be primitive recursive. h is *manageable* iff there is a primitive recursive $\sigma : V \to \mathrm{On}$ such that

$$x \in h(y) \to \sigma(x) < \sigma(y).$$

(Hence the relation $x \in h(y)$ is well founded.)

Lemma 1.2.8. *Let h be manageable. Let $g : V^{n+2} \to V$ be primitive recursive. Then $f : V^{n+1} \to V$ is primitive recursive, where:*

$$f(y,\vec{x}) = g(y,\vec{x},\langle f(z,\vec{x}) | z \in h(y)\rangle).$$

Proof. Let σ be as in the above definition. Let $|x| = \mathrm{lub}\{|y| | y \in h(x)\}$ be the rank of x in the relation $y \in h(x)$. Then $|x| \le \sigma(x)$. Set:

$$\Theta(z,\vec{x},u) = \bigcup_{\substack{y\in u \\ h(y)\subset \mathrm{dom}(z)}} \{\langle g(y,\vec{x},z \upharpoonright h(y)), y\rangle | y \in u \wedge h(y) \subset \mathrm{dom}(z)\}.$$

By induction on α, if u is h–closed (i.e. $x \in u \to h(x) \subset u$), then:

$$\Theta^{\alpha}(\emptyset, \vec{x}, u) = \langle f(y, \vec{x}) | y \in u \wedge |y| < \alpha \rangle$$

Set $\tilde{h}(v) = v \cup \bigcup_{z \in v} h(z)$. Then $\tilde{h}^{\alpha}(\{y\})$ in h–closed for $\alpha \geq |y|$. Hence:

$$f(y, \vec{x}) = \Theta^{\sigma(y)+1}(\emptyset, \vec{x}, \tilde{h}^{\sigma(y)}(\{y\}))(y).$$

QED (Lemma 1.2.8)

Corresponding to §1 Lemma 1.1.17 we have:

Lemma 1.2.9. *Let $u \in H_\omega$. The constant function $f(x) = u$ is primitive recursive.*

Proof: By $\in$–induction on u. QED

As we shall see, the constant function $f(x) = \omega$ is not primitive recursive, so the analogue of §1 Lemma 1.1.18 fails.

In place of §1 Lemma 1.1.19 we get:

Lemma 1.2.10. *The class* Fin *and the function $f(x) = \mathbb{P}_\omega(x)$ are primitive recursive in the parameter ω.*

Proof: Let f be primitive recursive such that $f(0, x) = \{\{z\} | z \in x\}$, $f(n+1, x) = \{u \cup v | \langle u, v \rangle \in f(n, x)^2\}$. Then $\mathbb{P}_\omega(x) = \bigcup_{n \in \omega} f(n, x)$. But then:

$$x \in \text{Fin} \leftrightarrow \bigvee n \in \omega \bigvee g \in \bigcup_{n < \omega} \mathbb{P}^n_\omega(x \times \omega) g : n \leftrightarrow x.$$

QED

Corollary 1.2.11. *The constant function $f(x) = H_\omega$ is primitive recursive in the parameter ω.*

Proof: $H_\omega = \bigcup_{n < \omega} \mathbb{P}^n_\omega(\emptyset)$. QED

Corresponding to Lemma 1.1.21 of §1 we have:

Lemma 1.2.12. *The function* Def(u) *is primitive recursive in the parameter ω.*

The proof involves carrying out the proof of §1 Lemma 1.1.21 (which we also omitted) while ensuring that the relevant classes and functions are primitive recursive. We give not further details here (though filling in the details can be an arduous task). A fuller account can be found in [PR] or [AS].

Hence:

Corollary 1.2.13. *The function $f(\alpha) = L_\alpha$ is primitive recursive in ω.*

Similarly:

Lemma 1.2.14. *The function $f(\alpha, x) = L_\alpha(x)$ is primitive recursive in ω.*

Lemma 1.2.15. *Let $A \subset V$ be primitive recursive in the parameter p. Then $f(\alpha) = L_\alpha^A$ is primitive recursive in p.*

One can generalize the notion primitive recursive to *primitive recursive in the class $A \subset V$* (or *in the classes $A_1, \ldots, A_n \subset V$*).

We define:

Definition 1.2.6. Let $A_1, \ldots, A_n \subset V$. The function $f : V^n \to V$ is *primitive recursive in $A_1, \ldots, A_n$* iff it is obtained by successive applications of the schemata (i) – (vi) together with the schemata:

$$f(x) = X_{A_i}(x)(i = 1, \ldots, n).$$

A relation R is primitive recursive in $A_1, \ldots, A_n$ iff

$$R = \{\langle \vec{x} \rangle | f(\vec{x}) \neq 0\}$$

for a function f which is primitive recursive in $A_1, \ldots, A_n$.

It is obvious that all of the previous results hold with "primitive recursive in $A_1, \ldots, A_n$" in place of "primitive recursive".

By induction on the defining schemata of f we can show:

Lemma 1.2.16. *Let f be primitive recursive in $A_1, \ldots, A_n$, where each A_i is primitive recursive in $B_1, \ldots, B_m$. Then f is primitive recursive in $B_1, \ldots, B_m$.*

The proof is by induction on the defining schemata leading from $A_1, \ldots, A_n$ to f. The details are left to the reader. It is clear, however, that this proof is *uniform* in the sense that the schemata which give in f from $B_1, \ldots, B_m$ are not dependent on $B_1, \ldots, B_m$ or $A_1, \ldots, A_n$, but only on the schemata which lead from $A_1, \ldots, A_n$ to f and the schemata which led from $B_1, \ldots, B_m$ to $A_i (i = 1, \ldots, n)$.

This will be made more precise in §1.2.2

1.2.2 PR Definitions

Since primitive recursive functions are proper classes, the foregoing discussion must ostensibly be carried out in second order set theory. However, we can translate it into ZF by talking about *primitive recursive definitions*. By a primitive recursive definition we mean a finite sequence of equations of the form (i) – (vi) such that:

- The function variable on the left side does not occur in a previous equation in the sequence
- every function variable on the right side occurs previously on the left side with the same number of argument places.

We assume that the language in which we write these equation has been *arithmetized* — i.e. formulae, terms, variables etc. have been identified in a natural way with elements of ω (or at least H_ω).

Every primitive recursive definition s defines a function F_s. If $s = \langle s_0, \ldots, s_{n-1}\rangle$, then $F_s = F_s^{n-1}$, where F_s^i interprets the leftmost function variable of s_i. This is defined in a straightforward way. If e.g. s_i is "$f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$" and g was leftmost in s_j, then we get

$$F^i(y, \vec{x}) = \bigcup_{z \in y} F^j(z, \vec{x}).$$

Let PD be the class of primitive recursive definitions. In order to define $\{\langle x, s\rangle | s \in PD \wedge x \in F_s\}$ in ZF we proceed as follows:

Let $s = \langle s_0, \ldots, s_{n-1}\rangle \in PD$. Let M be any admissible structure. By induction we can then define $\langle F_s^{i,M} | i < n\rangle$ where F_s^i a function on M^{n_i} (n_i being the number of argument places). By admissibility we know that F_s^i exists and is defined on all of M^{n_i}. We then set: $F_s^M = F_s^{n-1,M}$. This defines the set $\langle F_i^M | s \in PD\rangle$. If $M \subseteq M'$ and M' is also admissible, it follows by an emy induction on $i < n$ that $F^{i,M} = F^{i,M'} \upharpoonright M$. Hence $F_s^M \subset F_s^{M'}$. We can then set:

$$F_s = \bigcup \{F_s^M | M \text{ is admissible}\}.$$

Note that by §1, each F_s^M has a uniform Σ_1 definition φ_s which defines F_s^M over *every* admissible M. It follows that φ_s defines F_s in V. Thus we have won an important absoluteness result: Every primitive recursive function has a Σ_1 definition which is absolute in all inner models, in all generic extensions of V, and indeed, in all admissible structures $M = \langle |M|, \in\rangle$. This absoluteness phenomenon is perhaps the main reason for using the theory of primitive recursive functions in set theory. Carol Karp was the first to notice the phenomenon — and to plumb its depths. She proved results going well beyond what I have stated here, showing for instance that the canonical Σ_1 definition can be so chosen, that $F_s \restriction M$ is the function defined over M by φ_s whenever M is transitive and closed under primitive recursive function. She also improved the characterization of such M: Call an ordinal α *nice* if it is closed under each of the function:

$$f_0(\alpha,\beta) = \alpha + \beta; f_1(\alpha,\beta) = \alpha \cdot \beta, f_2(\alpha,\beta) = \alpha^\beta \ldots \text{ etc.}$$

(More precisely: $f_{i+1}(\alpha,\beta) = \tilde{f}_i^\beta(\alpha)$ for $i \geq 1$, where $\tilde{f}_i(\alpha) = f_i(\alpha,\alpha)$, $g^\beta(\alpha)$ is defined by: $g^0(\alpha) = \alpha$, $g^{\beta+1}(\alpha) = g(g^\beta(\alpha))$, $g^\lambda(\alpha) = \sup_{v<\lambda} g^v(\alpha)$ for limit λ.) She showed that L_α is primitive recursively closed iff α is nice. Moreover, $L_\alpha[A_1,\ldots,A_n]$ is closed under functions primitive recursive in $A_1,\ldots,A_n$ iff α is nice.

Primitive recursiveness in classes $A_1,\ldots,A_n$ can also be discussed in terms of primitive recursive definitions. To this end we appoint new designated function variable $\dot{a}_i (i = 1,\ldots,n)$, which will be interpreted by $X_{A_i} (i = 1,\ldots,n)$. By a *primitive recursive definition in* $\dot{a}_1,\ldots,\dot{a}_n$ we mean a sequence of equation having either the form (i) – (vi), in which $\dot{a}_1,\ldots,\dot{a}_n$ do not appear, or the form

(*) $f(x_1,\ldots,x_p) = \dot{a}_i(x_j) (i = 1,\ldots,n, j = 1,\ldots,p)$

We impose our previous two requirements on all equations not of the form (*).

If $s = \langle s_0,\ldots,s_{n-1}\rangle$ is a pr definition in $\dot{a}_1,\ldots,\dot{a}_n$, we successively define $F_s^{i,A_1,\ldots,A_n} (i < n)$ as before, setting $F_s^{i,\vec{A}}(x_1,\ldots,x_p) = X_{A_i}(x_j)$ if s_i has the form (*). We again set $F_s^{\vec{A}} = F_s^{n-1,\vec{A}}$. The fact that $\{\langle x,s\rangle | x \in F_s^{\vec{A}}\}$ is uniformly $\langle V,\in,A_1,\ldots,A_n\rangle$ definable is shown essentially as before:

Given an admissible $M = \langle |M|,\in,a_1,\ldots,a_n\rangle$ we define $F_s^{i,M}, F_s^M = F_s^{n-1,M}$ as before, restricting to M. The existence of the total function $F_s^{i,M}$ follows as before by admissibility. Admissibility also gives a canonical Σ_1 definition φ_s such that

$$y = F_s^M(\vec{x}) \leftrightarrow M \models \varphi_s[y,\vec{x}].$$

(Thus F_s^M is uniformly Σ_1 regardless of M.) If M, M' are admissibles of the same type and $M \subseteq M'$ (i.e. M is structurally included in M'), then $F_s^M = F_s^{M'} \restriction M$. Thus we can let $F^{A_1,\ldots,A_n}s$ be the union of all F_s^M such that $M = \langle |M|,\in,A_1 \cap |M|,\ldots,A_n \cap |M|\rangle$ is admissible. φ_s then defines $F_s^{\vec{A}}$ over $\langle V,\vec{A}\rangle$. (Here, Karp refined the construction so as to show that

$F_s^{\vec{A}} \upharpoonright M = F_s^M$ whenever $M = \langle |M|, \in, A_1 \wedge |M|, \ldots, A_n \cap |M| \rangle$ is transitive and closed under function primitive recursive in $A_1, \ldots, A_n$. It can also be shown that $M = \langle |M|, \in, a_1, \ldots, a_n \rangle$ is closed under functions primitive recursive in $a_1, \ldots, a_n$ iff $|M|$ is primitive recursive closed and M is amenable, (i.e. $x \cap A_i \in M$ for all $x \in M$, $v = 1, \ldots, n$).

A full account of these results can be found in [PR] or [AS].

We can now state the uniformity involved in Lemma 2.2.19: Let $A_i \subset V$ be primitive recursive in $B_1, \ldots, B_n$ with primitive recursive def s_i in $b_1, \ldots, b_m (i = 1, \ldots, m)$. Let f be primitive recursive in $A_1, \ldots, A_n$ with primitive recursive definition s in $\dot{a}_1, \ldots, \dot{a}_n$. Then f is primitive recursive in $B_1, \ldots, B_n$ by a primitive recursive definition s' in $\dot{b}_1, \ldots, \dot{b}_m$. s' is *uniform* in the sense that it depends only on $s_1, \ldots, s_n$ and s, not on $B_1, \ldots, B_m$. In fact, the induction on the schemata in s implicitly describes an algorithm for a function

$$s_1, \ldots, s_m, s \mapsto s'$$

with the following property: Let $B_1, \ldots, B_m$ be any classes. Let s_i define g_i from $\vec{B}(i = 1, \ldots, n)$. Set: $A_i = \{x | g_i(x) \neq 0\}$ in $i = 1, \ldots, n$. Let f be the function defined by s from $\vec{A}$. Then s' defines f from $\vec{B}$.

Note $\langle H_\omega, \in \rangle$ is an admissible structure; hence $F_s \upharpoonright H_\omega = f_s^{H_\omega}$. This shows that the constant function ω is not primitive recursive, since $\omega \notin H_\omega$. It can be shown that $f : \omega \to \omega$ is primitive recursive in the sense of ordinary recursion theory iff

$$f^*(x) = \begin{cases} f(x) \text{ if } x \in \omega \\ 0 \text{ if not} \end{cases}$$

is primitive recursive over H_ω. Conversely, there is a primitive recursive map $\sigma : H_\omega \leftrightarrow \omega$ such that $f : H_\omega \to H_\omega$ is primitive recursive over H_ω iff $\sigma f \sigma^{-1}$ is primitive recursive in sense of ordinary recursion theory.

1.3 Ill founded ZF^- models

We now prove a lemma about arbitrary (possibly ill founded) models of ZF^- (where the language of ZF^- may contain predicates other than $\in$). Let $\mathbb{A} = \langle a, \underset{\mathbb{A}}{\in}, B_1, \ldots, B_n \rangle$ be such a model. For $X \subset A$ we of course write $\mathbb{A} | X = \langle X, \underset{\mathbb{A}}{\in} \cap X^2, \ldots \rangle$. By the *well founded core* of $\mathbb{A}$ we mean the set of all $v \in \mathbb{A}$ such that $\underset{\mathbb{A}}{\in} \cap C(x)^2$ is well founded, where $C(x)$ is the closure of $\{x\}$ under $\in_{\mathbb{A}}$. Let wfc($\mathbb{A}$) be the restriction $\mathbb{A}|C$ of $\mathbb{A}$ to its well founded core C. Then wfc($\mathbb{A}$) is a well founded structure satisfying the axiom of extensionality, and is, therefore, isomorphic to a transitive structure. Hence $\mathbb{A}$ is isomorphic to a structure $\mathbb{A}'$ such that $\text{wfc}(\mathbb{A}')$ is transitive (i.e. $\text{wfc}(\mathbb{A}') = \langle \mathbb{A}', \in, m\rangle$ where $\mathbb{A}'$ is transitive). We call such $\mathbb{A}'$ *grounded*, defining:

Definition 1.3.1. $\mathbb{A} = \langle A, \in^{\mathbb{A}}, \ldots\rangle$ is *grounded* iff $\text{wfc}(\mathbb{A})$ is transitive.

(**Note** Elsewhere we have called these models "solid" instead of "grounded". We avoid that usage here, however, since *solidity* — in quite another sense — is an important concept in inner model theory.)

By the argument just given, every consistent set of sentences in ZF^- has a grounded model. Clearly

(1) $\omega \subset \text{wfc}(\mathbb{A})$ if $\mathbb{A}$ is grounded.

For any ZF^- model $\mathbb{A}$ we have:

(2) If $x \in \mathbb{A}$ and $\{z | z \in^{\mathbb{A}} x\} \subset \text{wfc}(\mathbb{A})$, then $x \in \text{wfc}(\mathbb{A})$.

Proof: $C(x) = \{x\} \cup \bigcup\{C(z) | z \in^{\mathbb{A}} x\}$. QED

By Σ_0–absoluteness we have:

(3) Let $\mathbb{A}$ be grounded. Let φ be Σ_0 and let $x_1, \ldots, x_n \in \text{wfc}(\mathbb{A})$. Then

$$\text{wfc}(\mathbb{A}) \models \varphi[\vec{x}] \leftrightarrow \mathbb{A} \models \varphi[\vec{x}].$$

By $\in$–induction on $x \in \text{wfc}(\mathbb{A})$ it follows that the rank function is absolute:

(4) $\text{rn}(x) = \text{rn}^{\mathbb{A}}(x)$ for $x \in \text{wfc}(\mathbb{A})$ if $\mathbb{A}$ is grounded.

The converse also holds:

(5) Let $\text{rn}^{\mathbb{A}}(x) \in \text{wfc}(\mathbb{A})$. Then $x \in \text{wfc}(\mathbb{A})$.

Proof: Let $r = \text{rn}^{\mathbb{A}}(x)$. Then r is an ordinal by (3). Assume that r is the least counterexample. Then $\text{rn}^{\mathbb{A}}(z) < r$ for $z \in^{\mathbb{A}} x$. Hence $\{z | z \in^{\mathbb{A}} x\} \subset \text{wfc}(\mathbb{A})$ and $x \in \text{wfc}(\mathbb{A})$ by (2).

Contradiction! QED

We now prove:

Lemma 1.3.1. *Let $\mathbb{A}$ be grounded. Then* wfc($\mathbb{A}$) *is admissible.*

Proof: Axiom (1) and axiom (2) (Σ_0–subsets) follow trivially from (3). We verify the axiom of Σ_0 collection. Let $R(x, y)$ e $\underline{\Sigma}_0(\text{wfc}(\mathbb{A}))$. Let $u \in \text{wfc}(\mathbb{A})$ such that $\bigwedge x \in u \bigvee y R(x, y)$. It suffices to show:

Claim: $\bigvee v \bigwedge x \in u \bigvee y \in v R(x, y)$.

Let R' be $\underline{\Sigma}_0(\mathbb{A})$ by the same definition in the same parameters as R. Then $R = R' \cap \text{wfc}(\mathbb{A})^2$ by (3). If $\mathbb{A} = \text{wfc}(\mathbb{A})$, there is nothing to prove, so suppose not. Then there is $r \in \text{On}^{\mathbb{A}}$ such that $r \notin \text{wfc}(\mathbb{A})$. Hence

$$\mathbb{A} \models rn(y) < r \text{ for all } y \in \text{wfc}(\mathbb{A})$$

by (4). Hence there is an $r \in \text{On}^{\mathbb{A}}$ such that

(6) $\bigwedge x \in u \bigvee y(R'(x, y) \wedge \mathbb{A} \models rn(y) < r)$

Since $\mathbb{A}$ models ZF^-, there must be a least such r. But then:

(7) $r \in \text{wfc}(\mathbb{A})$.

Since by (2) there would otherwise be an r' such that $\mathbb{A} \models r' < r$ and $r' \notin \text{wfc}(\mathbb{A})$. Hence (6) holds for r', contradicting the minimality of r.

QED (7)

But there is w such that

(8) $\bigwedge x \in u \bigvee y \in w(R'(x, y) \wedge rn(y) < r)$.

Let $\mathbb{A} \models v = \{y \in w | \wedge rn(y) < r\}$. Then $rn^{\mathbb{A}}(v) \leq r$. Hence $rn^{\mathbb{A}}(v) \in \text{wfc}(\mathbb{A})$ and $v \in \text{wfc}(\mathbb{A})$ by (5). But:

$$\bigwedge x \in u \bigvee y \in vRxy.$$

QED (Lemma 1.3.1)

As immediate corollaries we have:

Corollary 1.3.2. *Let $\delta = \text{On} \cap \text{wfc}(\mathbb{A})$. Then $L_\delta(u)$ is admissible whenever $u \in \text{wfc}(\mathbb{A})$.*

Corollary 1.3.3. $L_\delta^A = \langle L_\delta[A], A \cap L_\delta[A] \rangle$ *is admissible whenever* $A \in \underline{\Sigma}_\omega(\mathbb{A})$ *(since $\langle \mathbb{A}, A \rangle$ is a ZF^- model.*

Note It is clear from the proof of lemma 1.3.1 that we can replace ZF^- by KP (Kripke–Platek set theory). In this form lemma 1.3.1 is known as *Ville's Lemma*.

1.4 Barwise Theory

Jon Barwise worked out the syntax and model theory of certain infinitary (but M–finite) languages in countable admissible structures M. In so doing, he created a powerful and flexible tool for set theory, which we shall utilize later in this book. In this chapter we give an introduction to Barwise's work.

1.4.1 Syntax

Let M be admissible. Barwise developed a first order theory in which arbitrary M–finite conjunction and disjunction are allowed. The predicates, however, have only a (genuinely) finite number of argument places and there are no infinite strings of quantifiers. In order that the notion "M–finite" have a meaning for the symbols in our language, we must "arithmetize" the language — i.e. identify its symbols with objects in M. There are many ways of doing this. For the sake of definitness we adopt a specific arithmetization of M–finitary first order logic:

Predicates: For each $x \in M$ and each n such that $1 \leq n < \omega$ we appoint an n–ary predicate $P_x^n =: \langle 0, \langle n, x \rangle \rangle$.

Constants: For each $x \in M$ we appoint a constant $c_x =: \langle 1, x \rangle$.

Variables: For each $x \in M$ we appoint a variable $v_x =: \langle 2, x \rangle$.

Note The set of variables must be M–infinite, since otherwise a single formula might exhaust all the variables.

We let P_0^2 be the identity predicate $\dot{=}$ and also reserve P_1^2 as the $\in$–predicate ($\dot{\in}$).

By a *primitive formula* we mean $Pt_1 \dots t_n =: \langle 3, \langle P, t_1, \dots, t_n \rangle \rangle$ where P is an n–ary predicate and $t_1, \dots, t_n$ are variables or constants.

We then define:

$$\neg\varphi =: \langle 4, \varphi\rangle, (\varphi \vee \psi) =: \langle 5, \langle\varphi, \psi\rangle\rangle,$$
$$(\varphi \wedge \psi) =: \langle 6, \langle\varphi, \psi\rangle\rangle, (\varphi \rightarrow \psi) =: \langle 7, \langle\varphi, \psi\rangle\rangle,$$
$$(\varphi \leftrightarrow \psi) =: \langle 8, \langle\varphi, \psi\rangle\rangle, \bigwedge v\varphi = \langle 9, \langle v, \varphi\rangle\rangle,$$
$$\bigvee v\varphi = \langle 10, \langle v, \varphi\rangle\rangle.$$

The infinitary conjunctions and disjunctions are

$$\bigwedge\!\!\!\bigwedge f =: \langle 11, f\rangle, \bigvee\!\!\!\bigvee f =: \langle 12, f\rangle.$$

The set Fml of *first order M–formulae* is then the smallest set X which contains all primitive formulae, is closed under $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and such that

- If v is a variable and $\varphi \in X$, then $\bigwedge v\varphi \in X$ and $\bigvee v\varphi \in X$.
- If $f = \langle\varphi_i | i \in I\rangle \in M$ and $\varphi_i \in X$ for $i \in I$, then $\bigwedge\!\!\!\bigwedge f \in X$ and $\bigvee\!\!\!\bigvee f \in X$.

(In this case we also write:

$$\bigwedge\!\!\!\bigwedge_{i\in I} \varphi_i =: \bigwedge\!\!\!\bigwedge f, \bigvee\!\!\!\bigvee_{i\in I} \varphi_i =: \bigwedge\!\!\!\bigwedge f.$$

If B is a set of formulae we may also write: $\bigwedge\!\!\!\bigwedge B$ for $\bigwedge\!\!\!\bigwedge_{\varphi\in B} \varphi$.)

Proof: It turns out that the usual syntactical notions are $\Delta_1(M)$, including: Fml, $Const$ (set of constants), Vbl (set of variables), $Sent$ (set of all sentences), as are the functions:

> $Fr(\varphi) =$ The set of free variables in φ
> $\varphi(^v/t) \simeq$ the result of replacing occurences of the variable v by t (where $t \in Vbl \cup Const$), as long as this can be done without a new occurence of t being bound by a quantifier in φ (if t is a variable).

That Vbl, $Const$ are Δ_1 (in fact Σ_0) is immediate. The characteristic function X of Fml is definable by a recursion of the form:

$$X(x) = G(x, \langle X(z) | z \in TC(x))$$

where $G : M^2 \rightarrow M$ is Δ_1. (This is an instance of the recursion schema in §1 Lemma 1.1.16. We are of course using the fact that any proper subformula of φ lies in $TC(\varphi)$.)

Now let $h(\varphi)$ be the set of immediate subformulae of φ (e.g. $h(\neg\varphi) = \{\varphi\}$, $h(\bigwedge\!\!\!\bigwedge_{i\in I}\varphi_i) = \{\varphi_i | i \in I\}$, $h(\bigwedge v\varphi) = \{\varphi\}$ etc.) Then h satisfies the condition in §1 Lemma 1.1.16. It is fairly easy to see that

$$Fr(\varphi) = G(\varphi, \langle F(x) | x \in h(\varphi)\rangle)$$

where G is a Σ_1 function defined on Fml. Then $Sent = \{\varphi | Fr(\varphi) = \emptyset\}$.

To define $\varphi({}^v/t)$ we first define it on primitive formulae, which is straightforward. We then set:

$$(\varphi \wedge \psi)({}^v/t) \simeq (\varphi({}^v/t) \wedge \psi({}^v/t)) \text{ (similarly for } \wedge, \rightarrow, \leftrightarrow)$$

$$\neg\varphi({}^v/t) \simeq \neg(\varphi({}^v/t))$$

$$(\bigwedge\!\!\!\bigwedge_{i\in I}\varphi_i)({}^v/t) \simeq \bigwedge\!\!\!\bigwedge_{i\in I}(\varphi_i({}^v/t)) \text{ similarly for } \bigvee\!\!\!\bigvee.$$

$$(\bigwedge u\varphi)({}^v/t) \simeq \begin{cases} \bigwedge u\varphi \text{ if } u = v \\ \bigwedge u(\varphi({}^v/t)) \text{ if } u \neq v, t \quad \text{(similarly for } \bigvee) \\ \text{otherwise undefined} \end{cases}$$

This has the form:

$$\varphi({}^v/t) \simeq G(\varphi, v, t\langle X({}^v/t) | X \in h(\varphi)\rangle),$$

where G is $\Sigma_1(M)$. The domain of the function $f(\varphi, v, t) = \varphi({}^v/t)$ is $\Delta_1(M)$, however, so f is M–recursive.

(We can then define:

$$\varphi({}^{v_1,\ldots,v_n}/t_1, \ldots, t_n) = \varphi({}^{v_1}/w_1)\ldots({}^{v_n}/w_n)({}^{w_1}/t_1)\ldots({}^{w_n}/t_n)$$

where $v_1, \ldots, v_n$ is a sequence of distinct variables and $w_1, \ldots, w_n$ is any sequence of distinct variables which are different from $v_1, \ldots, v_n$, $t_1, \ldots, t_n$ and do not occur bound or free in φ. We of cours follow the usual conventions, writing $\varphi(t_1, \ldots, t_n)$ for $vp({}^{v_1,\ldots,v_n}/t_1, \ldots, t_n)$, taking $v_1, \ldots, v_n$ as known.)

M–finite predicate logic has the axioms:

- all instances of the usual propositional logic axiom schemata (enough to derive all tautologies with the help of modus ponens).
- $\bigwedge\!\!\!\bigwedge_{i\in U}\varphi_i \rightarrow \varphi_j,\ \varphi_j \rightarrow \bigvee\!\!\!\bigvee_{i\in U}\varphi_i\ (j \in U \in M)$
- $\bigwedge x\varphi \rightarrow \varphi({}^x/t),\ \varphi({}^x/t) \rightarrow \bigvee x\varphi$

- $x \dot{=} y \to (\varphi(x) \leftrightarrow \varphi(y))$

The *rules of inference* are:

- $\frac{\varphi, \varphi \to \psi}{\psi}$ (modus ponens)
- $\frac{\varphi \to \psi}{\varphi \to \bigwedge x \psi}$ if $x \notin Fr(\varphi)$
- $\frac{\psi \to \varphi}{\bigvee x \psi \to \varphi}$ if $x \notin Fr(\varphi)$
- $\frac{\varphi \to \psi_i (i \in u)}{\varphi \to \bigwedge\!\!\!\bigwedge \psi_i}$ $(u \in M)$
- $\frac{\psi_i \to \varphi (i \in u)}{\bigvee\!\!\!\bigvee \psi_i \to \varphi}$ $(u \in M)$

We say that φ is *provable* from a set of sentences A iff φ is in the smallest set which contains A and the axioms and is closed under the rules of inference. We write $A \vdash \varphi$ to mean that φ is provable from A. $\vdash \varphi$ means the same as $\emptyset \vdash \varphi$.

However, this definition of provability cannot be stated in the 1st order language of M and rather misses the point which is that a provable formula should have an M–finite proof. This, as it turns out, will be the case whenever A is $\underline{\Sigma}_1(M)$. In order to state and prove this, we must first formalize the notion of proof. Because we have not assumed the axiom of choice to hold in our admissible structure M, we adopt a somewhat unorthodox concept of proof:

Definition 1.4.1. By a *proof from A* we mean a sequence $\langle p_i | i < \alpha \rangle$ such that $\alpha \in \mathrm{On}$ and for each $i < \alpha$, $p_i \subset Fml$ and whenever $\psi \in p_i$, then either $\psi \in A$ or ψ is an axiom or ψ follows from $\bigcup_{h<i} p_n$ by a single application of one of the rules.

Definition 1.4.2. $p = \langle p_i | i < \alpha$ is a *proof of φ from A* iff p is a proof from A and $\varphi \in \bigcup_{i<\alpha} p_i$.

(**Note** that this definition does *not* require a proof to be M–finite.)

It is straightforward to show that φ is provable iff it has a proof. However, we are more interested in M–finite proofs. If A is $\Sigma_1(M)$ in a parameter q, it follows easily that $\{p \in M | p \text{ is a proof from } A\}$ is $\Sigma_1(M)$ in the same parameter. A more interesting conclusion is:

Lemma 1.4.1. *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is an M–finite proof of φ from A.*

Proof: ($\leftarrow$) trivial. We prove ($\rightarrow$)

Let $X =$ the set of φ such that there is $p \in M$ which proves φ from A.

Claim: $\{\varphi | A \vdash \varphi\} \subset X$.

Proof: We know that $A \subset X$ and all axioms lie in X. Hence it suffices to show that X is closed under the rules of proof. This must be demonstrated rule by rule. As an example we show:

Claim: Let $\varphi \rightarrow \psi_i$ be in X for $i \in u$. Then $\varphi \rightarrow \bigwedge\!\!\!\bigwedge_{i \in U} \varphi \in X$.

Proof: Let $P(p, \varphi)$ mean: p is a proof of φ from A. Then P is $\underline{\Sigma}_1(M)$. We have assumed:

(1) $\wedge i \in u \bigvee_p \; P(p, \varphi \rightarrow \psi_i)$.
Now let $P(p_i, x) \leftrightarrow \bigvee z P'(z, p_i, x)$ where P' is Σ_0. We than have:

(2) $\bigwedge i \in u \bigvee p \bigvee z P'(z, p, \varphi \rightarrow \psi_i)$.
Hence there is $v \in M$ with:

(3) $\bigwedge i \in u \bigvee p, z \in v P'(z, p, \varphi \rightarrow \psi_i)$.
Set: $w = \{p \in v | \bigvee i \in u \bigvee z \in v P'(z, p, \varphi \rightarrow \psi_i)\}$
Set: $\alpha = \bigcup_{p \in w} \text{dom}(p)$. For $i < \alpha$ set:

$$q_i = \bigcup \{p_i | p \in w \wedge i \in \text{dom}(p)\}$$

Then $q = \langle q_i | i < \alpha \rangle \in M$ is a proof.

But then $q^\frown \{\bigwedge\!\!\!\bigwedge_{i \in U} \psi_i\}$ is a proof of $\bigwedge\!\!\!\bigwedge_{i \in U} \psi_i$. Hence $\bigwedge\!\!\!\bigwedge_{i \in U} \psi_i \in X$.

QED (Lemma 1.4.1)

From this we get the *M–finiteness lemma*:

Lemma 1.4.2. *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is $a \subset A$ such that $a \in M$ and $a \vdash \varphi$.*

Proof: ($\leftarrow$) is trivial. We prove ($\rightarrow$). Let $p \in M$ be a proof of φ from A. Set:

$a =$ the set of ψ such that for some $i \in \text{dom}(p)$, $\psi \in p_i$ and ψ is neither an axiom nor follows from $\bigcup_{I < i} p_i$ by an application of a single rule.

Then $a \subset A$, $a \in M$, and p is a proof of φ from a. QED (Lemma 1.4.2)

Another consequence of Lemma 1.4.1 is:

Lemma 1.4.3. *Let A be $\Sigma_1(M)$ in q. Then $\{\varphi | A \vdash \varphi\}$ is $\Sigma_1(M)$ in the same parameter (uniformly in the Σ_1 definition of A).*

Proof: $\{\varphi | A \vdash \varphi\} = \{\varphi | \bigvee p \in m \ p$ proves φ from $A\}$.

Corollary 1.4.4. *Let A be $\Sigma_1(M)$ in q. Then "A is consistent" is $\Pi_1(M)$ in the same parameter (uniformly in the Σ_1 definition of A).*

"p proves φ from u" is uniformly $\Sigma_i(M)$. Hence:

Lemma 1.4.5. *$\{\langle u, \varphi\rangle | u \in m \wedge u \vdash \varphi\}$ is uniformly $\Sigma_1(m)$.*

Corollary 1.4.6. *$\{\langle u \in M | u$ is consitent$\}$ is uniformly $\Pi_1(m)$.*

Note. Call a proof p *strict* iff $\overline{\overline{P}}_i = 1$ for $i \in \text{dom}(p)$. This corresponds to the more usual notion of proof. If M satisfies the axiom of choice in the form: Every set is enumerable by an ordinal, then Lemma 1.4.1 holds with "strict proof" in place of "proof". We leave this to the reader.

1.4.2 Models

We will not normally employ all of the predicates and constants in our M–finitary first order logic, but cut down to a smaller set of symbols which we intend to interpret in a model. Thus we define a *language* to be a set $\mathbb{L}$ of predicates and constants. By a *model* of $\mathbb{L}$ we mean a structure:

$$\mathbb{A} = \langle |\mathbb{A}|, \langle t^{\mathbb{A}} | t \in \mathbb{L}\rangle\rangle$$

such that $|\mathbb{A}| \neq \emptyset$, $P^{\mathbb{A}} \subset |\mathbb{A}|^n$ whenever P is an n–ary predicate, and $c^{\mathbb{A}} \in |\mathbb{A}|$ whenever c is a constant. By a *variable assignment* we mean a map of f of the variables into $\mathbb{A}$. The *satisfaction relation* $\mathbb{A} \models \varphi[f]$ is defined in the usual way, where $\mathbb{A} \models [f]$ means that the formula φ becomes true in $\mathbb{A}$ if the free variables of φ are interpreted by the assignment f. We leave the definition to the reader, remarking only that:

$$\mathbb{A} \models \bigwedge\!\!\!\bigwedge_{i \in u} \varphi_i[f] \leftrightarrow \bigwedge i \in u \ \mathbb{A} \models \varphi_i[f]$$
$$\mathbb{A} \models \bigvee\!\!\!\bigvee_{i \in u} \varphi_i[f] \leftrightarrow \bigvee i \in u \ \mathbb{A} \models \varphi_i[f]$$

We adopt the usual conventions of model theory, writing $\mathbb{A} = \langle |\mathbb{A}|, t_1^{\mathbb{A}}, \ldots\rangle$ if we think of the predicates and constants of $\mathbb{L}$ as being arranged in a fixed sequence $t_1, t_2, \ldots$. Similarly, if $\varphi = \varphi(v_1, \ldots, v_n)$ is a formula in which at most the variables $v_1, \ldots, v_n$ occur free, we write $\mathbb{A} \models \varphi[a_1, \ldots, a_n]$ for:

$$\mathbb{A} \models \varphi[f] \text{ where } f(v_i) = a_i \text{ for } i = 1, \ldots, n.$$

If φ is a sentence we write: $\mathbb{A} \models \varphi$. If A is a set of sentences, we write $\mathbb{A} \vdash A$ to mean: $\mathbb{A} \models \varphi$ for all $\varphi \in A$.

Proof: The *correctness theorem* says that if A is a set of $\mathbb{L}$ sentences and $\mathbb{A} \models A$, then A is consistent. (We leave this to the reader.)
Barwise's Completeness Theorem says that the converse holds whenever our admissible structure is countable:

Theorem 1.4.7. *Let M be a countable admissible structure. Let $\mathbb{L}$ be an M–language and let A be a set of statements in $\mathbb{L}$. If A is consistent in M–finite predicate logic, then $\mathbb{L}$ has a model $\mathbb{A}$ such that $\mathbb{A} \models A$.*

Proof: (Sketch)
We make use of the following theorem of Rasiowa and Sikorski: Let $\mathbb{B}$ be a Boolean algebra. Let $X_i \subset \mathbb{B}(i < \omega)$ such that the Boolean union $\bigcup X_i = b_i$ exists in the sense of $\mathbb{B}$. Then $\mathbb{B}$ has an ultrafilter U such that

$$b_i \in U \leftrightarrow X_i \cap U \neq \emptyset \text{ for } i < \omega.$$

(Proof. Successively choose $c_i(i < \omega)$ by: $c_0 = 1$, $c_{i+1} = c_i \cap b \neq 0$, where $b \in X_i \cup \{\neg b_i\}$. Let $\overline{U} = \{a \in \mathbb{B} | \bigvee i c_i \subset a\}$. Then $\overline{U}$ is a filter and extends to an ultrafilter on $\mathbb{B}$.)

Extend the language $\mathbb{L}$ by adding an M–infinite set C of new constants. Call the extended language $\mathbb{L}^*$. Set:

$$[\varphi] =: \{\psi | A \vdash (\psi \leftrightarrow \varphi)\}$$

for $\mathbb{L}^*$–sentences φ. Then

$$\mathbb{B} =: \{[\varphi] | \varphi \in Sent_{\mathbb{L}^*}\}$$

in the Lindenbaum algebra of $\mathbb{L}^*$ with the defining equations:

$$[\varphi] \cup [\psi] = [\varphi \vee \psi], [\varphi] \cap [\psi] = [\varphi \wedge \psi], \neg[\varphi] = [\neg\varphi]$$
$$\bigcup_{i \in U} [\varphi_i] = [\bigwedge\!\!\!\bigwedge_{i \in U} \varphi_i](i \in u), \bigcap_{i \in U} [\varphi_i] = [\bigwedge\!\!\!\bigwedge_{i \in U} \varphi_i](i \in u)$$
$$\bigcup_{c \in C} [\varphi(c)] = [\bigvee v \varphi(v)], \bigcap_{c \in C} [\varphi(c)] = [\bigwedge v \varphi(v)].$$

The last two equations hold because the constants in C, which do not occur in the axiom A, behave like free variables. By Rasiowa dn Sikorski there is then an ultrafilter U on $\mathbb{B}$ which respects the above operations. We define a model $\mathbb{A} = \langle |\mathbb{A}|, \langle t^{\mathbb{A}} | t \in \mathbb{L} \rangle \rangle$ as follows: For $c \in C$ set $[c] =: \{c' \in C | [c = c'] \in U\}$. If $p \in \mathbb{L}$ is an n–place predicate, set:

$$P^{\mathbb{A}}([c_1], \ldots, [c_n]) \leftrightarrow: [Pc_1, \ldots, c_n] \in U.$$

If $t \in \mathbb{L}$ is a constant, set:

$$t^{\mathbb{A}} = [c] \text{ where } c \in C, [t = c] \in U.$$

A straightforward induction then shows:

$$\mathbb{A} \models \varphi[[c_1], \ldots, [c_n] \leftrightarrow [\varphi(c_1, \ldots, c_n)] \in U$$

for formulae $\varphi = \varphi(v_1, \ldots, v_n)$ with at most the free variables $v_1, \ldots, v_n$. In particular, $\mathbb{A} \models \varphi \leftrightarrow [\varphi] \in U$ for $\mathbb{L}^*$–statements φ. Hence $\mathbb{A} \models A$.

QED (Theorem 1.4.7)

Combining the completeness theorem with the M–finiteness lemma, we get the well known *Barwise compactness theorem*:

Corollary 1.4.8. *Let M be countable. Let $\mathbb{L}$ be a language. Let A be a $\underline{\Sigma}_1(M)$ set of sentences in $\mathbb{L}$. If every M–finite subset of $\mathbb{A}$ has a model, then so does A.*

1.4.3 Applications

Definition 1.4.3. By a *theory* or *axiomatized language* we mean a pair $\mathbb{L} = \langle \mathbb{L}_0, A \rangle$ such that $\mathbb{L}_0$ is a language and A is a set of $\mathbb{L}_0$–sentences. We say that $\mathbb{A}$ *models* $\mathbb{L}$ iff $\mathbb{A}$ is a model of $\mathbb{L}_0$ and $\mathbb{A} \models A$. We also write $\mathbb{L} \vdash \varphi$ for: ($\varphi \in Fml_{\mathbb{L}_0}$ and $A \vdash \varphi$). We say that $\mathbb{L} = \langle \mathbb{L}_0, A \rangle$ is $\Sigma_1(M)$ (in p) iff $\mathbb{L}_0$ is $\Delta_1(M)$ (in p) and A is $\Sigma_1(M)$ (in p). Similarly for: $\mathbb{L}$ is $\Delta(M)$ (in p).

We now consider the class of axiomazized languages containing a fixed predicate $\dot{\in}$, the special constants $\underline{x} (x \in M)$ (we can set e.g. $\underline{x} = \langle 1, \langle 0, x \rangle \rangle$), and the *basic axioms*:

- Extensionality
- $\bigwedge v (v \dot{\in} \underline{x} \leftrightarrow \bigvee\!\!\bigvee_{z \in x} v \dot{=} \underline{z})$ for $x \in M$.

(Further predicates, constants, and axioms are allowed of course.) We call any such theory an "$\in$–theory". Then:

Lemma 1.4.9. *Let $\mathbb{A}$ be a grounded model of an $\in$–theory $\mathbb{L}$. Then $\underline{x}^{\mathbb{A}} = x \in \mathrm{wfc}(\mathbb{A})$ for $x \in M$.*

In an $\in$–theory $\mathbb{L}$ we often adopt the set of axioms ZFC$^-$ (or more precisely ZFC$^-_{\mathbb{L}}$). This is the collection of all $\mathbb{L}$–sentences φ such that φ is the universal quantifier closure of an instance of the ZFC$^-$ axiom schemata — but does *not* contain infinite conjunctions or disjunctions. (Hence the collection of all subformulae is finite.) (Similarly for ZF^-, ZFC, ZF.)

(**Note** If we omit the sentences containing constants, we get a subset $B \subset$ ZFC$^-$ which is equivalent to ZFC$^-$ in $\mathbb{L}$. Since each element of B contain at most finitely many variables, we can restrict further to the subset B' of sentences containing only the variables $v_i (i < \omega)$. If $\omega \in M$ and the set of predicates in $\mathbb{L}$ is M–finite, then B' will be M–finite. Hence ZFC$^-$ is equivalent in $\mathbb{L}$ to the statement $\bigwedge\!\!\!\bigwedge B'$.)

We now bring some typical applications of $\in$–theories. We say that an ordinal α is *admissible in* $a \subset \alpha$ iff $\langle L_\alpha[a], \in, a\rangle$ is admissible.

Lemma 1.4.10. *Let $\alpha > \omega$ be a countable admissible ordinal. Then there is $a \subset \omega$ such that α is the least ordinal admissible in a.*

This follows straightforwardly from:

Lemma 1.4.11. *Let M be a countable admissible structure. Let $\mathbb{L}$ be a consistent $\underline{\Sigma}_1(M)$ $\in$–theory such that $\mathbb{L} \vdash ZF^-$. Then $\mathbb{L}$ has a grounded model $\mathbb{A}$ such that $\mathbb{A} \neq \mathrm{wfc}(\mathbb{A})$ and $\mathrm{On} \cap \mathrm{wfc}(\mathbb{A}) = \mathrm{On} \cap M$.*

We first show that lemma 1.4.11 implies lemma 1.4.10. Take $M = L_\alpha$. Let $\mathbb{L}$ be the M–theory with:

Predicate: $\dot{\in}$

Constants: $\underline{x}(x \in M), \dot{a}$

Axioms: Basic axioms $+$ZFC$^-$ $+\underline{\beta}$ is not admissible in $\dot{a}$

Then $\mathbb{L}$ is consistent, since $\langle H_{\omega_1}, \in, a\rangle$ is a model, where a is any $a \subset \omega$ codes a well ordering of type $\geq \alpha$. Let $\mathbb{L}$ be a grounded model of $\mathbb{L}$ such that $\mathrm{wfc}(\mathbb{A}) \neq \mathbb{A}$ and $\mathrm{On} \wedge \mathrm{wfc}(\mathbb{A}) = \alpha$. Then $\mathrm{wfc}(\mathbb{A})$ is admissible by §3. Hence so is $L_\alpha[a]$ where $a = \dot{a}^{\mathbb{A}}$. QED

Note This is a very typical application in that Barwise theory hands us an ill founded model, but our interest is entirely concentrated on its well founded part.

Note Persuing this method a bit further we can use lemma 1.4.11 to prove: Let $\omega < \alpha_0 < \ldots < \alpha_{n-1}$ be a sequence of countable admissible ordinals. There is $a \subset \omega$ such that $\alpha_i =$ the i–theory $\alpha < \omega$ which is admissible in $a(1 = 0, \ldots, n-1)$.

We now prove lemma 1.4.11 by modifying the proof of the completeness theorem. Let $\Gamma(v)$ be the set of formulae: $v \in \text{On}$, $v > \underline{\beta}(\beta \in \text{On} \wedge M)$. Add an M–infinite (but $\underline{\Delta}_1(M)$) set E of new constants to $\mathbb{L}$. Let $\mathbb{L}'$ be $\mathbb{L}$ with the new constants and new axioms: $\Gamma(e)$ $(e \in E)$. Then $\mathbb{L}'$ is consistent, since any M–finite subset of the axioms can be modeled in an arbitrary grounded model $\mathbb{A}$ of $\mathbb{L}$ by interpreting the new constants as sufficiently large elements of α. As in the proof of completeness we then add a new class C of constants which is not M–finite. We assume, however, that C is $\Delta_1(M)$. We add no further axioms, so the elements of C behave like free variables. The *iv* extended language $\mathbb{L}''$ is clearly $\underline{\Sigma}_1(M)$.

Now set:

$$\Delta(v) =: \{v \notin \text{On}\} \cup \bigcup_{\beta \in M} \{v \leq \underline{\beta}\} \cup \bigcup_{e \in E} \{e < v\}.$$

Claim Let $c \in C$. Then $\bigcup\{[\varphi] | \varphi \in \Delta(c)\} = 1$ in the Lindenbaum algebra of $\mathbb{L}''$.

Proof: Suppose not. Then there is ψ such that $A \vdash \varphi \to \psi$ for all $\varphi \in \Delta(c)$ and $A \cup \{\neg\psi\}$ is consistent, where $\mathbb{L}'' = \langle \mathbb{L}''_0, A \rangle$. Pick an $e \in E$ which does not occur in ψ. Let A^* be the result of omitting the axioms $\Gamma(e)$ from A. Then $A^* \cup \{\neg\psi\} \cup \Gamma(e) \vdash c \leq e$. By the finiteness lemma there is $\beta \in M$ such that $A^* \cup \{\neg\psi\} \cup \{\underline{\beta} \leq e\} \vdash c \leq e$. But e behaves here like a free variable, so $A^* \cup \{\neg\psi\} \vdash c \leq \underline{\beta}$. But $A \supset A^*$ and $A \cup \{\neg\psi\} \vdash \underline{\beta} < c$. Hence $A \cup \{\neg\psi\} \vdash \underline{\beta} < \underline{\beta}$ and $A \cup \{\neg\psi\}$ is inconsistent.
Contradiction! QED (Claim)

Now let U be an ultrafilter on the Lindenbaum algebra of $\mathbb{L}''$ what respects both two operations listed in the proof of the completeness theorem and the unions $\bigcup\{[\varphi] | \varphi \in \Delta(c)\}$ for $c \in C$. Let $X = \{\varphi | [\varphi] \in U\}$. Then as before, $\mathbb{L}''$ has a grounded model $\mathbb{A}$, all of whose elementes have the form $c^{\mathbb{A}}$ for $a \subset\in C$ and such that:

$$\mathbb{A} \models \varphi \text{ iff } \varphi \in X$$

for $\mathbb{L}''$–statements φ. But then for each $x \in A$ we have either $x \notin \text{On}_{\mathbb{A}}$ or $x < \beta$ for a $\beta \in \text{On} \cap M$ or $e^{\mathbb{A}} < v$ for all $e \in E$. In particular, if $x \in \text{On}_{\mathbb{A}}$ and $x > \beta$ for all $\beta \in \text{On} \cap M$, then there is $e^{\mathbb{A}} < x$ in $\mathbb{A}$. But $\beta < e^{\mathbb{A}}$ for all $\beta \in \text{On} \cap M$. Hence $\text{On}_{\mathbb{A}} \setminus \text{On}_M$ has no minimal element in $\mathbb{A}$.

QED (Lemma 1.4.11)

Another typical application is:

Lemma 1.4.12. *Let W be an inner model of* ZFC. *Suppose that, in W, U is a normal measure on κ. Let $\tau > u$ be regular in W. Set: $M = \langle H_\tau^W, U \rangle$. Assume that M is countable in V. Then for any $\alpha \subseteq u$ there is $\overline{M} = \langle \overline{H}, \overline{U} \rangle$ such that*

- $\overline{M} \models \overline{U}$ *is a normal measure on* $\overline{\kappa}$ *for a* $\overline{\kappa} \in \overline{M}$
- $\overline{M}$ *iterates to M in α many steps.*

(Hence $\overline{M}$ is iterable, since M is.)

Proof: The case $\alpha = 0$ is trivial, so assume $\alpha > 0$. Let δ be least such that $L_\delta(M)$ is admissible. Let $\mathbb{L}$ be the $\in$–theory on $L_\delta(M)$ with:

Predicate: $\dot{\in}$

Constants: $\underline{x}(x \in L_\delta(M)), \dot{M}$

Axiom:
- Basic axioms + ZFC$^-$
- $\dot{M} = \langle \dot{H}, \dot{U} \rangle \models$ (ZFC$^-$ +$\dot{U}$ is a normal measure on a $\kappa < \dot{H}$)
- $\dot{M}$ iterates to $\underline{M}$ in $\underline{\alpha}$ many steps.

It will suffice to show:

Claim $\mathbb{L}$ is consistent.

We first show that the claim implies the theorem. Let $\mathbb{A}$ be a grounded model of $\mathbb{L}$. Then $\mathbb{L}_\delta(M) \subset \mathrm{wfc}(\mathbb{A})$. Hence $M, \overline{M} \in \mathrm{wfc}(\mathbb{A})$, where $\overline{M} = \dot{M}^{\mathbb{A}}$. But then in $\mathbb{A}$ there is an iteration $\langle \overline{M}_i | i \leq \alpha \rangle$ of $\overline{M}$ to M. By absoluteness $\langle \overline{M}_i | i \leq \alpha \rangle$ really is such an iteration. QED

We now prove the claim.

Case 1 $\alpha < \kappa$
Iterate $\langle W, U \rangle$ α many times, getting $\langle W_i, U_i \rangle (i \subseteq \alpha)$ with iteraton maps $\pi_{i,j}$. Then $\pi_{0,\alpha}(\alpha) = \alpha$. Set $M_i = \pi_{0,1}(M)$. Then $\langle M_i | i \leq \alpha \rangle$ is an iteration of M with iteration maps $\pi_{i,j} \upharpoonright M_i$. But $M_\alpha = \pi_{0,\alpha}(M)$. Hence $\langle H_{\kappa^+}, M \rangle$ models $\pi_{0,\alpha}(\mathbb{L})$. But then $\pi_{0,\alpha}(\mathbb{L})$ is consistent. Hence so is $\mathbb{L}$. QED

Case 2 $\alpha = \kappa$
Iterate $\langle W, U \rangle$ β many times, where $\pi_{0,\beta}(\kappa) = \beta$. Then $\langle M_i | i \leq \beta \rangle$ iterates M to M_β in β many steps. Hence $\langle H_{\kappa^+}, M \rangle$ models $\pi_{0,\beta}(\mathbb{L})$. Hence $\pi_{0,\beta}(\mathbb{L})$ is consistent and so is $\mathbb{L}$. QED (Lemma 1.4.12)

Barwise theory is useful in situations where one is given a transitive structure Q and wishes to find a transitive structure $\overline{Q}$ with similar properties inside an inner model. Another tool, which is often used in such situations, is Schoenfield's lemma, which, however, requires coding Q by a real. Unsurprizingly, Schoenfield's lemma can itself be derived from Barwise theory. We first note the well known fact that every Σ^1_2 condition on a real is equivalent to a $\Sigma_1(H_{\omega_1})$ condition, and conversely. Thus it suffices to show:

Lemma 1.4.13. *Let $H_{\omega_1} \models \varphi[a], a \subset \omega$, where φ is Σ_1. Then:*

$$H_{\omega_1} \models \varphi[a] \text{ in } L(a).$$

Proof: Let $\varphi = \bigvee z\psi$, where ψ is Σ_0. Let $H_{\omega_1} \models \psi[z,a]$ where $\mathrm{rn}(z) = \delta < \alpha < \omega_1$ and α is admissible in a. Let $\mathbb{L}$ be the language on $L_\alpha(a)$ with:

Predicate: $\dot{\in}$

Constants: $\underline{x}(x \in L_\alpha(a))$

Axioms: Basic acioms $+\mathrm{ZFC}^- + \bigvee z(\psi(z,\underline{a}) \wedge \mathrm{rn}(z) = \underline{\delta})$.

Then $\mathbb{L}$ is consistent, since $\langle H_{\omega_1}, a\rangle$ is a model. We cannot necessarily chose α such that it is countable in $L(a)$, however. Hence, working in $L(a)$, we apply a Skolem–Löwenheim argument to $L_\alpha(a)$, getting countable $\overline{\alpha}, \overline{\delta}, \pi$ such that $\pi : L_{\overline{\alpha}}(a) \prec L_\alpha(a)$ and $\pi(\overline{\delta}) = \delta$. Let $\overline{\mathbb{L}}$ be defined from $\overline{\delta}$ over $L_{\overline{\alpha}}(a)$ as $\mathbb{L}$ was defined from δ over $L_\alpha(a)$. Then $\overline{\mathbb{L}}$ is consistent by corollary 1.4.4. Since $L_{\overline{\alpha}}(a)$ is countable in $L(a)$, $\overline{\mathbb{L}}$ has a grounded model $\mathbb{A} \in L(a)$. But then there is $z \in \mathbb{A}$ such that $\mathbb{A} \models \psi[z,a]$ and $rn^{\mathbb{A}}(z) = \overline{\delta}$. Thus $rn(z) = \overline{\beta} \in \mathrm{wfc}(\mathbb{A})$ and $z \in \mathrm{wfc}(\mathbb{A})$. Thus $\mathrm{wfc}(\mathbb{A}) \models \psi[z,a]$, where $\mathrm{wfc}(\mathbb{A}) \subset H_{\omega_1}$ in $L(a)$. Hence $H_{\omega_1} \models \varphi[a]$ in $L(a)$. QED

Chapter 2

Basic Fine Structure Theory

2.1 Introduction

Fine structure theory arose from the attempt to describe more precisely the way the constructable hierarchy grows. There are many natural natural questions. We know for instance by Gödel's condensation lemma that there are countable γ such that L_γ models $\text{ZFC}^- + \omega_1$ exists. This means that some $\beta < \gamma$ is a cardinal in L_γ but not in L. Hence there is a subset $b \subset \beta$ lying in L but not in L_γ. Hence there must be a least $\alpha > \gamma$ such that such a subset lies in $L_{\alpha+1} = \text{Def}(L_\alpha)$. What happens there, and what do such α look like? It turns out that there is then a $\underline{\Sigma}_\omega(L_\alpha)$ injection of L_α into β, and that α can be anything — even a successor ordinal.

In chapter 1 we developed an elaborate body of methods for dealing with admissible structures. In order to deal with questions like the above ones, we must try to adapt these methods to an arbitrary L_α. A key concept in this endeavor is that of *amenability*:

Definition 2.1.1. A transitive structure $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ is *amenable* iff $A_i \cap x \in M$ for all $x \in M$, $i = 1, \ldots, n$.

Thus, as stated at the end of chapter 1, §1.1, an $\alpha \geq \omega$ is strongly admissible iff $\langle L_\alpha, A \rangle$ is amenable for all $\underline{\Sigma}_1(L_\alpha)$ sets A. Using this as a starting point, we sketch (omitting all details!) the fine structural proof that if $b \subset \beta < \alpha$ and $b \in L_{\alpha+1} \setminus L_\alpha$, then there is a $\underline{\Sigma}_\omega(L_\alpha)$ injection of L_α into β. Suppose, first, that b is $\underline{\Sigma}_1(L_\alpha)$. Then $\beta \geq \rho^0$, where ρ^0 is the projectum of L_α. But as stated in chapter 1, §1.1, there is then a $\underline{\Sigma}_1(L_\alpha)$ injection f^0 of L_α into ρ^0, which proves the result. Now suppose that b is $\underline{\Sigma}_2(L_\alpha)$ but not $\underline{\Sigma}_1(L_2)$ and that $\beta < \rho^0$. By the existens of f^0 there is a $\underline{\Sigma}_1(L_\alpha)$ set $A^0 \subset \rho^0$ which completely codes L_α. $N^0 = \langle L_{\rho_0}, A^0\rangle$ is then amenable and b is $\underline{\Sigma}_1(N^0)$. Thus $\beta \geq \rho^1$, where ρ^1 is the projectum of N^0. However, N^0 is so much like an L_α that there is a $\underline{\Sigma}_1(N^0)$ injection f^1 of N^0 into ρ^1. Thus $f^1 \circ f^0$ is a $\underline{\Sigma}_\omega(L_\alpha)$ injection into $\rho^1 \geq \beta$. If b is $\underline{\Sigma}_3(L_\alpha)$ but not $\underline{\Sigma}_2(L_\alpha)$ and $\beta < \rho^1$, we go one step further, forming $N^1 = \langle J_{\rho_1}, A^1\rangle$ which codes N^0 and note that b is now $\underline{\Sigma}_1(N^1)$ etc. Note that, since $\alpha \geq \rho^0 \geq \rho^1, \ldots$, the sequence of ρ^i must stabilize at some point.

The first proof of the above result was due to Hilary Putnam and did not use the full fine structure analysis we have just outlined. However, our analysis yielded many new insights; giving for instance the first proof that L_α is $\underline{\Sigma}_n$ uniformizable for all $n \geq 1$. (I.e. every $\underline{\Sigma}_n$ relation is uniformizable by a $\underline{\Sigma}_n$ function.)

Not long afterwards fine structure theory was used to prove some deep global properties of L, such as:

$$L \models \Box_\beta \text{ for all infinite cardinals } \beta.$$

It was also used to prove the covering lemma for L. That, in turn, led to extended versions of fine structure theory which could be used to analyze larger inner models, in which some large cardinals could be realized. (Here, however, the fine structure theory was needed not only to analyze the inner model, but even to define it in the first place.)

Carrying out the above analysis of L requires a very fine study of definability over an arbitrary L_α. In order to achieve this, however, one must overcome some formidable technical obstacles which arise from Gödel's definition of the constructible hierarchy: At successors α, L_α is not even closed under ordered pairs, let alone other basic set functions like unit set, crossproduct etc. One solution is to employ the theory of *rudimentary functions* in an auxiliary role. These functions, which were discovered by Gandy and Jensen, are exactly the functions which are generated by the schemata for primitive recursive functions when the recursion schema is omitted. (Cf. the remark following chapter 1, §2, Lemma 1.1.4). If $\text{rn}(x_i) < \gamma$ for $i = 1, \ldots, n$ and f is rudimentary, then $\text{rn}(f(x_1, \ldots, x_n)) < \gamma + \omega$. All reasonable "elementary" set theoretic functions are rudimentary. If α is a limit ordinal, then L_α is closed under rudimentary functions. If α is a successor, then closing L_α under rudimentary functions yields a transitive structure L^*_α of rank $\alpha+\omega$. It then turns out that every $\underline{\Sigma}_\omega(L^*_\alpha)$ definable subset of L_α is already $\underline{\Sigma}_\omega(L^*_\alpha)$, and conversely. Hence we can, in effect, replace the rather weak definability theory of L_α by the rather nice definability theory of L^*_α. (This method was used in [JH], except that L^*_α was given a different but equivalent definition, since the rudimentary functions were not yet known.) It turns out that if N is transitive and rudimentarily closed, and $\text{Rud}(N)$ is defined to be the closure of $N \cup \{N\}$ under rudimentary functions, then $\mathbb{P}(N) \cap \text{Rud}(N) = \text{Def}(N)$. This suggests an alternative version of the constructible hierarchy in which every level is rudimentarily closed. We shall index this hierarchy by the class Lm of limit ordinals, setting:

$$J_\omega = H_\omega = \text{Rud}(\emptyset)$$

$$J_{\alpha+\omega} = \text{Rud}(J_\alpha) \text{ for } \alpha \in \text{Lm}$$

$$J_\lambda = \bigcup_{\nu<\lambda} J_\nu \text{ for } \lambda \text{ a limit p.t. of Lm.}$$

(**Note** Setting $J = \bigcup_\alpha J_\alpha$, we have: $J = L$ in fact $J_\alpha = L_\alpha$ whenever α is pr closed.)

(**Note** This indexing was introduced by Sy Friedman. In [FSC] we indexed by *all* ordinals, so that our $J_{\omega\alpha}$ corresponds to the J_α of [FSC]. The usage in [FSC] has been followed by most authors. Nonetheless we here adopt Friedman's usage, which seems to us more natural, since we then have: $\alpha = \text{rn}(J_\alpha) = \text{On} \cap J_\alpha$.)

In the following section we develop the theory of rudimentary functions.

2.2 Rudimentary Functions

Definition 2.2.1. $f : V^n \to V$ is a *rudimentary* (rud) *function* iff it is generated by successive applications of schemata (i) – (v) in the definition of *primitive recursive* in chapter 1, §2.

A relation $R \subset V^n$ is rud iff there is a rud function f such that: $R\vec{x} \leftrightarrow f(\vec{x}) = 1$. In chapter 1, §1.2 we established that:

Lemma 2.2.1. *Lemmas 1.2.1 – 1.2.4 of chapter 1, §1.2 hold with* 'rud' *in place of 'pr'.*

(**Note** Our definition of 'rud function', like the definition of 'pr function' is ostensibly in second order set theory, but just as in chapter 1, §1.2 we can work in ZFC by talking about rud *definitions*. The notion of rud definition is defined like that of pr definition, except that instances of schema (vi) are not allowed. As before, we can assign to each rud definition s a rud function $F_s : V^n \to V$ with the property that $F_s^M = F_s \upharpoonright M$ whenever M is admissible and $F_s^M : M^n \to M$ is the function on M defined by s. But then if M is transitive and closed under rud functions, it follows by induction on the length of s that there is a unique $F_s^M = F_s \upharpoonright M$.)

A rudimentary function can raise the rank of its arguments by at most a finite amount:

Lemma 2.2.2. *Let $f : V^n \to V$ be* rud. *Then there is $p < \omega$ such that*

$$f(\vec{x}) \subset \mathbb{P}^p(TC(x_1 \cup \ldots \cup x_n)) \text{ for all } x_1, \ldots, x_n.$$

(Hence $\mathrm{rn}(f\vec{x}) \leq \max\{\mathrm{rn}(x_1), \ldots, \mathrm{rn}(x_n)\} + p$ *and* $\bigcup^p f(\vec{x}) \subset TC(x_1 \cup \ldots \cup x_n)$*.)*

Proof: Call any such p *sufficient* for f. Then if p is sufficient, so is every $q \geq p$. By induction on the defining schemata for f, we prove that f has a sufficient p. If f is given by an initial schema, this is trivial. Now let $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$. Let p be sufficient for h and q be sufficient for $g_i (i = 1, \ldots, m)$. It follows easily that $p + q$ is sufficient for f. Now let $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$, where p is sufficient for g. It follows easily that p is sufficient for f. QED

By lemma 2.2.1 and chapter 1 lemma 1.2.3 (i) we know that every Σ_0 relation is rud. We now prove the converse. In fact we shall prove a stronger result. We first define:

Definition 2.2.2. $f : V^n \to V$ is *simple* iff whenever $R(z, \vec{x})$ is a Σ_0 relation, then so is $R(f(\vec{x}), \vec{y})$.

The simple functions are obviously closed under composition. The simplicity of a function f is equivalent to the conjunction of the two conditions:

(i) $x \in f(\vec{y})$ is Σ_0

(ii) If $A(z, \vec{u})$ is Σ_0, then $\bigwedge z \in f(\vec{x}) A(z, u)$ is Σ_0,

for given these we can verify by induction on the Σ_0 definition of R that $R(f(\vec{x}), \vec{y})$ is Σ_0.
But then:

Lemma 2.2.3. *All* rud *functions are simple.*

Proof: Using the above facts we verify by induction on the defining schemata of f that f is simple. The proof is left to the reader. QED

In particular:

Corollary 2.2.4. *Every* rud *function* f *is* Σ_0 *as a relation. Moreover* $f \upharpoonright U$ *is uniformly* $\Sigma_0(U)$ *whenever* U *is transitive and* rud *closed.*

Corollary 2.2.5. *Every* rud *relation is* Σ_0.

In chapter 1, §2 we relativized the concept 'pr' to 'pr in $A_1, \ldots, A_n$'. We can do the same thing with 'rud'.

Definition 2.2.3. Let $A_i \subset V(i = 1, \ldots, m)$. $f : V^n \to V$ is *rudimentary in* $A_1, \ldots, A_n$ (rud *in* $A_1, \ldots, A_n$) iff it is obtained by successive applications of the schemata (i) – (v) and:

$$f(x) = \chi_A(x) \ (i = 1, \ldots, n)$$

where χ_A is the characteristic function of A.

Lemma 1.1.1 and 1.1.2 obviously hold with 'rud in $A_1, \ldots, A_n$' in place of 'rud'. Lemma 2.2.3 and its corollary do *not* hold, however, since e.g. the relation $\{x\} \in A$ is not Σ_0 in A.

However, we do get:

Lemma 2.2.6. *If* f *is* rud *in* $A_1, \ldots, A_n$*, then*

$$f(\vec{x}) = f_0(\vec{x}, A_1 \cap f_1(\vec{x}), \ldots, A_n \cap f_n(\vec{x}))$$

where $f_0, f_1, \ldots, f_n$ *are* rud *functions.*

Proof: We display the proof for the case $n = 1$. Let f be rud in A. By induction on the defining schemata for f we show:

$$f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x})) \text{ where } f_0, f_1 \text{ are rud}.$$

Case 1 f is given by schemata (i) – (iii). This is trivial.

Case 2 $f(x) = X_A(x)$. Then

$$f(x) = \left\{ \begin{array}{l} 1 \text{ if } A \cap \{x\} \neq \emptyset \\ 0 \text{ if not} \end{array} \right\} = f'(x, A \cap \{x\})$$

where f' is rud. QED (Case 2)

Case 3 $f(\vec{x}) = g(h^1(\vec{x}), \ldots, h^m(\vec{x}))$. Let

$$g(\vec{z}) = g_0(\vec{z}, A \cap g_1(\vec{z}))$$
$$h^i(\vec{x}) = h_0^i(\vec{x}, A \cap h_1^i(\vec{z}))(i = 1, \ldots, m)$$

where g_0, g_1, h_0^i, h_1^i are rud. Set:

$$\begin{aligned}
\tilde{g}(\vec{z}, u) &= g_0(\vec{z}, u \cap g_1(\vec{z})) \\
\tilde{h}^i(\vec{x}, u) &= h_0^i(\vec{x}, u \cap h_1^i(\vec{x})) \\
\tilde{f}(\vec{x}, u) &= \tilde{g}(\tilde{h}^1(\vec{x}, u), \dots, \tilde{h}^m(\vec{x}, u), u) \\
k(\vec{x}) &= g_1(\vec{h}_1(\vec{x})) \cup \bigcup_{i=1}^{m} h_1^i(\vec{x}).
\end{aligned}$$

Then $f(\vec{x}) = \tilde{f}(\vec{x}, A \cap k(\vec{x}))$, where $\tilde{f}, k$ are rud. This follows from the facts:

$$\tilde{h}^i(\vec{x}, A \cap v) = h_0^i(\vec{x}, A \cap h_1^i(\vec{x})) = h^i(\vec{x}) \text{ if } h_1^i(\vec{x}) \subset v$$

$$\tilde{g}^i(\vec{z}, A \cap v) = g_0(\vec{z}, A \cap z) \text{ if } g_1(\vec{z}) \subset v.$$

QED (Case 3)

Case 4 $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$. Let $g(z, \vec{x}) = g_0(z, \vec{x}, A \cap g_1(z, \vec{x}))$. Set

$$\begin{aligned}
\tilde{g}(z, \vec{x}, u) &= g_0(z, \vec{x}, u \cap g_1(z, \vec{x})) \\
\tilde{f}(y, \vec{x}, u) &= \bigcup_{z \in y} \tilde{g}(z, \vec{x}, u) \\
k(y, \vec{x}) &= \bigcup_{z \in y} g_1(z, \vec{x})
\end{aligned}$$

Then $f(y, \vec{x}) = \tilde{f}(y, \vec{x}, A \cap k(y, \vec{z}))$ where $\tilde{f}, k$ are rud.

QED (Lemma 2.2.6)

Definition 2.2.4. X is *rudimentarily closed* (rud closed) iff it is closed under rudimentary functions. $\langle M, A_1, \dots, A_n \rangle$ is rud closed iff M is closed in functions rudimentary in $A_1, \dots, A_n$.

If $M = \langle |M|, A_1, \dots, A_n \rangle$ is transitive and rud closed, then it is amenable, since it is closed under $f(x) = x \cap A$. By lemma 2.2.6 we then have:

Corollary 2.2.7. *Let $M = \langle |M| A_1, \dots, A_n \rangle$ be transitive. M is rud closed iff it is amenable and $|M|$ is rud closed.*

Corresponding to corollary 2.2.4 we have:

Corollary 2.2.8. *Every function f which is* rud *in A is Σ_1 in A as a relation. Moreover $f \upharpoonright U$ is $\Sigma_1(\langle U, A \cap U \rangle)$ by the same Σ_1 definition whenever $\langle U, A \cap U \rangle$ is transitive and rud closed. (Similarly for "*rud *in $A_1, \dots, A_n$".)*

Proof: Let $f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x}))$ where f_0, f_1 are rud. Then:

$$y = f(\vec{x}) \leftrightarrow \bigvee u \bigvee z (y = f_0(\vec{x}, z) \wedge u = f_1(\vec{x}) \wedge z = A \cap u).$$

QED (Corollary 2.2.8)

In chapter 1 §2.2 we extended the notion of "pr definition" so as to deal with functions pr in classes $A_1, \ldots, A_n$. We can do the same for rudimentary functions:

We appoint new designated function variables $\dot{a}_1, \ldots, \dot{a}_n$ and define the set of rud *definition in* $a_1, \ldots, a_n$ exactly as before, except that we omit the schema (vi). Given $A_1, \ldots, A_n$ we can, exactly as before, assign to each rud definition s in $\dot{a}_1, \ldots, \dot{a}_n$ a function $F_s^{A_1,\ldots,A_n}$ are then exatly the functions rud in $A_1, \ldots, A_n$. Since lemma 2.2.6 (and with it corollary 2.2.8) is proven by induction on the defining schemata, its proof implicitly defines an algorithm which assigns to each s as Σ_1 formula φ_s which defines $F_s^{\vec{A}}$.

Corresponding to chapter 1 §1 Lemma 1.1.13 we have:

Lemma 2.2.9. *Let f be* rud *in $A_1, \ldots, A_n$, where each A_i is* rud *in $B_1, \ldots, B_m$. Then f is* rud *in $B_1, \ldots, B_m$.*

The proof is again by induction on the defining schemata. It shows, in fact that f is *uniformly* rud in $\vec{B}$ in the sense that its rud definition from $\vec{B}$ depends only on its rud definition from $\vec{A}$ and the rud definition of A_i from $\vec{B}$ $(i = 1, \ldots, n)$.

We also note:

Lemma 2.2.10. *Let $\pi : \overline{M} \to_{\Sigma_0} M$, where $\overline{M}, M$ are* rud *closed. Then π preserves rudimentarily in the following sense: Let $\overline{f}$ be defined from the predicates of $\overline{M}$ by the* rud *definition s. Let f be defined from the predicates of M by s. Then $\pi(\overline{f}(\vec{x})) = f(\pi(\vec{x}))$ for $x_1, \ldots, x_n \in \overline{M}$.*

Proof: Let φ_s be the canonical Σ_1 definition. Then $\overline{M} \models \varphi_s[y, \vec{x}] \to M \models \varphi_s[\pi(y), \pi(\vec{x})]$ by Σ_0–preservation. QED (Lemma 2.2.10)

We now define:

Definition 2.2.5.
rud(U) =: The closure of U under rud functions
rud$_{A_1,\ldots,A_n}(U)$ =: The closure of U under functions rud in $A_1, \ldots, A_n$

(Hence rud$(U) = $ rud$_\emptyset(U)$.)

Lemma 2.2.11. *If U is transitive, then so is* rud(U).

Proof: Let $W = \text{rud}(U)$. Let $Q(x)$ mean: $TC(\{x\}) \subset W$. By induction on the defining schemata of f we show:

$$(Q(x_1) \wedge \ldots \wedge Q(x_n)) \rightarrow Q(f(x_1, \ldots, x_n))$$

for $x_1, \ldots, x_n \in W$. The details are left to the reader. But $x \in U \rightarrow Q(x)$ and each $z \in W$ has the form $f(\vec{x})$ where f is rud and $x_1, \ldots, x_n \in U$. Hence $TC(\{z\}) \subset W$ for $z \in W$. QED

The same proof shows:

Corollary 2.2.12. *If U is transitive, then so is* $\text{rud}_{\vec{A}}(U)$.

Using Corollary 2.2.12 and Lemma 2.2.3 we get:

Lemma 2.2.13. *Let U be transitive and $W = \text{rud}(U)$. Then the restriction of any $\underline{\Sigma}_0(W)$ relation to U is $\underline{\Sigma}_0(U)$.*

Proof: Let R be $\underline{\Sigma}_0(W)$. Let $R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p})$ where R' is $\Sigma_0(W)$ and $p_1, \ldots, p_n \in W$. Let $p_i = f_i(\vec{z})$, where f_i is rud and $z_1, \ldots, z_n \in U$. Then for $x_1, \ldots, x_m \in U$:

$$\begin{aligned} R(\vec{x}) &\leftrightarrow R'(\vec{x}, \vec{f}(\vec{z})) \\ &\leftrightarrow R''(\vec{x}, \vec{z}) \end{aligned}$$

where R'' is $\Sigma_0(U)$, by lemma 2.2.3. QED (Lemma 2.2.13)

We now define:

Definition 2.2.6. Let U be transitive.

$$\begin{aligned} &\text{Rud}(U) =: \text{rud}(U \cup \{U\}) \\ &\text{Rud}_{\vec{A}}(U) =: \text{rud}_{\vec{A}}(U \cup \{U\}) \end{aligned}$$

Then $\text{Rud}(U)$ is a proper transitive extension of U. By Lemma 2.2.13:

Corollary 2.2.14. $\text{Def}(U) = \mathbb{P}(U) \cap \text{Rud}(U)$ *if $U \neq \emptyset$ is transitive.*

Proof: If $A \in \text{Def}(U)$, then A is $\underline{\Sigma}_0(U \cup \{U\})$. Hence $A \in \text{Rud}(U)$. Conversely, if $A \in \text{Rud}(U)$, then A is $\underline{\Sigma}_0(U \cup \{U\})$ by lemma 1.1.7. It follows easily that $A \in \text{Def}(U)$. QED (Corollary 2.2.14)

[**Note** To see that $A \in \text{Def}(U)$, consider the $\in$–language augmented by a new constant $\dot{U}$ which is interpreted by U. We assign to every Σ_0 formula φ in this language a first order formula φ' not containing $\dot{U}$ such that for all $x_1, \ldots, x_n \in U$:

$$U \cup \{U\} \models \varphi[\vec{x}] \leftrightarrow U \models \varphi'[\vec{x}].$$

(Here x_i is taken to interpret v_i where $v_1, \ldots, v_n$ is an arbirtrarily chosen of distinct variables, including all variables which occur free in φ.) We define φ' by induction on φ. For primitive formulae we set first:

$$(v \in w)' = v \in w, (v \in \dot{U})' = v = v,$$
$$(\dot{U} \in v)' = v \neq v, (\dot{U} \in \dot{U}) = \bigvee v \; v \neq v.$$

For sentential combinations we do the obvious thing:

$$(\varphi \wedge \psi)' = (\varphi' \wedge \psi'), (\neg\varphi)' = \neg\varphi',$$

etc. Quantifiers are treated as follows:

$$(\bigwedge v \in w\varphi)' = \bigwedge v \in w\varphi'$$
$$(\bigwedge v \in \dot{U}\varphi)' = \bigwedge v\varphi'.]$$

Given finitely many rud functions $s_1, \ldots, s_p$ we say that they constitute a *basis* for the rud function iff every rud function is obtainable by successive application of the schemata:

- $f(x_1, \ldots, x_n) = x_j$ $(j = 1, \ldots, n)$
- $f(\vec{x}) = s_i(g_1(\vec{x}), \ldots, g_m(\vec{x})$ $(i = 1 \ldots, p)$

Note that if $s_1, \ldots, s_n$ is a basis, then $\text{rud}(U)$ is simply the closure of U under the finitely many functions $s_1, \ldots, s_p$. We shall now prove the *Basis Theorem*, which says that the rud functions possess a finite basis. We first define:

Definition 2.2.7. $(x, y) =: \{\{x\}, \{x, y\}\}; (x) = x$,
$(x_1, \ldots, x_n) = (x_1, (x_2, \ldots, x_n))$ for $n \geq 2$.

(Note: Our "official" notation for n–tuples is $\langle x_1, \ldots, x_n \rangle$. However, we have refrained from specifying its definition. Thus we do not know whether $(\vec{x}) = \langle \vec{x} \rangle$.)

We also set:

Definition 2.2.8.

$$x \otimes y = \{(z, w) | z \in x \wedge w \in y\}$$
$$\text{dom}^*(x) = \{z | \bigvee y(y, z) \in x\}$$
$$x^* z = \{y | (y, z) \in x\}$$

Theorem 2.2.15. *The following functions form a basis for the* rud *function:*

$$
\begin{aligned}
F_0(x,y) &= \{x,y\}\\
F_1(x,y) &= x\setminus y\\
F_2(x,y) &= x\otimes y\\
F_3(x,y) &= \{(u,z,v)|z\in x\wedge (u,v)\in y\}\\
F_4(x,y) &= \{(u,v,z)|z\in x\wedge (u,v)\in y\}\\
F_5(x,y) &= \bigcup x\\
F_6(x,y) &= \mathrm{dom}^*(x)\\
F_7(x,y) &= \{(z,w)|z,w\in x\wedge z\in w\}\\
F_8(x,y) &= \{x^*z|z\in y\}
\end{aligned}
$$

Proof: The proof stretches over several subclaims. Call a function f *good* iff it is obtainable from $F_0,\ldots,F_8$ by successive applications of the above schemata. Then every good function is rud. We must prove the converse. We first note:

Claim 1 The good functions are closed under composition — i.e. if $g,h_1,\ldots,h_n$ are good, then so is $f(\vec{x})=g(\vec{h}(\vec{x}))$.

Proof: Set $G=$ the set of good function $g(y_1,\ldots,y_v)$ such that whenever $h_i(\vec{x})$ is good for $i=1,\ldots,r$, then so is $f(\vec{x})=g(\vec{h}(\vec{x}))$. By a straightforward induction on the defining schemata it is easily shown that all good functions are in G. QED (Claim 1)

Claim 2 The following functions are good:

$$
\begin{aligned}
&\{x,y\},x\setminus y,x\otimes y,x\cup y=\bigcup\{x,y\},\\
&x\cap y=x\setminus(x\setminus y),\{x_1,\ldots,x_n\}=\{x_1\}\cup\ldots\cup\{x_n\},\\
&C_n(u)=u\cup\bigcup u\cup\ldots\cup\overbrace{\bigcup\ldots\bigcup}^{n}u,(x_1,\ldots,x_n)
\end{aligned}
$$

(since $(x_1,\ldots,x_n)$ is obtained by iteration of F_0.) By an $\in$*–formula* we mean a first oder formula containing only $\dot{\in}$ as a non logical predicate. If $\varphi=\varphi(v_1,\ldots,v_n)$ is any $\in$–formula in which at most the distinct variables $(v_1,\ldots,v_n)$ occur free, set:

$$t_\varphi(u)=:\{(x_1,\ldots,x_n)|\vec{x}\in u\wedge\langle u,\in\rangle\models\varphi[\vec{x}]\}.$$

(**Note** We follow the usual convention of suppressing the list of variables.)

(**Note** Recall our convention that $\vec{x}\in u$ means that $x_i\in u$ for $i=1,\ldots,n$.) Then t_φ is *rud*. We claim:

Claim 3 t_φ is good for every $\in$–formula φ.

Proof:

(1) It holds for $\varphi = v_i \in v_j$ $(1 \le i < j \le n)$

Proof: For $i = 2, 3$ set:

$$F_i^0(u,w) = w,\ F_i^{m+1}(u,w) = F_i(u, F_i^m(u,w))$$

then $F_i^m u$ is good for all m. For $m \ge 1$ we have:

$$\begin{aligned} F_2^m(u,w) &= \{(x_1,\dots,x_m,z) | \vec{x} \in u \wedge z \in w\} \\ F_3^m(u,w) &= \{(y,x_1,\dots,x_m,z) | \vec{x} \in u \wedge (y,z) \in w\} \end{aligned}$$

We also set

$$\begin{aligned} u^{(m)} &= \{(x_1,\dots,x_m) | \vec{x} \in u\} \\ &= F_2^{m-1}(u,u) \end{aligned}$$

If $j = n$, then

$$\begin{aligned} t_\varphi(u) &= \{(x_1,\dots,x_m) | \vec{x} \in u \wedge x_i \in x_j\} \\ &= F_2^{i-1}(u, F_3^{n-i-1}(u, F_7(u,u))). \end{aligned}$$

Now let $n > j$. Noting that:

$$F_4(u^{(m)}, w) = \{(y,z,x_1,\dots,x_m) | \vec{x} \in u \wedge (y,z) \in w\},$$

we have:

$$t_\varphi(u) = F_2^{i-1}(u, F^{j-i-1}(u, F_4(u^{(n-j)}, F_7(u,u)))).$$

QED (1)

(2) It holds for $\varphi = v_i \in v_i$.

Proof: $t_\varphi(w) = \emptyset = w \setminus w$.

(3) If it holds for $\varphi = \varphi(v_1,\dots,v_n)$, then for $\neg\varphi$.

Proof:

$$t_{\neg\varphi}(w) = (w^{(n)} \setminus t_\varphi(w)).$$

QED (3)

(4) If it holds for φ, ψ, then for $\varphi \wedge \psi$, $\varphi \vee \psi$. (Hence for $\varphi \to \psi$, $\varphi \leftrightarrow \psi$ by (3).)

Proof:

$$\begin{aligned} t_{\varphi\vee\psi}(w) &= t_\varphi(w) \cup t_\psi(w) = \bigcup\{t_\varphi(w), t_\psi(w)\} \\ t_{\varphi\wedge\psi}(w) &= t_\varphi(w) \cap t_\psi(w),\ \text{where } x \wedge y = (x \setminus (x \setminus y)). \end{aligned}$$

QED (4)

(5) If it holds for $\varphi = \varphi(u, v_1, \ldots, v_n)$, then for $\bigwedge u\varphi$, $\bigvee_v \varphi$.

Proof:

$$t_{\bigvee u\varphi}(w) = F_6(t_\varphi, t_\varphi)_i \text{ hence}$$
$$t_{\bigwedge u\varphi}(w) = t_{\neg\bigvee u\neg\varphi}(w) \text{ by (3)}$$

QED (5)

(6) It holds for $\varphi = v_i = v_j$ $(i, j \leq n)$.

Proof: Let $\psi(v_1, \ldots, v_n) = \bigwedge z(z \in v_i \leftrightarrow z \in v_j)$. Then for $(\vec{x}) \in U^{(n)}$ we have:

$$(\vec{x}) \in t_\psi(u \cup \bigcup u) \leftrightarrow x_i = x_j,$$

since $x_i, x_j \subset (u \cup \bigcup u)$. Hence

$$t_\varphi(u) = u^{(n)} \cap t_\psi(u \cup \bigcup u).$$

QED (6)

(7) It holds for $\varphi = v_j \in v_i$ $(i < j)$

Proof:

$$v_j \in v_i \leftrightarrow \bigvee u(u = v_j \wedge u \in v_i).$$

We apply (6), (5) and (4). QED (7)

But then if $\varphi(v_1, \ldots, v_n) = Qu_1, \ldots Qu_n\psi(\vec{u}, \vec{v})$ is any formula in prenex normal form, we apply (1), (2), (6), (7) and (3), (4) to see that t_ψ is good. But then t_φ is good by iterated applications of (5). QED (Claim 3)

In our application we shall use the function t_φ only for Σ_0 formulae φ. We shall make strong use of the following well known fact, which can be proven by induction on n.

Fact Let $\varphi = \varphi(v_1, \ldots, v_m)$ be a Σ_0 formula in which at most n quantifiers occur. Let u be any set and let $x_1, \ldots, x_m \in u$. Then $V \models \varphi[\vec{x}] \leftrightarrow C_n(u) \models \varphi[\vec{x}]$.

Definition 2.2.9. Let $f : V^n \to V$ be rud. f is *verified* iff there is a good $f^* : V \to V$ such that $f''U^n \subset f^*(U)$ for all sets u. We then say that f^* *verifies* f.

Claim 4 Every verified function is good.

Proof: Let f be verified by f^*. Let φ be the Σ_0 formula: $y = f(x_1, \ldots, x_n)$. For sufficient n we know that for any set u we have:

$$y = f(\vec{x}) \leftrightarrow (y, \vec{x}) \in t_\varphi(C_n(u \cup f^*(u)))$$
$$\text{for } y, \vec{x} \in u \cup f^*(u).$$

Define a good function F by:

$$F(u) =: (f^*(u) \otimes u^{(n)}) \cap t_\varphi(C_n(U \cup f^*(u))).$$

Then $F(u)$ is the set of $(f(\vec{x}), \vec{x})$ such that $\vec{x} \in u$. In particular, if $u = \{x_1, \ldots, x_n\}$, then:

$$F_8(F(\{\vec{x}\}), \{(\vec{x})\}) = \{f(\vec{x})\}$$

and $f(\vec{x}) = \bigcup F_8(F(\{\vec{x}\}), \{(\vec{x})\})$. QED (Claim 4)

Thus it remains only to prove:

Claim 5 Every rud function is verified.

Proof: We proceed by induction on the defining schemata of f.

Case 1 $f(\vec{x}) = x_i$

Take $f^*(u) = u = u \setminus (u \setminus u)$.

Case 2 $f(\vec{x}) = x_i \setminus x_j$

Let φ be the Σ_0 formula $z \in x \setminus y$. For sufficient n we have:

$$z \in x \setminus y \leftrightarrow C_n(u \cup \bigcup u) \models z \in x \setminus y$$

for $z, x, y \in u \cup \bigcup u$. But if $x, y \in u$, then $x \setminus y \subset \bigcup u$. Hence:

$$(x, y, z) \in t_\varphi(C_n|u \cup \bigcup u)) \leftrightarrow z \in x \setminus y$$

for all $x, y \in u$ and all z.

Hence:

$$f''u^n = \{x \setminus y | x, y \in u\} \subset F_8(t_\varphi(C_n(u \cup \bigcup u)), u^{(z)}).$$

QED (Case 2)

Case 3 $f(\vec{x}) = \{x_i, x_j\}$

Then $f''u^n = \{\{x, y\} | x, y \in u\} = \bigcup u^{(2)}$. QED (Case 3)

Case 4 $f(\vec{x}) = g(\vec{h}(\vec{x}))$

Let h_i^* verify h_i and g^* verify g. Then $f^*(u) = g^*(\bigcup_i h_i^*(u))$ verifies f.

QED (Case 4)

Case 5 $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$. Let g^* verify g. Let $\varphi = \varphi(w, y\vec{x})$ be the Σ_0 formula: $\bigvee z \in y \; w \in g(z, \vec{x})$. For sufficient n we have:

$$\bigvee z \in y \; w \in g(z, \vec{x}) \leftrightarrow (w, y, \vec{x}) \in t_\varphi(C_n(u \cup \bigcup g^*(u)))$$

for all $w, y, \vec{x} \in u \cup \bigcup g^*(u)$.

Set $F(u) = t_\varphi(C_n(u \cup \bigcup g^*(u)))$. Then $g(z, \vec{x}) \subset \bigcup g^*(u)$ whenever $y, \vec{x} \in u$ and $z \in y$. Hence

$$F(u)^*(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$$

for $y, \vec{x} \in U$. Hence

$$f''u^{n+1} \subset F_8(F(u), u^{(n+1)}).$$

QED (Theorem 2.2.15)

Combining Theorem 2.2.15 with Lemma 2.2.6 we get:

Corollary 2.2.16. *Let $A_1, \ldots, A_n \subset V$. Then $F_0, \ldots, F_8$ together with the functions $a_i(x) = x \cap A_i (i = 1, \ldots, n)$ form a basis for the functions which are rudimentary in $A_1, \ldots, A_n$.*

Let $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$. '$F_M$' denotes the satisfaction relation for M and '$\models^{\Sigma_n}_M$' denotes its restriction to Σ_n formulae. We can make good use of the basis theorem in proving:

Lemma 2.2.17. *$\models^{\Sigma_0}_M$ is uniformly $\Sigma_1(M)$ over transitive* rud *closed $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$.*

Proof: We shall prove it for the case $n = 1$, since the extension of our proof to the general case is then obvious. We are then given: $M = \langle |M|, \in, A \rangle$. By a *variable evaluation* we mean a function e which maps a finite set of variables of the M–language into $|M|$. Let E be the set of such evaluations. If $e \in E$, we can extend it to an evaluation e^* of all variables by setting:

$$e^*(v) = \begin{cases} e(v) \text{ if } v \in \text{dom}(e) \\ \emptyset \text{ if not} \end{cases}$$

$\models_M \varphi[e]$ then means that φ becomes true in M if each free variable v in φ is interpreted by $e^*(v)$.

We assume, of course, that the first order language of M has been "arithmetized" in a reasonable way — i.e. the syntactic objects such as formulae and variables have been identified with elements of H_ω in such a way that the basic syntactic relations and operations becom recursive. (Without this the assertion we are proving would not make sense.) In particular the set Vbl of variables, the set Fml of formulae, and the set Fml_0 of Σ_0–formulae are all recursive (i.e. $\Delta_1(H_\omega)$). We first note that every $\Sigma_0(M)$ relation is rud, or equivalently:

(1) Let φ be Σ_0. Let $v_1, \ldots, v_n$ be a sequence of distinct variables containing all variables occuring free in φ. There is a function f uniformly rud in A such that

$$\models_M \varphi[e] \leftrightarrow f(e^*(v_1), \ldots e^*(v_n)) = 1$$

for all $e \in E$.

Proof: By induction on φ. We leave the details to the reader.

QED (1)

The notion A–*good* is defined like "good" except that we now add the function $F_9(x, y) = x \cap A$ to our basis. By Corollary 2.2.16 we know that every function rud in A is A–good. We now define in H_ω an auxiliary term language whose terms represent the A–good function. We first set: $\dot{F}_i(x, y) =: \langle i, \langle x, y\rangle\rangle$ for $i = 0, \ldots, 9$: $\dot{x} = \langle 10, x\rangle$. The set Tm of *Terms* is then the smallest set such that

- $\dot{v}$ is a term whenever $v \in Vbl$
- If t, t' are terms, then so is $\dot{F}_i(t, t')$ for $i = 0, \ldots, 9$.

Applying the methods of Chapter 1 to the admissible set H_ω it follows easily that the set Tm is recursive (i.e. $\Delta_1(H_\omega)$). Set

$C(t) \simeq$: The smallest set C such that the term $t \in C$ and C is closed under subterms (i.e. $\dot{F}_i(s, s') \in C \to s, s' \in C$).

Then $C(t) \in H_\omega$ for $t \in Tm$, and the function $C(t)$ is recursive (hence $\Delta_1(H_\omega)$). Since Vbl is recursive, the function
$Vbl(t) \simeq: \{v \in Vbl | \dot{v} \in C(t)\}$ is recursive.

We note that:

(2) Every recursive relation on H_ω is uniformly $\Sigma_1(M)$.

Proof: It suffices to note that: H_ω is uniformly $\Sigma_1(M)$, since

$$x \in H_\omega \leftrightarrow \bigvee f \bigvee u \bigvee n \varphi(f, u, n, x)$$

where φ is the Σ_0 formula: f is a function $\wedge$ u is transitive
$\wedge n \in \omega \wedge f : n \leftrightarrow u \wedge x \in u$. QED (2)

Given $e \in E$ we recursively define an evaluation $\langle \overline{e}(t) | t \in Tm\rangle$ by:

$$\overline{e}(\dot{v}) = e^*(v) \text{ for } v \in Vbl$$
$$\overline{e}(\dot{F}_i(t, s)) = F_i(\overline{e}(t), \overline{e}(s)).$$

Then:

(3) $\{\langle y, e, t\rangle | e \in E \wedge t \in Tm \wedge y = \overline{e}(t)\}$ is uniformly $\Sigma_1(M)$.

Proof: Let $e \in E$, $t \in Tm$. Then $y = \overline{e}(t)$ can be expressed in M by:

$$\bigvee g \bigvee u \bigvee v (u = C(t) \wedge v = Vbl(t) \wedge \varphi(y, e, u, v, y, t))$$

where φ is the Σ_0 formula:

$$\begin{aligned}
&(g \text{ is a function} \wedge \mathrm{dom}(g) = u \wedge \bigwedge x \in v\ x \in u \\
&\qquad \wedge \bigwedge x \in v((x \in \mathrm{dom}(e) \wedge g(\dot{x}) = e(x)) \vee \\
&\qquad\qquad \vee (x \notin \mathrm{dom}(e) \wedge g(\dot{x}) = \emptyset)) \\
&\quad \wedge \bigwedge_{i=0}^{9} \bigwedge t, s, i \in u (t = \dot{F}_i(s, s') \rightarrow \\
&\qquad\qquad \rightarrow g(t) = F_i(g(s), y(s'')) \\
&\quad \wedge y = g(t))
\end{aligned}$$

QED (3)

(4) Let $f(x_1, \ldots, x_n)$ be A–good. Let $v_1, \ldots, v_n'$ be any sequence of distinct variables. There is $t \in Tm$ such that

$$f(e^*(v_1), \ldots, e^*(v_n)) = \overline{e}(t)$$

for all $e \in E$.

Proof: By induction on the defining schemata of f. If $f(\vec{x}) = x_i$, we take $t = \dot{v}_i$. If $e^*(\vec{v})) = \overline{e}(s_i)$ for $e \in \mathbb{E}(i = 0, 1)$, and $f(\vec{x}) = F_i(g_0(\vec{x}), g_1(\vec{x}))$, we set $t = \dot{F}_i(s_0, s_1)$. Then

$$\overline{e}(t) = F_i(\overline{e}(s_0), \overline{e}(s_1)) = F_i(g_0(\vec{x}), g_1(\vec{x})) = f(\vec{x}).$$

QED (4)

But then:

(5) Let φ be a Σ_0 formula. There is $t \in Tm$ such that $M \models \varphi[e] \leftrightarrow \overline{e}(t) = 1$ for all $e \in E$.

Proof: Let $v_1, \ldots, v_n$ be a sequence of distinct variables containing all variables which occur free in φ. Then

$$M \models \varphi[e] \leftrightarrow M \models \varphi[e^*(v_1), \ldots, e^*(v_n)]$$

for all $e \in E$. Set

$$(*)\ f(\vec{x}) = \begin{cases} 1 \text{ if } M \models \varphi[\vec{x}] \\ 0 \text{ if not.} \end{cases}$$

Then f is rudimentary, hence A–good. Let $t \in Tm$ such that

$$(**)\ f(e^*(v_1), \ldots, e^*(v_n)) = \overline{e}(t).$$

Then: $M \models \varphi[e] \leftrightarrow \overline{e}(t) = 1$. QED (6)

(5) is, however, much more than an existence statement, since our proofs are *effective*: Clearly we can effectively assign to each Σ_0 formula φ a sequence $v(\varphi) = \langle v_1, \ldots, v_n \rangle$ of distinct variables containing all variables which occur free in φ. But the proof that the f defined by $(*)$ is rud in fact implicity defines a rud definition D_φ such that D_φ defines such an $f = f_{D_\varphi}$ over any rud closed $M = \langle M, \in, A \rangle$. The proof that f is A–good is by induction on the defining schemata and implicitly defines a term $t = T_\varphi$ which satisfies $(**)$ over any rud closed M. Thus our proofs implicitly describe an algorithm for the function $\varphi \mapsto T_\varphi$. Hence this function is recursive, hence uniformly $\Sigma_1(M)$. But then Σ_0 satisfaction can be defined over M by:

$$M \models \varphi[e] \leftrightarrow: \overline{e}(T_\varphi) = 1.$$

QED (Lemma 2.2.17)

Corollary 2.2.18. *Let $n \geq 1$. $\models_M^{\Sigma_n}$ is uniformly $\Sigma_n(M)$ for transitive* rud *closed structures $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$.*

(We leave this to the reader.)

2.2.1 Condensation

The *condensation lemma* for rud closed sets $U = \langle U, \in \rangle$ reads:

Lemma 2.2.19. *Let $U = \langle U, \in \rangle$ be transitive and* rud *closed. Let $X \prec_{\Sigma_1} U$. Then there is an isomorphism $\pi : \overline{U} \stackrel{\sim}{\longleftrightarrow} X$, where $\overline{U}$ is transitive and* rud *closed. Moreover, $\pi(f(\vec{x})) = f(\pi(\vec{x}))$ for all* rud *functions f.*

Proof: X satisfies the extensionality axiom. Hence by Moztowski's isomorphism theorem there is $\pi : \overline{U} \stackrel{\sim}{\longleftrightarrow} X$, where $\overline{U}$ is transitive. Now let f be rud and $x_1, \ldots, x_n \in \overline{U}$. Then there is $y' \in X$ such that $y' = f(\pi(\vec{x}))$, since $X \prec_{\Sigma_1} U$. Let $\pi(y) = y'$. Then $y = f(\vec{x})$, since the condition '$y = f(\vec{x})$' is Σ_0 and π is Σ_1–preserving. QED (Lemma 2.2.19)

The condensation lemma for rud closed $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ is much weaker, however. We state it for the case $n = 1$.

Lemma 2.2.20. *Let $M = \langle |M|, \in, A \rangle$ be transitive and* rud *closed. Let $X \prec_{\Sigma_0} M$. There is an isomorphism $\pi : \overline{M} \stackrel{\sim}{\longleftrightarrow} X$, where $\overline{M} = \langle |\overline{M}|, \in, \overline{A} \rangle$ is transitive and* rud *closed. Moreover:*

(a) $\pi(\overline{A} \cap x) = A \cap \pi(x)$

(b) *Let* f *be* rud *in* A. *Let* f *be characterized by:* $f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x}))$, *where* f_0, f_1 *are* rud. *Set:* $\overline{f}(\vec{x}) =: f_0(\vec{x}, \overline{A} \cap f_1(\vec{x}))$. *Then:*

$$\pi(\overline{f}(\vec{x})) = f(\pi(\vec{x})).$$

The proof is left to the reader.

2.3 The J_α hierarchy

We are now ready to introduce the alternative to Gödel's constructible hierarchy which we had promised in §1. We index it by ordinals from the class Lm of limit ordinals.

Definition 2.3.1.

$$\begin{aligned} J_\omega &= \mathrm{Rud}(\emptyset) \\ J_{\beta+\omega} &= \mathrm{Rud}(J_\beta) \text{ for } \beta \in \mathrm{Lm} \\ J_\lambda &= \bigcup_{\gamma<\lambda} J_\gamma \text{ for } \lambda \text{ a limit point of } \mathrm{Lm} \end{aligned}$$

It can be shown that $L = \bigcup\limits_\alpha J_\alpha$ and, indeed, that $L_\alpha = J_\alpha$ for a great many α (fr. ins. pr closed α). Note that $J_\omega = L_\omega = H_\omega$.

By §2 Corollary 2.2.14 we have:

$$\mathbb{P}(J_\alpha) \cap J_{\alpha+\omega} = \mathrm{Def}(J_\alpha),$$

which pinpoints the resemblance of the two hierarchies. However, we shall not dwell further on the relationship of the two hierarchies, since we intend to consequently employ the J–hierarchy in the rest of this book. As usual, we shall often abuse notation by not distinguishing between J_α and $\langle J_\alpha, \in \rangle$.

Lemma 2.3.1. $\mathrm{rn}(J_\alpha) = \mathrm{On} \cap J_\alpha = \alpha$.

Proof: By induction on $\alpha \in \mathrm{Lm}$. For $\alpha = \omega$ it is trivial. Now let $\alpha = \beta + \omega$, where $\beta \in \mathrm{Lm}$. Then $\beta = \mathrm{On} \cap J_\beta \in \mathrm{Def}(J_\beta) \subset J_\alpha$. Hence $\beta + n \in J_\alpha$ for $n < \omega$ by rud closure. But $\mathrm{rn}(J_\alpha) \leq \beta + \omega = \alpha$ since J_α is the rud closure of $J_\alpha \cup \{J_\alpha\}$. Hence $\mathrm{On} \cap J_\alpha = \alpha = \mathrm{rn}(J_\alpha)$.

If α is a limit point of Lm the conclusion is trivial. QED (Lemma 2.3.1)

To make our notation simpler, define

Definition 2.3.2. $\mathrm{Lm}^* =$ the limit points of Lm.

It is sometimes useful to break the passage from J_α to $J_{\alpha+\omega}$ into ω many steps. Any way of doing this will be rather arbitrary, but we can at least do it in a uniform way. As a preliminary, we use the basis theorem (§2 Theorem 2.2.15) to prove:

Lemma 2.3.2. *There is a* rud *function* $s : V \to V$ *such that for all* U*:*

(a) $U \subset s(U)$

(b) $\mathrm{rud}(U) = \bigcup_{n<\omega} s^n(U)$

(c) If U *is transitive, so is* $s(U)$*.*

Proof: Define rud functions $G_i (i = 0, 1, 2, 3)$ by:

$$G_0(x, y, z) = (x, y)$$
$$G_1(x, y, z) = (x, y, z)$$
$$G_2(x, y, z) = \{x, (y, z)\}$$
$$G_3(x, y, z) = x^* y$$

Set:

$$s(U) =: U \cup \bigcup_{i=0}^{9} F_i^U U^2 \cup \bigcup_{i=0}^{3} G_i^U U^3.$$

(a) is then immediate, (b) is immediate by the basis theorem. We prove (c).

Let $a \in s(U)$. We claim: $a \subset s(U)$. There are 14 cases: $a \in U$, $a = F_i(x, y)$ for an $i = 0, \ldots, 8$, where $x, y \in U$, and $a = G_i(x, y, z)$ where $x, y, z \in U$ and $i = 0, \ldots, 3$. Each of the cases is quite straightforward. We give some example cases:

- $a = F(x, y) = x \otimes y$. If $z \in a$, then $z = (x', y')$ where $x' \in x,\ y' \in y$. But then $x', y' \in U$ by transitivity and $z = G_0(x', y', x') \in s(U)$.
- $a = F_3(x, y) = \{(w, z, v) | z \in x \wedge (u, v) \in y\}$. If $a' = (w, z, v) \in a$, then $w, z, v \in U$ by transitivity and $a' = G_1(w, z, v) \in s(U)$.
- $a = F_8(x, y)$. If $a' \in a$, then $a' = x^* z$ where $z \in y$. Hence $z \in U$ by transitivity and $a' = G_3(x, z, z) \in s(U)$.
- $a = G_0(x, y, z) = \{\{x\}, \{x, y\}\}$. Then $a \subset F_0'' U^2 \subset s(U)$.
- $a = G_1(x, y, z) = (x, y, z) = \{\{x\}, \{x, (y, z)\}\}$. Then $\{x\} = F_0(x, x) \in s(U)$ and $\{x, (y, z)\} = G_2(x, y, z) \in s(U)$. QED (Lemma 2.3.2)

If we then set:

Definition 2.3.3. $S(U) = s(U \cup \{U\})$ we get:

Corollary 2.3.3. *S is a* rud *function such that*

(a) $U \cup \{U\} \subset S(U)$

(b) $\bigcup_{n<\omega} S^n(U) = \text{Rud}(U)$

(c) If U is transitive, so is $S(U)$.

We can then define:

Definition 2.3.4.

$$\begin{aligned} &S_0 = \emptyset \\ &S_{\nu+1} = S(S_\nu) \\ &S_\lambda = \bigcup_{\nu<\lambda} S_\nu \text{ for limit } \lambda. \end{aligned}$$

Obviously then: $J_\gamma = S_\gamma$ for $\gamma \in \text{Lm}$. (It would be tempting to simply define $J_\nu = S_\nu$ for all $\nu \in \text{On}$. We avoid this, however, since it could lead to confusion: At successors ν the models S_ν do not have very nice properties. Hence we retain the convention that whenever we write J_α we mean α to be a limit ordinal.)

Each J_α has Σ_1 knowledge of its own genesis:

Lemma 2.3.4. *$\langle S_\nu | \nu < \alpha \rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof: $y = S_\nu \leftrightarrow \bigvee f(\varphi(f) \wedge y = f(\nu))$, where $\varphi(f)$ is the Σ_0 formula:

$$\begin{aligned} &f \text{ is a function } \wedge \text{dom}(f) \in \text{On} \wedge f(0) = \emptyset \\ &\wedge \bigwedge \xi \in \text{dom}(f)(\xi + 1 \in \text{dom}(f) \to f(\xi + 1) = S(f(\xi))) \\ &\wedge \bigwedge \lambda \in \text{dom}(f|(\lambda \text{ is a limit } \to f(\lambda) = \bigcup f''\lambda). \end{aligned}$$

Thus it suffices to show that the existence quantifier can be restricted to J_α — i.e.

Claim $\langle S_\nu | \nu < \tau \rangle \in J_\alpha$ for $\tau < \alpha$.

Case 1 $\alpha = \omega$ is trivial.

Case 2 $\alpha = \beta + \omega$, $\beta \in \text{Lm}$.
Then $\langle S_\nu | \nu < \beta \rangle \in \text{Def}(J_\beta) \subset J_\alpha$. Hence $S_\beta = \bigcup_{\nu<\beta} S_\nu \in J_\alpha$. By rud closure it follows that $S_{\beta+n} \in J_\alpha$ for $n \subset w$. Hence $S \upharpoonright \nu \in J_\alpha$ for $\nu < \alpha$. QED (Case 2)

Case 3 $\alpha \in \mathrm{Lm}^*$.

This case is trivial since if $\nu < \beta \in \alpha \cap \mathrm{Lm}$. Then $S \upharpoonright \nu \in J_\beta \subset J_\alpha$.

QED (Lemma 2.3.4)

We now use our methods to show that each J_α has a uniformly $\Sigma_1(J_\alpha)$ well ordering. We first prove:

Lemma 2.3.5. *There is a* rud *function* $w : V \to V$ *such that whenever* r *is a well ordering of* u*, then* $w(u, r)$ *is a well ordering of* $s(u)$ *which end extends* r.

Proof: Let r_2 be the r–lexicographic ordering of u^2:

$$\langle x, y\rangle r_2 \langle z, w\rangle \leftrightarrow (xrz \vee (x = z \wedge yrw)).$$

Let r_3 be the r–lexicographic ordering of u^3. Set:

$$u_0 = u,\ u_{1+i} = F_i'' u^2 \text{ for } i = 0, \ldots, 8,\ u_{10+i} = G_i'' u^3 \text{ for } i = 0, \ldots, 3.$$

Define a well ordering w_i of u_i as follows: $w_0 = r$, For $i = 0, \ldots, 9$ set

$$\begin{aligned} & x w_{1+i} y \leftrightarrow \bigvee a, b \in u^2 (x = F_i(a) \wedge y = F_i(b) \wedge \\ & \wedge a r_2 b \wedge \bigwedge a' \in u^2 (a' r_2 a \to x \neq F_i(a')) \wedge \\ & \wedge \bigwedge b' \in u^2 (b' r_2 b \to y \neq F_i(b'))) \end{aligned}$$

For $i = 0, \ldots, 3$ let w_{10+i} have the same definitions with G_i in place of F_i and u^3, r_3 in place of u^2, r_2.

We then set:

$$\begin{aligned} w = w(u) = \{\langle x, y\rangle \in s(u)^2 | \bigvee_{i=0}^{13} ((x w_i y \wedge x, y \notin \bigcup_{h<i} u_n) \vee \\ \vee (x \in \textstyle\bigcup_{h<i} u_n \wedge y \notin \bigcup_{n<i} u_n))\} \end{aligned}$$

(where $\bigcup_{h<0} u_n = \emptyset$). QED (Lemma 2.3.5)

If r is a well ordering of u, then

$$r_u = \{\langle x, y\rangle | \langle x, y\rangle \in r \vee (x \in u \wedge y = u)\}$$

is a well ordering of $u \cup \{u\}$ which end extends r. Hence if we set:

Definition 2.3.5. $W(u, r) =: w(u \cup \{u\}, r_u)$.

We have:

Corollary 2.3.6. *W is a* rud *function such that whenever r is a well ordering of u, then $W(u,r)$ is a well ordering of $S(u)$ which end extends r.*

If we then set:

Definition 2.3.6.

$$<_{S_0} = \emptyset$$
$$<_{S_{\nu+1}} = W(S_\nu, <_{S_\nu})$$
$$<_{S_\lambda} = \bigcup_{\nu<\lambda} <_{S_\nu} \text{ for limit } \lambda,$$

it follows that $<_{S_\alpha}$ is a well ordering of S_α which end extends $<_{S_\nu}$ for all $\nu < \alpha$.

Definition 2.3.7. $<_\alpha = <_{J_\alpha} =: <_{S_\alpha}$ for $\alpha \in \text{Lm}$.

Then $<_\alpha$ is a well ordering of J_α for $\alpha \in \text{Lm}$.

By a close imitation of the proof of Lemma 2.3.4 we get:

Lemma 2.3.7. *$\langle <_{S_\nu} | \nu < \alpha\rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof:

$$y = <_{S_\nu} \leftrightarrow \bigvee f \bigvee g(\varphi(f) \wedge \psi(f,g) \wedge y = g(\nu))$$

where φ is as in the proof of Lemma 2.3.4 and ψ is the Σ_0 formula:

$$\begin{array}{l} g \text{ is a function } \wedge \text{dom}(g) = \text{dom}(f) \\ \wedge g(0 = \emptyset \wedge \bigwedge \xi \in \text{dom}(g) | \xi + 1 \in \text{dom}(g) \to \\ \to g(\xi+1) = W(f(\xi), g(\xi))) \\ \wedge \bigwedge \lambda \in \text{dom}(g)\ (\lambda \text{ is a limit} \to g(\lambda) = \bigcup g''\lambda). \end{array}$$

Just as before, we show that the existence quantifiers can be restricted to J_α. QED (Lemma 2.3.7)

But then:

Corollary 2.3.8. *$<_\alpha = \bigcup_{\nu<\alpha} <_{S_\nu}$ is a well ordering of J_α which is uniformly $\Sigma_1(J_\alpha)$. Moreover $<_\alpha$ end extends $<_\nu$ for $\nu \in \text{Lm}$, $\nu < \alpha$.*

Corollary 2.3.9. *u_α is uniformly $\Sigma_1(J_\alpha)$, where $u_\alpha(x) \simeq \{z | z <_\alpha x\}$.*

Proof:

$$y = u_\alpha(x) \leftrightarrow \bigvee \nu(x \in S_\nu \wedge y = \{z \in S_\nu | z <_{S_\nu} x\})$$

QED (Corollary 2.3.9)

Note We shall often write $<_{J_\alpha}$ for $<_\alpha$. We also write $<_\infty$ or $<_J$ or $<_L$ for $\bigcup_{\alpha\in\mathrm{On}} <_\alpha$. Then $<_L$ well orders L and is an end extension of $<_\alpha$.

We obtain a particularly strong form of Gödel's condensation lemma:

Lemma 2.3.10. *Let $X \prec_{\Sigma_1} J_\alpha$. Then there are $\overline{\alpha}, \pi$ such that $\pi : J_{\overline{\alpha}} \overset{\sim}{\longleftrightarrow} X$.*

Proof: By §2 Lemma 2.2.19 there is rud closed U such that U is transitive and $\pi :\overset{\sim}{\longleftrightarrow} X$. Note that the condition

$$S(f,\nu) \leftrightarrow: f = \langle S_\xi | \nu < \xi\rangle$$

is Σ_0, since:

$$\begin{aligned} S(f,\nu) \leftrightarrow &\ (f \text{ is a function } \wedge \\ &\wedge \mathrm{dom}(f) = \nu \wedge f(0) = \emptyset \text{ if } 0 < \nu \wedge \\ &\bigwedge \xi \in \mathrm{dom}(f)(\xi+1 \in \mathrm{dom}(f) \to \\ &\to f(\xi+1) = S(f(\xi)))). \end{aligned}$$

Let $\overline{\alpha} = \mathrm{On} \cap U$ and let $\overline{\nu} < \overline{\alpha}$. Let $\pi(\overline{\nu}) = \nu$. Then $f = \langle S_\xi | \xi < \nu\rangle \in X$ since $X \prec_{\Sigma_1} J_\alpha$. Let $\pi(\overline{f}) = f$. Then $\overline{f} = \langle S_\xi | \xi < \overline{\nu}\rangle$, since $S(\overline{f}, \overline{\nu})$. But then $J_{\overline{\alpha}} = \bigcup_{\xi < \overline{\alpha}} S_\xi \subset U$. But since π is Σ_1 preserving we know that

$$\begin{aligned} x \in U \to &\bigvee f, \nu \in U (S(f,\nu) \wedge x \in U f''\nu) \\ &\to x \in J_{\overline{\alpha}}. \end{aligned}$$

QED (Lemma 2.3.10)

Corollary 2.3.11. *Let $\pi : J_{\overline{\alpha}} : J_{\overline{\alpha}} \to_{\Sigma_1} J_\alpha$. Then:*

(a) $\nu < \tau \leftrightarrow \pi(\nu) < \pi(\tau)$ for $\nu, \tau < \overline{\alpha}$.

(b) $x <_L y \leftrightarrow \pi(x) <_L \pi(y)$ for $x, y \in J_{\overline{\alpha}}$.
Hence:

(c) $\nu \leq \pi(\nu)$ for $\nu < \overline{\alpha}$.

(d) $x \leq_L \pi(x)$ for $x \in J_{\overline{\alpha}}$.

Proof: (a), (b) follow by the fact that $< \cap J_\alpha^2$ and $<_L \cap J_\alpha^2 = <_\alpha$ are uniformly $\Sigma_1(J_\alpha)$. But if $\pi(\nu) < \nu$, then $\nu, \pi(\nu), \pi^2(\nu), \ldots$ would form an infinite decreasing sequence by (a). Hence (c) holds. Similarly for (d). QED (Corollary 2.3.11)

2.3.1 The J_α^A–hierarchy

Given classes $A_1, \ldots, A_n$ on can generalize the previous construction by forming the *constructible hierarchy* $\langle J_\alpha^{A_1,\ldots,A_n} | \alpha \in \Gamma\rangle$ *relativized to* $A_1, \ldots, A_n$. We have this far dealt only with the case $n = 0$. We now develop the case $n = 1$, since the generalization to $n > 1$ is then entirely straightforward. (Moreover the case $n = 1$ is sufficient for most applications.)

Definition 2.3.8. Let $A \subset V$. $\langle J_\alpha^A | \alpha \in \mathrm{Lm}\rangle$ is defined by:

$$\begin{aligned} J_\alpha^A &= \langle J_\alpha[A], \in, A \cap J_\alpha[A]\rangle \\ J_\omega[A] &= \mathrm{Rud}_A(\emptyset) = H_\omega \\ J_{\beta+\omega}[A] &= \mathrm{Rud}_A(J_\beta) \text{ for } \beta \in \mathrm{Lm} \\ J_\lambda[A] &= \bigcup_{\nu<\lambda} J_\nu[A] \text{ for } \lambda \in \mathrm{Lm}^* \end{aligned}$$

Note $A \cap J_\alpha[A]$ is treated as an unary predicate.

Thus every J_α^A is rud closed. We set

Definition 2.3.9.

$$\begin{aligned} L[A] = J[A] &= \bigcup_{\alpha\in\mathrm{On}} J_\alpha[A]; \\ L^A = J^A &= \langle L[A], \in, A \cap L[A]\rangle. \end{aligned}$$

Note that $J_\alpha[\emptyset] = J_\alpha$ for all $\alpha \in \mathrm{Lm}$.

Repeating the proof of Lemma 1.1.1 we get:

Lemma 2.3.12. $\mathrm{rn}(J_\alpha^A) = \mathrm{On} \cap J_\alpha^A = \alpha$.

We wish to break $J_{\alpha+\omega}^A$ into ω smaller steps, as we did with $J_{\alpha+\omega}$. To this end we define:

Definition 2.3.10. $S^A(u) = S(u) \cup \{A \cap u\}$.

Corresponding to Corollary 2.3.3 we get:

Lemma 2.3.13. S^A *is a function* rud *in* A *such that whenever* u *is transitive, then:*

(a) $u \cup \{u\} \cup \{A \cap u\} \subset S(u)$

(b) $\bigcup_{n<\omega} (S^A)^n(u) = \text{Rud}_A(u)$

(c) $S(u)$ is transitive.

Proof: (a) is immediate. (c) holds, since $S(u)$ is transitive, $a \subset S(u)$ and $A \cap u \subset u$. (b) holds since $S(u) \supset u$ is transitive and $A \cap u \subset u$. But if we set: $U = \underset{n<\omega}{\omega} (S^A)^n(u)$, then U is rud closed and $\langle U, A \cap U \rangle$ is amenable.
QED (Lemma 2.3.13)

We then set:

Definition 2.3.11.

$$S_0^A = \emptyset$$
$$S_{\alpha+1}^A = S^A(S_\alpha^A)$$
$$S_\lambda^A = \bigcup_{\nu<\lambda} S_\nu^A \text{ for limit } \lambda.$$

We again have: $J_\alpha[A] = S_\alpha^A$ for $\alpha \in \text{Lm}$. A close imitation of the proof of Lemma 2.3.4 gives:

Lemma 2.3.14. *$\langle S_\nu^A | r < \alpha \rangle$ is uniformly $\Sigma_1(J_\alpha^A)$.*

Proof: This is exactly as before except that in the formula $\varphi(f)$ we replace $S(f(\nu))$ by $S^A(f(\nu))$. But this is $\Sigma_0(J_\alpha^A)$, since:

$$x \in S^A(u) \leftrightarrow (x \in S(u) \vee x = A \cap u),$$

hence:

$$\begin{aligned} y = & S^A(u) \leftrightarrow \bigwedge z \in y\ z \in S^A(u) \\ & \wedge \bigwedge z \in S(u) z \in y \wedge \bigvee z \in y\ z = A \cap u. \end{aligned}$$

QED (Lemma 2.3.14)

We now show that J_α^A has a uniformly $\Sigma_1(J_\alpha^A)$ well ordering, which we call $<_\alpha^A$ or $<_{J_\alpha^A}$.

Set:

Definition 2.3.12.

$$\begin{aligned} W^A(u,r) = & \{\langle x,y \rangle | \langle x,y \rangle \in W(u,r) \vee \\ & (x \in S(u) \wedge y = A \cap u \notin S(u)\} \end{aligned}$$

If u is transitive and r well orders u, then $W^A(u,r)$ is a well ordering of $S^A(u)$ which end extends r.

We set:

Definition 2.3.13.

$$<_0^A = \emptyset$$
$$<_{\nu+1}^A = W^A(S_\nu^A, <_\nu^A)$$
$$<_\lambda^A = \bigcup_{\nu<\lambda} <_\nu^A \text{ for limit } < .$$

Then $<_\nu^A$ is a well ordering of S_ν^A which end extends $<_\xi^A$ for $\xi < \nu$. In particular $<_\alpha^A$ well orders J_α^A for $\alpha \in \Gamma$. We also write: $<_{J_\alpha^A} =: <_\alpha^A$. We set: $<_{L^A} = <_{J^A} = <_\infty^A =: \bigcup_{\nu<\infty} <_\nu^A$.

Just as before we get:

Lemma 2.3.15. *$\langle <_\nu^A | \nu < \alpha \rangle$ is uniformly $\Sigma_1(J_\alpha^A)$.*

The proof is left to the reader. Just as before we get:

Lemma 2.3.16. *$<_\alpha^A$ and $f(u) = \{z | z <_\alpha^A u\}$ are uniformly $\Sigma_1(J_\alpha^A)$.*

Up until now almost everything we proved for the J_α hierarchy could be shown to hold for the J_α^A hierarchy. The condensation lemma, however, is available only in a much weaker form:

Lemma 2.3.17. *Let $X \prec_{\Sigma_1} J_\alpha^A$. Then there are $\overline{\alpha}, \pi, \overline{A}$ such that $\pi : J_{\overline{\alpha}}^{\overline{A}} \overset{\sim}{\longleftrightarrow} X$.*

Proof: By Lemma 2.2.19 there is $\langle \overline{U}, \overline{A} \rangle$ such that $\pi : \langle \overline{U}, \overline{A} \rangle \overset{\sim}{\longleftrightarrow} X$ and $\langle \overline{U}, \overline{A} \rangle$ is rud closed. As before, the condition

$$S^A(f, \nu) \leftrightarrow f = \langle S_\xi^A | \nu < \xi \rangle$$

si Σ_0 in A. Now let $\overline{\nu} < \overline{\alpha}, \pi(\overline{\nu}) = \nu$. As before $f = \langle S_\xi | \xi < \nu \rangle \in X$. Let $\pi(\overline{f} = f$. Then $\overline{f} = \langle S_\xi^A | \xi < \overline{\nu} \rangle$, since $S^{\overline{A}}(\overline{f}, \overline{\nu})$. Then $J_{\overline{\alpha}}^{\overline{A}} \subset \bigcup_{\xi<\overline{\alpha}} S_\xi^{\overline{A}} \subset \overline{U}$. $U \subset J_{\overline{\alpha}}^{\overline{A}}$ then follows as before. QED (Lemma 2.3.17)

A sometimes useful feature of the J_α^A hierarchy is:

Lemma 2.3.18. *$x \in J_\alpha^A \rightarrow TC(x) \in J_\alpha^A$.*

(Hence $\langle TC(c) | x \in J_\alpha^A \rangle$ is $\Pi_1(J_\alpha^A)$ since $u = TC(x)$ is defined by:

$$u \textit{ is transitive } \wedge x \subset u \wedge \bigwedge v((v \textit{ is transitive } \wedge x \subset v) \rightarrow u \subset v)$$

Proof: By induction on α.

Case 1 $\alpha = \omega$ (trivial)

Case 2 $\alpha = \beta + \omega,\ \beta \in \text{Lim}$.
Then every $x \in J_\alpha^A$ has the form $f(\vec{z})$ where $z_1, \ldots, z_n \in J_\beta[A] \cup \{J_\beta[A]\}$ and f is rud in A. By Lemma 2.2.2 we have

$$\bigcup{}^p X \subset \bigcup_{i=1}^n TC(z_i) \subset J_\beta[A].$$

Hence $TC(x) = C_p(x) \cup TC(\bigcup_{i=1}^n TC(z_i)$, where $\langle TC(z) | z \in J_\beta[A]\rangle$ is J_β^A–definable, hence an element of J_α^A.

Case 3 $\alpha \in \text{Lm}^*$ (trivial). QED (Lemma 2.3.18)

Corollary 2.3.19. *If $\alpha \in \text{Lm}^*$, then $\langle TC(x) | x \in J_\alpha^A\rangle$ is uniformly $\Delta_1(J_\alpha^A)$.*

Proof: We have seen that it is $\Pi_1(J_\alpha^A)$. But $TC \upharpoonright J_\alpha^A \in J_\alpha^A$ for all $\beta \in \text{Lm} \cap \alpha$. Hence $u = TC(x)$ is definable in J_α^A by:

$$\begin{aligned}&\bigvee f (f \text{ is a function } \wedge \text{dom}(f) \text{ is transitive } \wedge u = f(x)\\ &\quad \wedge \bigwedge x \in \text{dom}(f) f(x) = x \cup \bigcup f^n x)\end{aligned}$$

QED (Corollary 2.3.19)

2.4 J–models

We can add further unary predicates to the structure $J_\alpha^{\vec{A}}$. We call the structure:

$$M = \langle J_\alpha^{A_1, \ldots, A_n}, B_1, \ldots, B_n\rangle$$

a *J–model* if it is amenable in the sense that $x \cap B_i \in J_\alpha^{\vec{A}}$ whenever $x \in J_\alpha^{\vec{A}}$ and $i = 1, \ldots, m$. The B_i are again taken as unary predicates. The *type* of M is $\langle n, m\rangle$. (Thus e.g. J_α has type $\langle 0, 0\rangle$, J_α^A has type $\langle 1, 0\rangle$, and $\langle J_\alpha, B\rangle$ has type $\langle 0, 1\rangle$.) By an abuse of notation we shall often fail to distinguish between M and the associated structure:

$$\hat{M} = \langle J_\alpha[\vec{A}], A'_1, \ldots, A'_n, B_1, \ldots, B_m\rangle$$

where $A'_i = A_i \cap J_\alpha[\vec{A}]$ $(i = 1, \ldots, n)$.
We may for instance write $\Sigma_1(M)$ for $\Sigma_1(\hat{M})$ or $\pi : N \to_{\Sigma_n} M$ for $\pi : \hat{N} \to_{\Sigma_n} \hat{M}$. (However, we cannot unambignously identify M with $\hat{M}$, since e.g. for $M = \langle J_\alpha^A, B\rangle$ we might have: $\hat{M} = J_\alpha^{A,B}$.)

In practice we shall usually deal with J models of type $\langle 1,1\rangle$, $\langle 1,0\rangle$, or $\langle 0,0\rangle$. In any case, following the precedent in earlier section, when we prove general theorem about J–models, we shall often display only the proof for type $\langle 1,1\rangle$ or $\langle 1,0\rangle$, since the general case is then straightforward.

Definition 2.4.1. If $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$ is a J–model and $\beta \leq \alpha$ in Lm, we set:

$$M|\beta =: \langle J_\beta^{\vec{A}}, B_1 \cap J_\alpha^{\vec{A}}, \dots, B_n \cap J_\alpha^{\vec{A}}\rangle.$$

In this section we consider $\Sigma_1(M)$ definability over an arbitrary $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$. If the context permits, we write simply Σ_1 instead of $\Sigma_1(M)$. We first list some properties which follow by rud closure alone:

- $\models_M^{\Sigma_1}$ is uniformly Σ_1, by Corollary 2.2.18 (**Note** 'Uniformly' here means that the Σ_1 definition is the same for any two M having the same type.)
- If $R(y, x_1, \dots, x_n)$ is a Σ_1 relation, then so is $\bigvee y R(y, x_1, \dots, x_n)$ (since $\bigvee y \bigvee z P(yz, \vec{x}) \leftrightarrow \bigvee u \bigvee y, z \in u P(y, z, \vec{x})$ where $R(y, \vec{x}) \leftrightarrow \bigvee z P(y, z, \vec{x})$ and P is Σ_0).

  By an n–ary $\Sigma_1(M)$ *function* we mean a partial function on M^n which is $\Sigma_1(M)$ as an $n+1$–ary relation.
- If R, R' are n–ary Σ_1 relations, then so are $R \cap R'$, $R \cup R'$. (Since e.g.

$$(\bigvee y P(y, \vec{x}) \wedge \bigvee P'(y, \vec{x})) \leftrightarrow \bigvee y y' (P(y, \vec{x}) \wedge P'(y', \vec{x})).)$$

- If $R(y_1, \dots, y_m)$ is an n–ary Σ_1 relation and $f_i(\vec{x})$ is an n–ary Σ_1 function for $i = 1, \dots, m$, then so is the n–ary relation

$$R(\vec{f}(\vec{x})) \leftrightarrow: \bigvee y_1, \dots, y_m (\bigwedge_{i=1}^{m} y_i = f_i(\vec{x}) \wedge R(\vec{y})).$$

- If $g(y_1, \dots, y_m)$ is an m–ary Σ_1 function and $f_i(\vec{x})$ is an n–ary Σ_1 function for then $h(\vec{x}) \simeq g(\vec{f}(\vec{x}))$ is an n–ary Σ_1 function. (Since $z = h(\vec{x}) \leftrightarrow \bigvee_{y_1, \dots, y_m} (\bigwedge_{i=1}^{m} y_i = f_i(\vec{x}) \wedge z = g(\vec{y})).$)

  Since $f(x_1, \dots, x_n) = x_i$ is Σ_1 function, we have:
- If $R(x_1, \dots, x_n)$ is Σ_1 and $\sigma : n \to m$, then

$$P(z_1, \dots, z_m) \leftrightarrow: R(z_{\sigma(1)}, \dots, z_{\sigma(n)})$$

  is Σ_1.

- If $f(x_1, \ldots, x_n)$ is a Σ_1 function and $\sigma : n \to m$, then the function:

$$g(z_1, \ldots, z_m) \simeq: f(z_{\sigma(1)}, \ldots, z_{\sigma n})$$

in Σ_1.

J–models have the further property that every binary Σ_1 relation is uniformizable by a Σ_1 function. We define

Definition 2.4.2. A relation $R(y, \vec{x})$ is *uniformized* by the function $F(\vec{x})$ iff the following hold:

- $\bigvee y R(y, \vec{x}) \to F(\vec{x})$ is defined
- If $F(\vec{x})$ is defined, then $R(F(\vec{x}), \vec{x})$

We shall, in fact, prove that M has a uniformly Σ_1 definable *Skolem function*. We define:

Definition 2.4.3. $h(i, x)$ is a Σ_1–*Solem function* for M iff h is a $\Sigma_1(M)$ partial map from $\omega \times M$ to M and, whenever $R(y, x)$ is a $\Sigma_1(M)$ relation, there is $i < \omega$ such that h_i uniformizes R, where $h_i(x) \simeq h(i, x)$.

Lemma 2.4.1. *M has a Σ_1–Skolem function which is uniformly $\Sigma_1(M)$.*

Proof: $\models_M^{\Sigma_1}$ is uniformly Σ_1. Let $\langle \varphi_i | i < \omega \rangle$ be a recursive enumeration of the Σ_1 formulae in which at most the two variables v_0, v_1 occur free. Then the relation:

$$T(i, y, z) \leftrightarrow: \models_M^{\Sigma_1} \varphi_i[y, x]$$

is uniformly Σ_1. But then for any Σ_1 relation R there is $i < \omega$ such that

$$R(y, x) \leftrightarrow T(i, y, x).$$

Since T is Σ_1, it has the form:

$$\bigvee z T'(z, i, y, x)$$

where T' is Σ_0. Writing $<_M$ for $<_\alpha^{\vec{A}}$, we define:

$$y = h(i, x) \leftrightarrow \bigvee z (\langle z, y \rangle \text{ is the } <_M \text{–least}$$
$$\text{pair } \langle z', y' \rangle \text{ such that } T'(z', i, y', x).$$

Recalling that the function $f(x) = \{z | z <_M x\}$ is Σ_1, we have:

$$\begin{aligned} y =& h(i, x) \leftrightarrow \bigvee z \bigvee u (T'(z, i, y, x) \wedge \\ & \wedge u = \{w | w <_n \langle z, y \rangle\} \wedge \\ & \wedge \bigwedge \langle z', y' \rangle \in u \neg T'(z, i, y, x)) \end{aligned}$$

QED 2.4.1

We call the function h defined above the *canonical* Σ_1 *Skolem function* for M and denote it by h_M. The existency of h implies that every $\Sigma_1(M)$ relation is uniformizable by a $\Sigma_1(M)$ function:

Corollary 2.4.2. *Let $R(y, x_1, \dots, x_n)$ be Σ_1. R is uniformizable by a Σ_1 function.*

Proof: Let h_i uniformize the binary relation

$$\{\langle y, z\rangle | \bigvee x_1 \dots x_n (R(y, \vec{x}) \wedge z = \langle x_1, \dots, x_n\rangle)\}.$$

Then $f(\vec{x}) \simeq: h_i(\langle\vec{x}\rangle)$ uniformizes R. QED

We say that a $\Sigma_1(M)$ function has a *functionally absolute* definition if it has a Σ_1 definition which defines a function over every J–model of the same type.

Corollary 2.4.3. *Every $\Sigma_1(M)$ function g has functionally absolute definition.*

Proof: Apply the construction in Corollary 2.4.2 to $R(y, \vec{x}) \leftrightarrow y = g(\vec{x})$. Then $f(x) \simeq: h_i(\langle\vec{x}\rangle)$ is functionally absolute since h_i is.

QED (Corollary 2.4.2)

Lemma 2.4.4. *Every $x \in M$ is $\Sigma_1(M)$ in parameters from* $\mathrm{On} \cap M$.

Proof: We must show: $x = f(\xi_1, \dots, \xi_n)$ where f is $\Sigma_1(M)$. If $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$, it obviously suffices to show it for the model $M' = J_\alpha^{\vec{A}}$. For the sake of simplicity we display the proof for J_α^A. (i.e. M has type $\langle 1, 0\rangle$). We proceed by induction on $\alpha \in \Gamma$.

Case 1 $\alpha = \omega$.

Then $J_\alpha = \mathrm{Rud}(\emptyset)$ and $x = f(\{0\})$ where f is rudimentary.

Case 2 $\alpha = \beta + \omega$, $\beta \in \mathrm{Lm}$.

Then $x = f(z_1, \dots, z_n, J_\beta^A)$ where $z_1, \dots, z_n \in J_\beta^A$ and f is rud in A. (This is meant to include the case: $n = 0$ and $x = f(J_\beta^A)$.) By the induction hypothesis there are $\vec{\xi} \in \beta$ such that $z_i = g_i(\vec{\xi})$ $(i = 1, \dots, n)$ and g_i is $\Sigma_1(J_\beta^A)$. For each i pick a functionally absolute Σ_1 definition for g_i and let g_i' be $\Sigma_1(J_\alpha^A)$ by the same definition. Then $z_i = g_i'(\vec{\xi})$ since the condition is Σ_1. Hence $x = f'(\vec{\xi}, \beta) = f(\vec{g}'(\xi, J_\beta^A)$ where f' is Σ_1. QED (Case 2)

Case 3 $\alpha \in \text{Lm}^*$.

Then $x \in J^A_\beta$ for a $\beta < \alpha$. Hence $x = f(\vec{\xi})$ where f is $\Sigma_1(J^A_\beta)$. Pick a functionally absolute Σ_1 definition of f and let f' be $\Sigma_1(J^A_\alpha)$ by the same definition. Then $x = f'(\vec{\xi})$. QED (Lemma 2.4.4)

But being Σ_1 in parameters from $\text{On} \cap M$ is the same as being Σ_1 in a finite subset of $\text{On} \cap M$:

Lemma 2.4.5. *Let $x = f(\vec{\xi})$ where f is $\Sigma_1(M)$. Let $a \subset \text{On} \cap M$ be finite such that $\xi_1, \ldots, \xi_n \in a$. Then $x = g(a)$ for a $\Sigma_1(M)$ function g.*

Proof: Set:

$$k_i(a) = \begin{cases} \text{the } i\text{–th element of } a \text{ in order} \\ \text{of size if } a \subset \text{On is finite} \\ \text{and card}(a) > i, \\ \text{undefined if not.} \end{cases}$$

Then k_i is $\Sigma_1(M)$ since:

$$y = k_i(a) \leftrightarrow \bigvee f \bigvee n < \omega(f : n \leftrightarrow a \wedge \bigwedge i, j < n(f(i) < f(j) \leftrightarrow i < j) \\ \wedge a \subset \text{On} \wedge y = f(i))$$

Thus $x = f(k_{i_1}(a), \ldots, k_{i_n}(a))$ where $\xi_l = k_{i_l}(a)$ for $l = 1, \ldots, n$.
QED (Lemma 2.4.5)

We now show that for every J–model M there is a $\underline{\Sigma}_1(M)$ partial map of $\text{On} \cap M$ onto M. As a preliminary we prove:

Lemma 2.4.6. *There is a partial $\underline{\Sigma}_1(M)$ map of* $\text{On} \cap M$ *onto* $(\text{On} \cap M)^2$.

Proof: Order the class of pairs On^2 by setting: $\langle \alpha, \beta \rangle <^* \langle \gamma, \delta \rangle$ iff $\langle \max(\alpha, \beta), \alpha, \beta \rangle$ is lexicographically less than $\langle \max(\gamma, \delta), \gamma, \delta \rangle$. This ordering has the property that the collection of predecessors of any pair form a set. Hence there is a function $p : \text{On} \to \text{On}^2$ which enumerates the pairs in order $<^*$.

Claim 1 $p \restriction \text{On}_M$ is $\Sigma_1(M)$.

Proof: If $M = \langle J^{\vec{A}}_\alpha, \vec{B} \rangle$, it suffices to prove it for $J^{\vec{A}}_\alpha$. To simplify notation, we assume: $M = J^A_\alpha$ for an $A \subset M$ (i.e. M is of type $\langle 1, 0 \rangle$.)

We know:

$$y = p(\nu) \leftrightarrow \bigvee f(\varphi(f) \wedge y = f(\nu))$$

where φ is the Σ_0 formula:

$$\begin{aligned}
& f \text{ is a function } \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge \\
& \wedge \bigwedge u \in \mathrm{rng}(f) \bigvee \beta, \gamma \in C_n(u) u = \langle \beta, \gamma \rangle \wedge \\
& \wedge \bigwedge \nu, \tau \in \mathrm{dom}(f)(\nu < \tau \leftrightarrow f(\nu) <^* f(\tau)) \\
& \wedge \bigwedge u \in \mathrm{rng}(f) \bigwedge \mu, \xi \leq \max(u)(\langle \mu, \xi \rangle <^* u \to \langle \mu, \xi \rangle \in \mathrm{rng}(f)).
\end{aligned}$$

Thus it suffices to show that the existence quantifier can be restricted to J^A_α — i.e. that $p \upharpoonright \xi \in J^A_\alpha$ for $\xi < \alpha$. This follows by induction on α in the usual way (cf. the proof of Lemma 2.3.14). QED (Claim 1)

We now proceed by induction on $\alpha = \mathrm{On}_M$, considering three cases:

Case 1 $p(\alpha) = \langle 0, \alpha \rangle$.
Then $p \upharpoonright \alpha$ maps α onto

$$\{u | u <_* \langle 0, \alpha \rangle\} = \alpha^2$$

and we are done, since $p \upharpoonright \alpha$ is $\Sigma_1(J^A_\alpha)$. (Note that ω satisfies Case 1.)

Case 2 $\alpha = \beta + \omega, \beta \in \mathrm{Lm}$ and Case 1 fails.
There is a $\Sigma_1(J^A_\alpha)$ bijection of β onto α defined by:

$$\begin{aligned}
& f(2n) = \beta + n \text{ for } n < \omega \\
& f(2n+1) = n \text{ for } n < \omega \\
& f(\nu) = \nu \text{ for } \omega \leq \nu < \beta
\end{aligned}$$

Let g be a $\underline{\Sigma}_1(J^A_\beta)$ partial map of β onto β^2. Set $(\langle \gamma_0, \gamma_1 \rangle)_i = \gamma_i$ for $i = 0, 1$.

$$g_i(\nu) \simeq (g(\nu))_i (i = 0, 1).$$

Then $f(\nu) \simeq \langle f g_0(\nu, f g_1(\nu) \rangle$ maps β onto α^2. QED (Case 2)

Case 3 The above cases fail.
Then $p(\alpha) = \langle \nu, \tau \rangle$, where $\nu, \tau < \alpha$. Let $\gamma \in \mathrm{Lm}$ such that $\max(\nu, \tau) < \gamma < \alpha$. Let g be a partial $\underline{\Sigma}_1(J^A_\alpha)$ map of γ onto γ^2. Then $g \in M, p^{-1}$ is a partial map of γ^2 onto α; hence $f = p^{-1} \circ g$ is a partial map of γ onto α. Set: $\tilde{f}(\langle \xi, \delta \rangle) \simeq \langle f(\xi), f(\delta) \rangle$ for ξ, δ, γ. Then $\tilde{f} g$ is a partial map of γ onto α^2. QED (Lemma 2.4.6)

We can now prove:

Lemma 2.4.7. *There is a partial $\underline{\Sigma}_1(M)$ map of On_M onto M.*

Proof: We again simplify things by taking $M = J^A_\alpha$. Let g be a partial map of α onto α^2 which is $\Sigma_1(J^A_\alpha)$ in the parameters $p \in J^A_\alpha$. Define "ordered pairs" of ordinals $< \alpha$ by:

$$(\nu, \tau) =: g^{-1}(\langle \nu, \tau \rangle).$$

We can then, for each $n \geq 1$, define "ordered n–tuples" by:

$$(\nu) =: \nu, (\nu_1, \ldots, \nu_n) = (\nu_1, (\nu_2, \ldots, \nu_n))(n \geq 2).$$

We know by Lemma 2.4.4 that every $y \in J^A_\alpha$ has the form: $y = f(\nu_1, \ldots, \nu_n)$ where $\nu_1, \ldots, \nu_n < \alpha$ and f is $\Sigma_1(J^A_\alpha)$. Define a function f^* by:

$$y = f^*(\tau) \leftrightarrow \bigvee \nu_1, \ldots, \nu_n(\tau = (\nu_1, \ldots, \nu_n) \wedge \\ \wedge y = f(\nu_1, \ldots, \nu_n)).$$

Then f^* is $\Sigma_1(J^A_\alpha)$ in p and $y \in f^{*\prime\prime}\alpha$. If we set: $h^*(i, x) \simeq h(i, \langle x, p \rangle)$, then each binary relation which is $\Sigma_1(J^A_\alpha)$ in p is uniformized by one of the functions $h^*_i(x) \simeq h^*(i, x)$. Hence $y = h^*(i, \gamma)$ for some $\gamma < \alpha$. Hence $J^A_\alpha = h^{*\prime\prime}(\omega \times \alpha)$. But, setting:

$$y = \hat{h}(\mu) \leftrightarrow \bigvee i, \nu(\mu = (i, \nu) \wedge y = h^*(i, \nu))$$

we see that $\hat{h}$ is $\Sigma_1(J^A_\alpha)$ in p and $y \in \hat{h}''\alpha$. Hence $J^A_\alpha = \hat{h}''\alpha$, where $\hat{h}$ is $\Sigma_1(J^A_\alpha)$ in p. QED (Lemma 2.4.7)

Corollary 2.4.8. *Let $x \in M$. There are $f, \gamma \in J^A_\alpha$ such that f maps γ onto x.*

Proof: We again prove it for $M = J^A_\alpha$. If $\alpha = \omega$ it is trivial since $J^A_\alpha = H_\omega$. If $\alpha \in \mathrm{Lm}^*$ then $x \in J^A_\beta$ for a $\beta < \alpha$ and there is $f \in J^A_\alpha$ mapping β onto J^A_β by Lemma 2.4.7. There remains only the case $\alpha = \beta + \omega$ where β is a limit ordinal. By induction on $n < \omega$ we prove:

Claim There is $f \in J^A_\alpha$ mapping β onto $S^A_{\beta+n}$. If $n = 0$ this follows by Lemma 2.4.7.

Now let $n = m + 1$.
Let $f : \beta \overset{\text{onto}}{\longrightarrow} S^A_{\beta+m}$ and define f' by $f'(0) = S^A_{\beta+m}, f'(n+1) = f(n)$ for $n < \omega$, $f'(\xi) = f(\xi)$ for $\xi \geq \omega$. Then f' maps β onto $U = S^A_{\beta+m} \cup \{S^A_{\beta+m}\}$ and $S^A_{\beta+m} = \bigcup\limits_{\delta=\beta}^{8} F_i''U^2 \cup \bigcup\limits_{i=0}^{3} G_i''U^3 \cup \{A \cap S^A_{\beta+m}\}$.

Set:

$g_i = \{\langle F_i(f'(\xi), f'(\zeta)), \langle i, \langle \xi, \zeta \rangle\rangle\rangle | \xi, \zeta < \beta\}$
for $i = 0, \ldots, 8$
$g_{8+i+1} = \{\langle G_i | f'(\xi), f'(\zeta), f'(\mu)), \langle 8 + i + 1, \langle \xi, \zeta, \mu \rangle\rangle | \xi, \zeta, \mu < \beta\}$
for $i = 0, \ldots, 3$
$g_{13} = \{\langle A \cap S^A_{\beta+m} \langle 13, \emptyset \rangle\rangle\}$

Then $g = \bigcup_{i=0}^{13} g_i \in J_\alpha^A$ is a partial map of J_β^A onto $S_{\beta+n}^A$ and $gh \in J_\alpha^A$ is a partial map of β onto S_β^A. QED (Corollary 2.4.8)

Define the *cardinal* of x in M by:

Definition 2.4.4. $\overline{\overline{x}} = \overline{\overline{x}}^M =$: the least γ such that some $f \in M$ maps γ onto x.

(**Note** this is a non standard definition of cardinal numbers. If M is e.g. *pr* closed, we get that there is $f \in M$ bijecting $\overline{\overline{x}}$ onto x.)

Definition 2.4.5. Let $X \subset M$. $h(X) = h_M(X) =$: The set of all $y \in M$ such that $y = f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in X$ and f is a $\Sigma_1(M)$ function

Since $\Sigma_1(M)$ functions are closed under composition, it follows easily that $Y = h(X)$ is closed under $\Sigma_1(M)$ functions.

By Corollary **??** we then have:

Lemma 2.4.9. *Let $Y = h(X)$. Then $M|Y \prec_{\Sigma_1} M$ where*

$$M|Y =: \langle Y, A_1 \cap Y, \ldots, A_n \cap Y, B_1 \cap Y, \ldots, B_m \cap Y \rangle.$$

(**Note** We shall often ignore the distinction between Y and $M|Y$, writing simply: $Y \prec_{\Sigma_1} M$.)

If f is a $\Sigma_1(M)$ function, there is $i < \omega$ such that $h(i, \langle \vec{x} \rangle) \simeq f(\vec{x})$. Hence:

Corollary 2.4.10. $h(X) = \bigcup_{n<\omega} h''(\omega \times X^n)$.

There are many cases in which $h(X) = h''(\omega \times X)$, for instance:

Corollary 2.4.11. $h(\{x\}) = h''(\omega \times \{x\})$.

Gödels pair function on ordinals is defined by:

Definition 2.4.6. $\prec \gamma, \delta \succ =: p^{-1}(\prec \gamma, \delta \succ)$, where p is the function defined in the proof of Lemma 2.4.6.

We can then define *Gödel n–tuples* by iterating the pair function:

Definition 2.4.7. $\prec \gamma \succ =: \gamma; \prec \gamma_1, \ldots, \gamma_n \succ =: \prec \gamma_1, \prec \gamma_2, \ldots, \gamma_n \succ\succ$ $(n \geq 2)$.

Hence any X which is closed under Gödel pairs is closed under the tuple–function. Imitating the proof of Lemma 2.4.7 we get:

Corollary 2.4.12. *If $Y \subset \mathrm{On}_M$ is closed under Gödel pairs, then:*

(a) $h(Y) = h''(\omega \times Y)$

(b) $h(Y \cup \{p\}) = h''(\omega \times (Y \times \{p\})$ *for* $p \in M$.

Proof: We display the proof of (b). Let $y \in h(Y \cup \{p\})$. Then $y = f(\gamma_1, \ldots, \gamma_n, p)$, where $\gamma_1, \ldots, \gamma_n \in Y$ and f is $\Sigma_1(M)$.

Hence $y = f^*(\langle \delta, p \rangle)$ where $\delta = \prec \gamma_1, \ldots, \gamma_n \succ$ and

$$y = f^*(z) \leftrightarrow \bigvee \gamma_1, \ldots, \gamma_n \bigvee p(z = \langle \prec \gamma_1, \ldots, \gamma_n \succ, p\rangle \wedge \\ \wedge y = f(\vec{\gamma}, p)).$$

Hence $y = h(i, \langle \delta, p \rangle)$ for some i. QED (Corollary 2.4.12)

Similarly we of course get:

Corollary 2.4.13. *If $Y \subset M$ is closed under ordered pairs, then:*

(a) $h(Y) = h''(\omega \times Y)$

(b) $h(Y \cup \{p\}) = h''(\omega \times (Y \times \{p\})$ *for* $p \in M$.

By Lemma 2.4.5 we easily get:

Corollary 2.4.14. *Let $Y \subset \mathrm{On}_M$. Then $h(Y) = h''(\omega \times \mathbb{P}_\omega(Y))$.*

In fact:

Corollary 2.4.15. *Let $A \subset \mathbb{P}_\omega(\mathrm{On}_M)$ be directed (i.e. $a, b \in A \rightarrow \bigvee c \in A\ a, b \subset c$). Let $Y = \bigcup A$. Then $h(Y) = h''(\omega \times A)$.*

By the condensation lemma we get:

Lemma 2.4.16. *Let $\pi : \overline{M} \rightarrow_{\Sigma_1} M$ where M is a J–model and $\overline{M}$ is transitive. Then $\overline{M}$ is a J–model.*

Proof: $\overline{M}$ is amenable by Σ_1 preservation. But then it is a J–model by the condensation lemma. QED (Lemma 2.4.16)

We can get a theorem in the other direction as well. We first define:

Definition 2.4.8. Let $\overline{M}, M$ be transitive structures. $\sigma : \overline{M} \to M$ *cofinally* iff σ is a structural embedding of $\overline{M}$ into M and $M = \bigcup \sigma''\overline{M}$.

Then:

Lemma 2.4.17. *If $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally. Then σ is Σ_1 preserving.*

Proof: Let $R(y, \vec{x})$ be $\Sigma_0(M)$ and let $\overline{R}(y, \vec{x})$ be $\Sigma_0(\overline{M})$ by the same definition. We claim:

$$\bigvee y R(y, \sigma(\vec{x})) \to \bigvee y \overline{R}(y, \vec{x})$$

for $x_1, \ldots, x_n \in \overline{M}$. To see this, let $R(y, \sigma(\vec{x}))$. Then $y \in \sigma(u)$ for a $u \in \overline{M}$. Hence $\bigvee y \in \sigma(u) R(y, \sigma(\vec{x}))$, which is a Σ_0 statement about $\sigma(u), \sigma(\vec{x})$. Hence $\bigvee y \in u \overline{R}(y, \vec{x})$. QED (Lemma 2.4.17)

Lemma 2.4.18. *Let $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally, where $\overline{M}$ is a J–model. Then M is a J–model.*

Proof: Let e.g. $\overline{M} = \langle J_{\overline{\alpha}}^{\overline{A}} \rangle, M = \langle U, A, \overline{B} \rangle$.

Claim 1 $U = J_\alpha^A$ where $\alpha = \mathrm{On}_M$.

> **Proof:** $y = S^{\overline{A}} \restriction \nu$ is a Σ_0 condition, so $\sigma(S^{\overline{A}} \restriction \nu) = S^A \restriction \sigma(\nu)$. But σ takes $\overline{\alpha}$ cofinally to α, so if $\xi < \alpha, \xi < \sigma(\nu)$, then $S_\xi^A (S^A \restriction \sigma(\nu))(\xi) \in U$. Hence $J_\alpha^A \subset U$. To see $U \subset J_\alpha^A$, let $x \in U$. Then $x \in \sigma(u)$ where $u \in J_{\overline{\alpha}}^{\overline{A}}$. Hence $u \subset S_\nu^{\overline{A}}$ and $x \in \sigma(S_\nu^{\overline{A}}) = S_{\sigma(\nu)}^A \subset J_\alpha^A$. QED (Claim 1)

Claim 2 M is amenable.

> Let $x \in S_{\sigma(\nu)}^A$. Then $\sigma(\overline{B} \cap S_\nu^{\overline{A}}) = B \cap S_{\sigma(\nu)}^A$ and $x \cap B = (B \cap S_\nu^A) \cap x \in U$, since S_ν^A is transitive. QED (Lemma 2.4.18)

Lemma 2.4.19. *Let $\overline{M}, M$ be J–models. Then $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally iff $\sigma : \overline{M} \to_{\Sigma_0} M$ and σ takes $\mathrm{On}_{\overline{M}}$ to On_M cofinally.*

Proof: $(\to)$ is obvious. We prove $(\leftarrow)$. The proof of $\sigma(S_\nu^{\overline{A}}) = S_{\sigma(\nu)}^A$ goes through as before. Thus if $x \in M$, we have $x \in S_\xi^A$ for some ξ. Let $\xi \leq \sigma(\nu)$. Then $x \in S_{\sigma(\nu)}^A = \sigma(S_\nu^{\overline{A}})$. QED (Lemma 2.4.19)

2.5 The Σ_1 projectum

2.5.1 Acceptability

We begin by defining a class of J–models which we call *acceptable*. Every J_α is acceptable, and we shall see later that there are many other naturally occurring acceptable structures. Accepability says essentially that if something dramatic happens to β at some later stage ν of the construction, then ν is, in fact, collapsed to β at that stage:

Definition 2.5.1. $J_\alpha^{\vec{A}}$ is *acceptable* iff for all $\beta \leq \nu < \alpha$ in Lm we have:

(a) If $a \subset \beta$ and $a \in J_{\nu+\omega}^{\vec{A}} \setminus J_\nu^{\vec{A}}$, then $\overline{\overline{\nu}} \leq \beta$ in $J_{\nu+\omega}^{\vec{A}}$.

(b) If $x \in J_\beta^{\vec{A}}$ and ψ is a Σ_1 condition such that $J_{\nu+\omega}^{\vec{A}} \models \psi[\beta, x]$ but $J_\nu^{\vec{A}} \not\models \psi[\beta, x]$, then $\overline{\overline{\nu}} \leq \beta$ in $J_{\nu+\omega}^{\vec{A}}$.
A J–model $\langle J_\alpha^{\vec{A}}, \vec{B}\rangle$ is *acceptable* iff $J_\alpha^{\vec{A}}$ is acceptable.

Note 'Acceptability' referred originally only to property (a). Property (b) was discovered later and was called 'Σ_1 acceptability'.

In the following we shall always suppose M to be acceptable unless otherwise stated. We recall that by Corollary 2.4.8 every $x \in M$ has a cardinal $\overline{\overline{x}} = \overline{\overline{x}}^M$. We call γ a cardinal in M iff $\gamma = \overline{\overline{\gamma}}$ (i.e. no smaller ordinal is mappable onto γ in M).

Lemma 2.5.1. *Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. Let $\gamma > \omega$ be a cardinal in M. Then:*

(a) $\gamma \in \mathrm{Lm}^*$

(b) $J_\gamma^A \prec_{\Sigma_1} J_\alpha^A$

(c) $x \in J_\gamma^A \to M \cap \mathbb{P}(x) \subset J_\gamma^A$.

Proof: We first prove (a). Suppose not. Then $\gamma = \beta + \omega$, where $\beta \in \mathrm{Lm}, \beta \geq \omega$. Then $f \in M$ maps β onto γ where: $f(2i) = i, f(2i+1) = \beta + i, f(\xi) = \xi$ for $\xi \geq \omega$.
Contradiction! QED (a)

If (b) were false, there would be ν such that $\gamma \leq \nu < \alpha$, and for some $x \in J_\gamma^A$ and some Σ_1 formula ψ we have:

$$J_{\nu+\omega}^A \models \psi[x], J_\nu^A \models \neg\psi[x].$$

But then $x \in J^A_\beta$ for some $\beta < \gamma$ in Lm. Hence $\overline{\overline{\gamma}} \leq \overline{\overline{\nu}} \leq \beta$.
Contradiction! QED (b)

To prove (c) suppose not. Then x is not finite. Let $\beta = \overline{\overline{x}}$ in J^A_γ. Then $\beta \geq \omega, \beta \in \text{Lm}$ by (a). Let $f \in J^A_\gamma$ map β onto x. Let $u \subset x$ such that $u \notin J^A_\gamma$. Then $v = f^{-1}{}''u \notin J^A_\gamma$. Let $\nu \geq \gamma$ such that $v \in J^A_{\nu+1} \setminus J^A_\nu$. Then $\gamma \leq \overline{\overline{\nu}} \leq \beta$.
Contradiction! QED (Lemma 2.5.1)

Remark We have stated and proven this lemma for M of type $\langle 1, 1\rangle$, since the extension to M of arbitrary type is self evident.

The most general form of GCH says that if $\mathbb{P}(x)$ exists and $\overline{\overline{x}} \geq \omega$, then $\overline{\overline{\mathbb{P}(x)}} = \overline{\overline{x}}^+$ (where α^+ is the least cardinal $> \alpha$).

As a corollary of Lemma 2.5.1 we have:

Corollary 2.5.2. *Let M, γ be as above. Let $a \in M, a \subset J^A_\gamma$. Then:*

(a) $\langle J^A_\gamma, a\rangle$ models the axiom of subsets and GCH.

(b) If γ is a successor cardinal in M, then $\langle J^A_\gamma, a\rangle$ models ZFC$^-$.

(c) If γ is a limit cardinal in M, then $\langle J^A_\gamma, a\rangle$ models Zermelo set theory.

Proof: (a) follows easily from Lemma 2.5.1 (c). (c) follows from (a) and rud closure of J^A_γ. We prove (b). We know that J^A_γ is rud closed and that the axiom of choice holds in the strong form: $\bigwedge x \bigvee \nu \bigvee f\ f$ maps ν onto x. We must prove the axiom of collection. Let $R(x, y)$ be $\underline{\Sigma}_\omega(J^A_\gamma)$ and let $u \in J^A_\gamma$ such that $\bigwedge x \in u \bigvee y R(x, y)$.

Claim $\bigvee \nu < \gamma \bigwedge x \in u \bigvee y \in J^A_\nu R(x, y)$. Suppose not.

Let $y = \beta^+$ in M. For each $\nu < \gamma$ there is a partial map $f \in M$ of β onto ν. But then $f \in J^A_\gamma$ since $f \subset \nu \times \beta \in J^A_\gamma$. Set f_ν — the $<_{J^A_\gamma}$ — least such f. For $x \in u$ set:

$$h(x) = \text{ the least } \mu \text{ such that } \bigvee y \in J^A_\mu R(y, x).$$

Then $\sup h''u = \gamma$ by our assumption. Define a partial map k on $u \times \beta$ by: $k(x, \xi) \simeq f_{h(x)}(\xi)$. Then k is onto γ. But $k \in M$, since k is $\underline{\Sigma}_1(J^A_\gamma)$. Clearly $\overline{\overline{u \times \beta}} = \beta$ in M, so $\overline{\overline{\gamma}} \leq \beta < \gamma$ in M.
Contradiction! QED (Corollary 2.5.2)

Corollary 2.5.3. *Let M, γ be as above. Then*

$$J^A_\gamma = H^M_\gamma =: \bigcup\{u \in M | u \textit{ is transitive } \cap \overline{\overline{u}} < \gamma \textit{ in } M\}.$$

Proof: Let $u \in M$ be transitive and $\overline{\overline{u}} < \gamma$ in M. It suffices to show that $u \in J^A_\gamma$. Let $\nu = \overline{\overline{u}} < \gamma$ in M. Let $f \in M$ map ν onto u. Set:

$$r = \{\langle \xi, \delta \rangle \in \nu^2 | f(\xi) \in f(\delta)\}.$$

Then $r \in J^A_\gamma$ by Lemma 2.5.1 (c), since $\nu^2 \in J^A_\gamma$. Let $\beta = \overline{\overline{\nu}}^+ =$ the least cardinal $> \nu$ in M. then J^A_β models ZFC$^-$ and $r, \nu \in J^A_\beta$. But then $f \in J^A_\beta \subset J^A_\gamma$, since f is defined by recursion on $r : f(x) = f''r''\{x\}$ for $x \in \nu$. Hence $u = \text{rng}(f) \in J^A_\gamma$. QED (Corollary 2.5.3)

Lemma 2.5.4. *If $\pi : \overline{M} \rightarrow_{\Sigma_1} M$ and M is acceptable, then so is $\overline{M}$.*

Proof: $\overline{M}$ is a J–model by §4. Let e.g. $M = J^A_\alpha, \overline{M} = J^{\overline{A}}_{\overline{\alpha}}$. Then $\overline{M}$ has a counterexample — i.e. there are $\overline{\nu} < \overline{\alpha}, \overline{\beta} < \overline{\nu}, \overline{a}$ such that card$(\overline{\nu}) > \overline{\beta}$ in $J_{\overline{\nu}+\omega}$ and either $\overline{a} \subset \overline{\beta}$ and $\overline{a} \in J^{\overline{\mathbb{A}}}_{\overline{\nu}+\omega} \setminus J^{A}_{\overline{\nu}}$ or else $\overline{a} \in J^{\overline{A}}_{\overline{\beta}}, J^{\overline{A}}_{\overline{\nu}+1} \models \psi[\overline{a}, \overline{\beta}]$ and $J^{\overline{A}}_{\overline{\nu}} \models \neg\psi[\overline{a}, \overline{\beta}]$, where ψ is Σ_1. But then letting $\pi(\overline{\beta}, \overline{\nu}, \overline{a}) = \beta, \nu, a$ it follows easily that β, ν, a is a counterexample in M.
Contradiction! QED (Lemma 2.5.4)

Lemma 2.5.5. *If $\pi : \overline{M} \rightarrow_{\Sigma_0} M$ cofinally and $\overline{M}$ is acceptable, then so is M.*

Proof: M is a J–model by §4. Let $M = J^A_\alpha, \overline{M} = J^A_\alpha$.

Case 1 $\overline{\alpha} = \omega$.
Then $\overline{M} = M = J^A_\omega, \pi = \text{id}$.

Case 2 $\overline{\alpha} \in \text{Lm}^*$.
Then 'M is acceptable' is a $\Pi_1(\overline{M})$ condition. But then $\alpha \in \text{Lm}^*$ and M must satisfy the same Π_1 condition.

Case 3 $\overline{a} = \overline{\beta} + \omega, \overline{\beta} \in \text{Lm}$.
Then $\alpha = \beta + \omega, \beta \in \text{Lm}$ and $\beta = \pi(\overline{\beta})$. Then $J^A_\beta = \pi(J^{\overline{A}}_{\overline{\beta}})$ is acceptable, so there can be no counterexample $\langle \delta, \nu, a \rangle \in J^A_\beta$.

We show that there can be no counterexample of the form $\langle \delta, \beta, a \rangle$. Let $\overline{\gamma} = \text{card}(\overline{\beta})$ in $\overline{M}$. The statement card$(\overline{\beta}) \leq \overline{\gamma}$ is $\Sigma_1(M)$. Hence card$(\beta) \leq \gamma = \pi(\overline{\gamma})$ in M. Hence there is no counterexample $\langle \delta, \beta, a \rangle$ with $\delta \geq \gamma$.

But since $\overline{M}$ is acceptable and $\overline{\gamma} \leq \overline{\beta}$ is a cardinal in $\overline{M}$, the following Π_1 statements hold in $\overline{M}$ by Lemma 2.5.1

$$\bigwedge \delta < \overline{\gamma} \bigwedge a \subset \delta a \in J^{\overline{A}}_{\overline{\gamma}}$$
$$\bigwedge \delta < \overline{\gamma} \bigwedge x \in J^{\overline{A}}_{\delta} (\bigvee y R(x, \delta) \to \bigvee y \in J^{\overline{A}}_{\overline{\gamma}})$$
where R is $\Sigma_0(\overline{M})$.

But then the corresponding statements hold in M. Hence $\langle \delta, \beta, a \rangle$ cannot be a counterexample for $\delta < \gamma$. QED (Lemma 2.5.5)

2.5.2 The projectum

We now come to a central concept of fine structure theory.

Definition 2.5.2. Let M be acceptable. The Σ_1*–projectum* of M (in symbols ρ_M) is the least $\rho \leq \text{On}_M$, such that there is a $\underline{\Sigma}_1(M)$ set $a \subset \rho$ with $a \notin M$.

Lemma 2.5.6. *Let $M = \langle J^A_\alpha, B \rangle, \rho = \rho_M$. Then*

(a) If $\rho \in M$, then ρ is cardinal in M.

(b) If D is $\underline{\Sigma}_1(M)$ and $D \subset J^A_\rho$, then $\langle J^A_\rho, D \rangle$ is amenable.

(c) If $u \in J^A_\rho$, there is no $\underline{\Sigma}_1(M)$ partial map of u onto J^A_ρ.

(d) $\rho \in \text{Lim}^*$

Proof:

(a) Suppose not. Then there are $f \in M$, $\gamma < \rho$ such that f maps γ onto ρ. Let $a \subset \rho$ be $\underline{\Sigma}_1(M)$ such that $a \notin M$. Set $\tilde{a} = f^{-1}{}''a$. Then $\tilde{a}$ is $\Sigma_1(M)$ and $\tilde{a} \subset \gamma$. Hence $\tilde{a} \in M$. But then $a = f''\tilde{a} \in M$ by rud closure.
Contradiction! QED (a)

(b) Suppose not. Let $u \in J^A_\rho$ such that $D \cap u \notin J^A_\rho$. We first note:

Claim $D \cap u \notin M$.

If $\rho = \alpha$ this is trivial, so let $\rho < \alpha$. Then ρ is a cardinal by (b) and by Lemma 2.5.1 we know that $\mathbb{P}(u) \cap M \subset J^A_\rho$. QED (Claim)

By Corollary 2.5.2 there is $f \in J^A_\rho$ mapping a $\nu < \rho$ onto $D \cap u$. Then $d = f^{-1u}(D \cap u)$ is $\underline{\Sigma}_1(M)$ and $d \subset \nu < \rho$. Hence $d \in M$. Hence $D \cap u = f''d \in M$ by rud closure. QED (b)

(c) Suppose not. Let f ba a counterexample. Set $a = \{x \in u | x \in \text{dom}(f) \wedge x \notin f(x)\}$. Then a is $\underline{\Sigma}_1(M)$, $a \subset u \in M$. Hence $a \in J^A_\rho$ by (b). Let $a = f(x)$. Then $x \in f(x) \leftrightarrow x \notin f(x)$.
Contradiction! QED (c)

(d) If not, then $\rho = \beta + \omega$ where $\beta \in \text{Lim}$. But then there is a $\underline{\Sigma}_1(M)$ partial map of β onto ρ, violating (c). QED (Lemma 2.5.6)

Remark This shows that we could have defined ρ to be least such that there is a $\underline{\Sigma}_1(M)$ set $a \subset J^A_\rho$ with $a \notin M$.

Remark We have again stated and proven the theorem for the special case $M = \langle J^A_\alpha, B\rangle$, since the general case is then obvious. We shall continue this practice for the rest of the book. A *good parameter* is a $p \in M$ which witnesses that $\rho = \rho_M$ is the projectum — i.e. there is $B \subset M$ which is $\Sigma_1(M)$ in p with $B \cap H^M_\rho \notin M$. But by §3 any $p \in M$ has the form $p = f(a)$ where f is a $\Sigma_1(M)$ function and a is a finite set of ordinals. Hence a is good if p is. For technical reasons we shall restrict ourselves to good parameters which ar finite sets of ordinals:

Definition 2.5.3. $P = P_M =:$ The set of $p \in [\text{On}_M]^{<\omega}$ which are good parameters.

Lemma 2.5.7. *If $p \in P$, then $p \setminus \rho_M \in P$.*

Proof: It suffices to show that if $\nu = \min(p)$ and $\nu < \rho$, then $p' = p \setminus (\nu+1) \in P$. Let B be $\Sigma_1(M)$ in p such that $B \cap H^M_\rho \notin M$. Let $B(x) \leftrightarrow B'(x,p)$ where B' is $\Sigma_1(M)$.

Set:

$$B^*(x) \leftrightarrow: \bigvee z \bigvee \nu (x = \langle z, \nu\rangle \wedge B'(z, p' \cup \{\nu\})).$$

Then $B^* \cap H_\rho \notin M$, since otherwise

$$B \cap H_\rho = \{x | \langle x, \nu\rangle \in B^* \cap H_\rho\} \in M.$$

Contradiction! QED (Lemma 2.5.7)

For any $p \in [\text{On}_M]^{<\omega}$ we define the *standard code T^p determined by p* as:

Definition 2.5.4.

$$T^p = T^p_M =: \{\langle i, x\rangle | \models_M \varphi_i[x,p]\} \cap H^M_{\rho_M}\}$$

where $\langle \varphi_i | i < \omega\rangle$ is a fixed recursive enumeration of the Σ_1–fomulae.

Lemma 2.5.8. $p \in P \leftrightarrow T^p \notin M$.

Proof:

($\leftarrow$) $T^p = T \cap H_p^M$ for a T which is $\Sigma_1(M)$ in p.

($\rightarrow$) Let B be $\Sigma_1(M)$ in p such that $B \cap H_p^M \notin M$. Then for some i:

$$B(x) \leftrightarrow \langle i, x\rangle \in T^p$$

for $x \in H_p^M$. Hence $T^p \notin M$. QED (Lemma 2.5.8)

A parameter p is *very good* if every element of M is Σ_1 definable from parameters in $\rho_M \cup \{p\}$. R is the set of very good parameters lying in $[\mathrm{On}_M]^{<\omega}$.

Definition 2.5.5. $R = R_M =$: the set of $r \in [\mathrm{On}_M]^{<\omega}$ such that $M = h_M(\rho_M \cup \{r\})$.

Note This is the same as saying $M = h_M(\rho_M \cup r)$, since

$$h(\rho \cup r) = h''(\omega \times [\rho \cup r]^\omega).$$

But $\rho \cup r = \rho \cup (r \setminus \rho)$. Hence:

Lemma 2.5.9. *If $r \in R$, then $r \setminus \rho \in R$. We also note:*

Lemma 2.5.10. $R \subset P$.

Proof: Let $r \in R$. We must find $B \subset M$ such that B is $\Sigma_1(M)$ in r and $B \cap H_\rho^M \notin M$. Set:

$$B = \{\langle i, x\rangle | \bigvee y\ y = h(i, \langle x, r\rangle) \wedge \langle i, x\rangle \notin y\}.$$

If $b = B \cap H_\rho^M \in M$, then $b = h(i, \langle x, r\rangle)$ for some i. Then $\langle i, x\rangle \in b \leftrightarrow \langle i, x\rangle \notin b$.
Contradiction! QED (Lemma 2.5.10)

However, R can be empty.

Lemma 2.5.11. *There is a function h^r uniformly $\Sigma_1(M)$ in r such that whenever $r \in R_M$, then $M = h^{r\prime\prime}\rho_M$.*

Proof: Let $x \in M$. Since $x \in h(\rho \cup \{r\})$ there is an f which is $\Sigma_1(M)$ n r such that $x = f(\xi_1, \ldots, \xi_n)$. But ρ is closed under Gödel pairs, so $x = f'(\prec \xi_1, \ldots, \xi_n \succ)$, where

$$x = f'(\xi) \leftrightarrow \bigvee \xi_1, \ldots, \xi_n(\xi = \prec \vec{\xi} \succ \wedge x = f(\vec{\xi})).$$

f' is $\Sigma_1(M)$ in r. Hence $x = h(i, \langle\langle\vec{\xi}\rangle, r\rangle)$ for some $i < \omega$. Set

$$x = h^r(\delta) \leftrightarrow \bigvee \xi \bigvee i < \omega(\delta = \langle i, \xi\rangle \wedge x = h(i, \langle \xi, r\rangle)).$$

Then $x = h^r(\langle i, \langle\vec{\xi}\rangle\rangle)$. QED (Lemma 2.5.11)

Lemma 2.5.11 explains why we called T^p a *code*: If $r \in R$, then T^r gives complete information about M. Thus the relation $\in' = \{\langle x, \tau\rangle | h^r(\nu) \in h^r(\tau)\}$ is rud in T^r, since $\nu \in' \tau \leftrightarrow \langle i, \langle \nu, \tau\rangle\rangle \in T^r$ for some $i < \omega$. Similarly, if $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$, then $A'_i = \{\nu | h^r(\nu) \in A_i\}$ and $B'_j = \{\nu | h^r(\nu) \in B_i\}$ are similary rud in T^r (as is, indeed, R' whenever R is a relation which is $\Sigma_1(M)$ in p). Note, too, that if $B \subset H_\rho^M$ is $\underline{\Sigma}(M)$, then B is rud in T^r. However, if $p \in P^1 \setminus R^1$, then T^p does not completely code M.

Definition 2.5.6. Let $p \in [\mathrm{On}_M]^{<\omega}$. Let $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$.

The *reduct of M by p* is defined to be

$$M^p =: \langle J_{\rho_M}^{\vec{A}}, T_M^p\rangle.$$

Thus M^p is an acceptable model which — if $p \in R_M$ — incorporates complete information about M.

The *downward extension of embeddings lemma* says:

Lemma 2.5.12. *Let $\pi : N \to_{\Sigma_0} M^p$ where N is a J–model and $p \in [\mathrm{On}_M]^{<\omega}$.*

(a) There are unique $\overline{M}, \overline{p}$ such that $\overline{M}$ is acceptable, $\overline{p} \in R_{\overline{M}}, N = \overline{M}^{\overline{p}}$.

(b) There is a unique $\tilde{\pi} \supset \pi$ such that $\tilde{\pi} : \overline{M} \to_{\Sigma_0} M$ and $\pi(\overline{p}) = p$.

(c) $\tilde{\pi} : \overline{M} \to_{\Sigma_1} M$.

Proof: We first prove the existence claim. We then prove the uniqueness claimed in (a) and (b).

Let e.g. $M = \langle J_\alpha^A, B\rangle, M^p = \langle J_\rho^A, T\rangle, N = \langle J_{\overline{\rho}}^{\overline{A}}, \overline{T}\rangle$. Set: $\tilde{\rho} = \sup \pi''\overline{\rho}$, $\tilde{M} = M^p|\tilde{\rho} = \langle J_{\tilde{\rho}}^A, \tilde{T}\rangle$ where $\tilde{T} = T \cap J_{\tilde{\rho}}^A$. Set $X = \mathrm{rng}(\pi)$, $Y = h_M(X \cup \{p\})$. Then $\tilde{\pi} : N \to_{\Sigma_0} \tilde{M}$ cofinally by §4.

(1) $Y \cap \tilde{M} = X$

Proof: Let $y \in Y \cap \tilde{M}$. Since X is closed under ordered pairs, we have $y = f(x, p)$ where $x \in X$ and f is $\Sigma_1(M)$. Then

$$\begin{aligned} y = f(x,p) &\leftrightarrow \models_M \varphi_i[\langle y, x\rangle, p] \\ &\leftrightarrow \langle i, \langle y, x\rangle\rangle \in \tilde{T}. \end{aligned}$$

Since $X \prec_{\Sigma_1} \tilde{M}$, there is $y \in X$ such that $\langle i, \langle y, x\rangle\rangle \in \tilde{T}$. Hence $y = f(x, \rho) \in X$. QED (1)

Now let $\tilde{\pi} : \overline{M} \tilde{\leftrightarrow} Y$, where $\overline{M}$ is transitive. Clearly $p \in Y$, so let $\tilde{\pi}(\overline{p}) = p$. Then:

(2) $\tilde{\pi} : \overline{M} \rightarrow_{\Sigma_1} M$, $\tilde{\pi} \upharpoonright N = \pi$, $\tilde{\pi}(\overline{p}) = p$.
But then:

(3) $\overline{M} = h_{\overline{M}}(N \cup \{\overline{p}\})$.
Proof: Let $y \in \overline{M}$. Then $\tilde{\pi}(y) \in Y = h_M{}''(\omega x(Xx\{p\}))$, since X is closed under ordered pairs. Hence $\tilde{\pi}(y) = h_M(i, \langle \pi(x), p\rangle)$ for an $x \in \overline{M}$. Hence $y = h_{\overline{M}}(i, \langle x, \overline{p})$. QED (3)

(4) $\overline{\rho} \geq \rho_{\overline{M}}$.
Proof: It suffices to find a $\Sigma_1(\overline{M})$ set b such that $b \subset N$ and $b \notin \overline{M}$. Set

$$b = \{\langle i, x\rangle \in \omega \times N | \bigvee y \quad (y = h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \\ \wedge \langle i, x\rangle \notin y)\}$$

If $b \in \overline{M}$, then $b = h_{\overline{M}}(i, \langle x, \overline{p}\rangle)$ for some $x \in N$. Hence

$$\langle i, x\rangle \in b \leftrightarrow \langle i, x\rangle \notin b.$$

Contradiction! QED (4)

(5) $\overline{T} = \{\langle i, x\rangle \in \omega \times N | \models_{\overline{M}} \varphi_i[i, \langle x, p\rangle]\}$.
Proof: $\overline{T} \subset \omega \times N$, since $\tilde{T} \subset \omega \times \tilde{M}$. But for $\langle i, x\rangle \in \omega \times N$ we have:

$$\begin{aligned} \langle i, x\rangle \in \overline{T} &\leftrightarrow \langle i, \pi(x)\rangle \in \tilde{T} \\ &\leftrightarrow M \models \varphi_i[\langle (x), p\rangle] \\ &\leftrightarrow \overline{M} \models \varphi_i[\langle x, p\rangle] \text{ by (2)} \end{aligned}$$

QED (5)

(6) $\overline{\rho} = \rho_{\overline{M}}$.
Proof: By (4) we need only prove $\overline{\rho} \leq \rho_{\overline{M}}$. It suffices to show that if $b \subset N$ is $\underline{\Sigma}_1(\overline{M})$, then $\langle J^{\overline{A}}_{\overline{\rho}}, b\rangle$ is amenable. By (3) b is $\Sigma_1(\overline{M})$ in $x, \overline{p}$ where $x \in \overline{N}$.
Hence

$$\begin{aligned} b &= \{z | \overline{M} \models \varphi_i[\langle z, x\rangle, \overline{p}]\} = \\ &= \{z | \langle i, z, x\rangle \in \overline{T}\} \end{aligned}$$

Hence b is rud in $\overline{T}$ where $N = \langle J^{\overline{A}}_{\overline{\rho}}, \overline{T}\rangle$ is amenable. QED (6)

But then $\overline{M} = h_{\overline{M}}(\overline{\rho} \cup \{\overline{\rho}\})$ by (3) and the fact that $h_{J^{\overline{A}}_{\overline{\rho}}}(\overline{\rho}) = J^{\overline{A}}_{\overline{\rho}}$. Hence

(7) $\overline{p} \in R_{\overline{M}}$.
By (6) we then conclude:

(8) $N = \overline{M}^{\overline{p}}$.
This proves the existence assertions. We now prove the uniqueness assertion of (a). Let $\hat{M}^{\hat{p}} = N$ where $\hat{p} \in R_{\hat{M}}$.
We claim: $\hat{M} = \overline{M}$, $\hat{p} = \overline{p}$.

Since the Skolem function is uniformly Σ_1 there is a $j < \omega$ such that

$$\begin{aligned} & h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \in h_{\hat{M}}(i, \langle y, \hat{p}\rangle \leftrightarrow \\ & \quad \leftrightarrow \hat{M} \models \varphi_j[\langle x, y\rangle, p] \leftrightarrow \langle j, \langle x, y\rangle\rangle \in \overline{T} \\ & \quad \leftrightarrow h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \in h_{\overline{M}}(i, \langle y, \overline{p}\rangle) \end{aligned}$$

Similarly:

$$h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \in \hat{A} \leftrightarrow h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \in \overline{A}$$
$$h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \in \hat{B} \leftrightarrow h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \in \overline{B}$$

where $\hat{M} = \langle J_{\hat{\alpha}}^{\hat{A}}, \hat{B}\rangle$, $\overline{M} = \langle J_{\overline{\alpha}}^{\overline{A}}, \overline{B}\rangle$. Then there is an isomorphism $\sigma : \hat{M} \overset{\sim}{\leftrightarrow} \overline{M}$ defined by $\sigma(h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \simeq h_{\overline{M}}(i, \langle x, \overline{p}\rangle)$ for $x \in N$. Clearly $\sigma(\hat{p}) = \overline{p}$. Hence $\sigma = \text{id}, \hat{M}, \overline{M}, \hat{p} = \overline{p}$, since $\overline{M}, \hat{M}$ are transitive.

We now prove (b). Let $\hat{\pi} \supset \pi$ such that $\hat{\pi} : \overline{M} \to_{\Sigma_0} M$ and $\hat{\pi}(\overline{p}) = p$. If $x \in N$ and $h_{\overline{M}}(i, \langle x, \overline{p}\rangle)$ is defined, it follows that:

$$\hat{\pi}(h_{\overline{M}}(i, \langle x, \overline{p})) = h_M(i, \langle \pi(x), p\rangle) = \tilde{\pi}(h_M(i, \langle x, \overline{p}\rangle)).$$

Hence $\hat{\pi} = \pi$. QED (Lemma 2.5.12)

If we make the further assumption that $p \in R_M$ we get a stronger result:

Lemma 2.5.13. *Let $M, N, \overline{M}, \pi, \overline{\pi}, p, \overline{p}$ be as above where $p \in R_M$ and $\pi : N \to_{\Sigma_l} M^p$ for an $l < \omega$. Then $\tilde{\pi} : \overline{M} \to_{\Sigma_{l+1}} M$.*

Proof: For $l = 0$ it is proven, so let $l \geq 1$ and let it hold at l. Let R be $\Sigma_{l+1}(M)$ if l is even and $\Pi_{l+1}(M)$ if l is odd. Let $\overline{R}$ have the same definition over $\overline{M}$. It suffices to show:

$$\overline{R}(\vec{x}) \leftrightarrow R(\tilde{\pi}(\vec{x})) \text{ for } x_1, \dots, x_n \in \overline{M}.$$

But:

$$R(\vec{x}) \leftrightarrow Q_1 y_1 \in M \dots Q_l y_l \in M R'(\vec{y}, \vec{x})$$

and

$$\overline{R}(\vec{x}) \leftrightarrow Q_1 y_1 \in \overline{M} \dots Q_l y_l \in \overline{M R}'(\vec{y}, \vec{x})$$

where $Q_1 \ldots Q_l$ is a string of alternating quantifiers, R' is $\Sigma_1(M)$, and $\overline{R}'$ is $\Sigma_1(\overline{M})$ by the same definition. Set

$$D =: \{\langle i, x\rangle \in \omega \times J^A_\rho | h_M(i, \langle x, p\rangle) \text{ is defined}\}$$
$$\overline{D} =: \{\langle i, x\rangle \in \omega \times J^{\overline{A}}_{\overline{\rho}} | h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \text{ is defined}\}.$$

Then D is $\Sigma_1(M)$ in p and $\overline{D}$ is $\Sigma_1(\overline{M})$ in $\overline{p}$ by the same definition. Then D is rud in T^p_M and $\overline{D}$ is rud in $T^{\overline{p}}_{\overline{M}}$ by the same definition, since for some $j < \omega$ we have:

$$x \in D \leftrightarrow \langle j, x\rangle \in T^p_M,\ x \in \overline{D} \leftrightarrow \langle j, x\rangle \in T^{\overline{p}}_{\overline{M}}.$$

Define h on D

$$k(\langle i, x\rangle) = h_M(i, \langle x, p\rangle);\ \overline{k}(\langle i, x\rangle) = h_{\overline{M}}(i, \langle x, \overline{p}\rangle).$$

Set:

$$P(\vec{w}, \vec{z}) \leftrightarrow (\vec{w}, \vec{z} \in D \wedge R'(k(\vec{w}), k(\vec{z}))$$
$$\overline{P}(\vec{w}, \vec{z}) \leftrightarrow (\vec{w}, \vec{z} \in \overline{D} \wedge \overline{R}'(\overline{k}(\vec{w}), \overline{k}(\vec{z}))$$

Then: as before, P is rud in T^p_M and $\overline{D}$ is rud in $T^{\overline{p}}_{\overline{M}}$ by the same definition. Now let $x_i = k(z_i)$ for $i = 1, \ldots, n$. Then $\tilde{\pi}(x_i) = k(\pi(z_i))$. But since π is Σ_l–preserving, we have:

$$\begin{aligned} \overline{R}(\vec{x}) &\leftrightarrow Q_1 w_1 \in \overline{D} \ldots Q_l w_l \in \overline{DP}(\vec{w}, \vec{z}) \\ &\leftrightarrow Q_1 w_1 \in D \ldots Q_l w_l \in DP(\vec{w}, \vec{z}) \\ &\leftrightarrow R(\tilde{\pi}(\vec{x})) \end{aligned}$$

QED (Lemma 2.5.13)

2.5.3 Soundness and iterated projecta

The reduct of an acceptable structure is itself acceptable, so we can take its reduct etc., yielding a sequence of reducts and nonincreasing projecta $\langle \rho^n_M | n < \omega\rangle$. this is the classical method of doing fine structure theory, which was used to analyse the constructible hierarchy, yielding such results as the $\square$ principles and the covering lemma. In this section we expound the basic elements of this classical theory. As we shall see, however, it only works well when our acceptable structures have a property called *soundness*. In this book we shall often have to deal with unsound structures, and will, therefore, take recourse to a further elaboration of fine structure theory, which is developed in §2.6.

It is easily seen that:

Lemma 2.5.14. *Let $p \in R_M$. Let B be $\underline{\Sigma}_1(M)$. Then $B \cap J^A_\rho$ is* rud *in parameters over M^p.*

Proof: Let B be Σ_1 in r, where $r = h_M(i, \langle v, p\rangle)$ and $\nu < \rho$. Then B is Σ_1 in ν, p. Let:

$$B(x) \leftrightarrow M \models \varphi_i[\langle x, \nu\rangle, p]$$

where $\langle \varphi_i | i < \omega\rangle$ is our canonical enumeration of Σ_1 formulae. Then:

$$x \in B \leftrightarrow \langle i, \langle x, \nu\rangle\rangle \in T^P$$

QED

It follows easily that:

Corollary 2.5.15. *Let $p, q \in R_M$. Let $D \subset J^A_\rho$. Then D is $\underline{\Sigma}_1(M^p)$ iff it is $\underline{\Sigma}_1(M^q)$.*

Assuming that $R_M \neq \emptyset$, there is then a uniquely defined *second projectum* defined by:

Definition 2.5.7. $\rho^2_M \simeq: \rho_{M^p}$ for $p \in R_M$.

We can then define:

$$R^2_M =: \text{ The set of } a \in [\mathrm{On}_M]^{<w} \text{ such that } a \in R_M \text{ and } a \cap \rho \in R_M a \setminus \rho.$$

If $R^2_M \neq \emptyset$ we can define the *second reduct:*

$$M^{2,a} =: (M^a)^{a \cap \rho^2} \text{ for } a \in R^2_M.$$

But then we can define the *third projectum:*

$$\rho^3 = \rho_{M^{2,a}} \text{ for } a \in R^2_M.$$

Carrying this on, we get R^n_M, $M^{n,a}$ for $a \in R^n_M$ and ρ^{n+1}, as long as $R^n_M \neq \emptyset$. We shall call M *weakly n–sound* if $R^n_M \neq \emptyset$.

The formal definitions are as follows:

Definition 2.5.8. Let $M = \langle J^A_\alpha, B\rangle$ be acceptable.

By induction on n we define:

- The set R^n_M of *very good n–parameters.*

- If $R^n_M \neq \emptyset$, we define the *n–th projectum* ρ^n_M.
- For all $a \in R^n_M$ the *n–th reduct* $M^{n,a}$.

We inductively verify:

* If $D \subset J^A_{\rho^n}$ and $a, b \in R^n$, then D is $\underline{\Sigma}_1(M^{n,a})$ iff it is $\underline{\Sigma}_1(M^{n,b})$.

Case 1 $n = 0$. Then $R^0 =: [\mathrm{On}_M]^{<\omega}, \rho^0 = \mathrm{On}_M, M^{0,a} = M$.

Case 2 $n = m + 1$. If $R^m = \emptyset$, then $R^n = \emptyset$ and ρ^n is undefined. Now let $R^m \neq \emptyset$. Since (*) holds at m, we can define

- $\rho^n =: \rho_{M^{m,a}}$ whenever $a \in R^m$.
- $R^n =:$ the set of $a \in [\alpha]^{<\omega}$ such that $a \in R^m$ and $a \cap \rho^m \in R_{M^{m,a}}$.
- $M^{n,a} =: (M^{m,a})^{a \cap \rho_m}$ for $a \in R^n$.

(**Note** It follows inductively that $a \setminus \rho^n \in R^n$ whenever $a \in R^n$.)

We now verify (*). It suffices to prove the direction $(\rightarrow)$. We first note that $M^{n,a}$ has the form $\langle J^A_{\rho n}, T\rangle$, where T is the restriction of a $\underline{\Sigma}_1(M^{m,a})$ set T' to $J^A_{\rho n}$. But then T' is $\underline{\Sigma}_1(M^{m,b})$ by the induction hypothesis. Hence T is rudimentary in parameters over $M^{n,b} = (M^{m,b})^{b \cap \rho^n}$ by Lemma 2.5.14.

Hence, if $D \subset J^A_{\rho n}$ is $\underline{\Sigma}_1(M^{n,a})$, it is also $\underline{\Sigma}_1(M^{n,b})$. QED

This concludes the definition and the verification of (*). Note that $R^1_M = R_M$, $\rho^1 = \rho^1_M$, and $M^{1,a} = M^a$ for $a \in R_M$.

We say that M is *weakly n–sound* iff $R^n_M \neq \emptyset$. It is *weakly sound* iff it is weakly n–sound for $n < \omega$. A stronger notion is that of *full soundness*:

Definition 2.5.9. M is *n–sound* (or *fully n– sound*) iff it is weakly n–sound and for all $i < n$ we have: If $a \in R^i$, then $P_{M^{i,a}} = R_{M^{i,a}}$.

Thus $R_M = P_M$, $R_{M^{1,a}} = P_{M^{1,a}}$ for $a \in P_M$ etc. If M is n–sound we write P^i_M for $R^i_M (i \leq n)$, since then: $a \in P^{i+1} \leftrightarrow (a \not\cap \rho^i \in P^i \wedge a \cap \rho^i \in \Gamma_{M^{i,a\cap\rho^i}}$ for $i < n$).

There is an alternative, but equivalent, definition of soundness in terms of *standard parameters*. in order to formulate this we first define:

Definition 2.5.10. Let $a, b \in [\mathrm{On}]^{<\omega}$.

$$a <_* b \leftrightarrow = \bigvee \mu(a \setminus \mu = b \setminus \mu \wedge \mu \in b \setminus a).$$

Lemma 2.5.16. *$<_*$ is a well ordering of $[\mathrm{On}]^{<\omega}$.*

Proof: It suffices to show that ever non empty $A \subset [\mathrm{On}]^{<\omega}$ has a unique $<_*$–minimal element. Suppose not. We derive a contradiction by defining an infinite descending chain of ordinals $\langle \mu_i | i < \omega \rangle$ with the properties:

- $\{\mu_0, \ldots, \mu_n\} \leq_* b$ for all $b \in A$.
- There is $b \in A$ such that $b \setminus \mu_n = \{\mu_0, \ldots, \mu_n\}$.

$\emptyset \notin A$, since otherwise $\emptyset$ would be the unique minimal element, so set: $\mu_0 = \min\{\max(b) | b \in A\}$. Given μ_n we know that $\{\mu_0, \ldots, \mu_n\} \notin A$, since it would otherwise be the $<_*$–minimal element. Set:

$$\mu_{n+1} = \min\{\max(b \cap \mu_n) | b \in A \cap b \setminus \mu_n = \{\mu_0, \ldots, \mu_n\}\}.$$

QED (Lemma 2.5.16)

Definition 2.5.11. The *first standard parameter* p_M is defined by:

$$p_M =: \text{ The } <_*\text{–least element of } P_M.$$

Lemma 2.5.17. *$P_M = P_M$ iff $p_M \in R_M$.*

Proof: ($\rightarrow$) is trivial. We prove ($\leftarrow$). Suppose not. Then there is $r \in P \setminus R$. Hence $p <_* r$, where $p = p_M$. Hence in M the statement:

(1) $V q <_* r \; r = h(i, \langle \nu, q \rangle)$

holds for some $i < \omega$, $\nu < p_M$. Form M^r and let $\overline{M}, \overline{r}, \pi$ be sucht that $\overline{M}^{\overline{r}} = M^r$, $\overline{\pi} \in R_{\overline{M}}$, $\pi : \overline{M} \rightarrow_{\Sigma_1} M$, and $\pi(\overline{r}) = r$. The statement (1) then holds of $\overline{r}$ in $\overline{M}$.

Let $\overline{q} \in \overline{M}$, $\overline{r} = h_{\overline{M}}(i, \overline{q})$ where $\overline{q} <_* \overline{r}$. Set $q = \pi(\overline{q})$. Then $r = h(i, q)$ in M, where $q <_* r$. Hence $q \in P_M$. But then $q \in R_M$ by the minimality of r. This impossible however, since

$$q \in \pi''\overline{M} = h_M(\rho_M \cup r) \neq M.$$

Contradiction! QED (Lemma 2.5.17)

Definition 2.5.12. The n–th *standard parameter* P^n_M is defined by induction on n as follows:

Case 1 $n = 0$. $p^0 = \emptyset$.

Case 2 $n = m + 1$. If $p^m \in R^m$
$p^n = p^m \cup P_{M^{m,p^m}}$

(**Note** that we always have: $p_N \cap \rho_N = \emptyset$ by $<_*$–minimality.)

If $p^m \notin R^m$, then p^n is undefined. By Lemma 2.5.17 it follows easily that:

Corollary 2.5.18. *M is n–sound iff p^n_M is defined and $p^n_M \in R^n_M$.*

This is the definition of soundness usually found in the literature.

Note that the sequences of projecta ρ^n will stabilize at some n, since it is monotonly non increasing. If it stabilizes at n, we have $R^{n+h} = R^n$ and $P^{n+h} = P^n$ for $h < \omega$.

By iterated application of Lemma 2.5.13 we get:

Lemma 2.5.19. *Let $a \in R^n_M$ and let $\overline{\pi} : N \to_{\Sigma_l} M^{na}$. Then there are $\overline{M}, \overline{a}$ and $\pi \supset \overline{\pi}$ such that $\overline{M}^{n\overline{a}} = M^{na}$, $\overline{a} \in R^n_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_{n+l+1}} M$ and $\pi(\overline{a}) = a$.*

We also have:

Lemma 2.5.20. *Let $a \in R^n_M$. There is an M–definable partial map of ρ^n onto M which is M–definable in the parameter a.*

Proof: By induction on n. The case $n = 0$ is trivial. Now let $n = m + 1$. Let f be a partial map of ρ^m onto M which is definable in $a \setminus \rho^m$. Let $N = M^{m, a\setminus\rho^n}$, $b = a \cap \rho^m$. Then $N = h_N(\rho^n \cup \{b\}) = h_N{}''(w \times (\rho^n \times \{b\}))$. Set:

$$g(\prec i, \nu \succ) \simeq: h_N(i, \langle \nu, b \rangle) \text{ for } \nu < \rho^n.$$

Then $N = g''\rho^n$. Hence $M = fg''\rho^n$, where fg is M–definable in a. QED

We have now developend the "classical" fine structure theory which was used to analyze L. Its applicability to L is given by:

Lemma 2.5.21. *Every J_α is acceptable and sound.*

Unfortunately, in this book we shall sometimes have to deal with acceptable structures which are not sound and can even fail to be weakly 1–sound. This means that the structure is not coded by any of its reducts. How can we deal with it? It can be claimed that the totality of reducts contains full information about the structure, but this totality is a very unwieldy object. In §2.6 we shall develop methods to "tame the wilderness".

We now turn to the proof of Lemma 2.5.21:

We first show:

(A) If J_α is acceptable, then it is sound.

Proof: By induction on n we show that J_α is n–sound. The case $n = 0$ is trivial. Now let $n = m + 1$. Let $p = p^m_M$. Let $q = p_{M^{m,p}} =$ The $<_*$–least $q \in P_{M^{m,p}}$.

Claim $q \in R_{M^{m,p}}$.

Suppose not. Let $X = h_{M^{m,p}}(\rho^n \cup q)$. Let $\overline{\pi} : N \overset{\sim}{\longleftrightarrow} X$, where N is transitive. Then $\overline{\pi} : N \to_{\Sigma_1} M^{np}$ and there are $\overline{M}, \overline{p}, \pi \supset \overline{\pi}$ such that $\overline{M}^{m\overline{p}} = M^{mp}$, $\overline{p} \in R^m_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_n} M$, and $\pi(\overline{p}) = p$. Then $\overline{M} = J_{\overline{\alpha}}$ for some $\overline{\alpha} \leq \alpha$ by the condensation lemma for L.

Let A be $\Sigma_1(M^{mp})$ in p such that $A \cap \rho^n_M \notin M^{m,p}$ Then $A \cap \rho^n_M \notin M$. Let $\overline{A}$ be $\Sigma_1(N)$ in $\overline{q} = \pi^{-1}(q)$ by the same definition. Then $A \cap \rho^n = \overline{A} \cap \rho^n$ is $J_{\overline{\alpha}}$ definable in $\overline{q}$. Hence $\overline{\alpha} = \alpha$, $\overline{M} = M$, since otherwise $A \cap \rho^n \in M$. But then $\pi = id$ and $N = \overline{M}^{m\overline{p}} = M^m$. But by definition: $N = h_{M^{m,p}}(\rho^n \cup q)$. Hence $q \in R_{M^{np}}$. QED

By induction on α we then prove:

(B) J_α is acceptable.

Proof: The case $\alpha = \omega$ is trivial. The case $\alpha \in \mathrm{Lim}^*$ is also trivial. There remains the case $\alpha = \beta + \omega$, where β is a limit ordinal. By the induction hypothesis J_β is acceptable, hence sound.

We first verify (a) in the definition of acceptability. Since J_β is acceptable, it suffices to show that if $\gamma \leq \beta$ and $a \in J_\alpha \setminus J_\beta$, then:

Claim $\overline{\overline{\beta}} \leq \gamma$ in J_α.

Suppose not. Since $\mathbb{P}(J_\beta) \cap J_\alpha = \mathrm{Def}(J_\beta)$, we show that that a is J_β–definable in a parameter r. We may assume w.l.o.g. that $r \in [\beta]^{<\omega}$. We may also assume that a is $\Sigma_n(J_\beta)$ in r for sufficiently large n. There is then, no partial map $f \in \mathrm{Def}(J_\beta)$ mapping γ onto β. Hence, by Lemma 2.5.20 we have $\gamma < \rho^n = \rho^n_{J_\beta}$ for all $n < \omega$.

Pick n big enough that a is $\Sigma_n(J_\beta)$ is r. Set: $p = p^n \cup r$ (where $p^n = p^n_{J_\beta}$). Then $p \in R^n$. Let $M = J_\beta$, $N = M^{np}$. Let $X = h_N(\gamma \cup q)$ where $q = p \cap \rho^n$. Let $\overline{\pi} : \overline{N} \overset{\sim}{\longleftrightarrow} X$, where $\overline{N}$ is transitive. Then $\overline{\pi} : \overline{N} \rightarrow_{\Sigma_1} N$ and hence there are $\overline{M}$, $\overline{p}$, $\pi \supset \overline{\pi}$ such that $\overline{M}^{n,\overline{p}} = \overline{N}$, $\overline{p} \in R^n_{\overline{M}}$, $\pi : \overline{M} \rightarrow_{\Sigma_n} M$, $\pi(\overline{p}) = p^n$. Hence $\overline{M} = J_{\overline{\beta}}$ for $\overline{\beta} \leq \beta$. Moreover, a is $\Sigma_n(\overline{M})$ in $\overline{p}$. Hence $\overline{\beta} = \beta$, since otherwise $a \in \mathrm{Def}(J_{\overline{\beta}}) \subset J_\beta$. But then $\pi = id$, $\overline{N} = N = h_N(\gamma \cup q)$. Hence $\gamma \geq \rho_N = \rho^{n+1}_M$.

Contradiction! QED (Claim)

This proves (a). We now prove (b) in the definition of "acceptable". Most of the proof will be a straightforward imitation of the proof of (a). Assume $J_\alpha \models \psi[x, \gamma]$, but $J_\beta \not\models \psi[x, \gamma]$, where $x \in J_\gamma$, $\gamma \leq \beta$ and ψ is Σ_1. As before we claim:

Claim $\overline{\overline{\beta}} \leq \gamma$ in J_α.

Suppose not. Then $\gamma < \beta$. Let $\psi = V \cup \varphi$ where φ is Σ_0. Let $J_\alpha \models \varphi(y, x, \gamma)$. Then $y = f(z, x, \gamma, J_\beta)$ where f is rud and $z \in J_\beta$. But

$$J_\alpha \models \varphi[f(z, x, \gamma, J_\beta), x, \beta]$$

reduces to:

$$J_\alpha \models \varphi'[z, x, \gamma, J_\beta]$$

where φ' is Σ_0. But then

$$J_\beta \cup \{J_\beta\} \models \varphi'[z, x, \gamma, J_\beta).$$

As we have seen in §2.3, this reduces to:

$$J_\beta \models \chi[z, x, \gamma]$$

where χ is a first order formula. Note that this reduction is *uniform*. Hence if $\gamma < \nu \leq \beta$, $z \in J_\nu$ and $J_\nu \models \chi[z, x, \gamma]$, it follows that $J_{\nu+\omega} \models \psi[x, \gamma]$. This means that $J_\nu \models \neg\chi'[x, \gamma]$ for $\gamma < \nu < \beta$, where $\chi = \chi(v_0, v_1, v_n)$ and $\chi' = V v_0 \chi$. We know that $\gamma < \rho^n_{J_\beta}$ for all n. Choose n such that χ' is Σ_n. Let $M = J_\beta$, $N : M^{n,p}$ when $P = P_N$. Let $X = h_N(\gamma + 1 \cup \{x\})$ and let $\overline{\pi} : \overline{N} \overset{\sim}{\longleftrightarrow} X$, where $\overline{N}$ is transitive. As before, there are $\overline{M}, \overline{p}, \pi \supset \overline{\pi}$ such that $\overline{M}^n \overline{p} = N$, $\pi : \overline{M} \rightarrow_{\Sigma_1} M$, and $\pi(\overline{p}) = p$. Let $\overline{M} = J_{\overline{\beta}}$. Then $J_{\overline{\beta}} \models \chi'(x, \gamma)$. Hence $\overline{\beta} = \beta$ and $\pi = \mathrm{id}$. Hence $N = h_N(\gamma + 1 \cup \{x\})$. Hence $\gamma \geq \rho^{n+1} = \rho_N$.

Contradiction! QED (Lemma 2.5.21)

2.6 Σ^*–theory

There is an alternative to the Levy hierarchy of relations on an acceptable structure $M = \langle J_\alpha^A, B\rangle$ which — at first sight — seems more natural. Σ_0, we recall, consists of the relation on M which are Σ_0 definable in the predicats of M. Σ_1 then consists of relations of the form $\bigvee yR(y, \vec{x})$ where R is Σ_0. Call these levels $\Sigma_0^{(0)}$ and $\Sigma_1^{(0)}$. Our next level in the new hierarchy, call it $\Sigma_0^{(1)}$, consists of relations which are "Σ_0 in $\Sigma_1^{(0)}$" — i.e. $\Sigma_0(\langle M, \vec{A}\rangle)$ where $A_1, \ldots, A_n$ are $\Sigma_1^{(0)}$. $\Sigma_1^{(1)}$ then consists of relations of the form $\bigvee yR(y, \vec{x})$ where R is $\Sigma_0^{(1)}$. $\Sigma_0^{(2)}$ then consists of relations which are Σ_0 in $\Sigma_1^{(1)} \ldots$ etc. By a $\underline{\Sigma}_i^{(n)}$ relation we of course mean a relation of the form

$$R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p}),$$

where $p_1, \ldots, p_m \in M$ and R' is $\Sigma_i^{(n)}(m)$. It is clear that there is natural class of $\Sigma_i^{(n)}$–formulae such that R is a $\Sigma_i^{(n)}$–relation iff it is defined by a $\Sigma_0^{(n)}$–formula. Thus e.g. we can define the $\Sigma_0^{(1)}$ formula to be the smallest set Σ of formulae such that

- All primitive formulae are in Σ.
- All $\Sigma_1^{(0)}$ formulae are in Σ.
- Σ is closed under the sentential operations $\vee, \rightarrow, \leftrightarrow, \neg$.
- If φ is in Σ, then so are $\bigwedge v \in u\varphi$, $\bigvee \in u\varphi$ (where $v \neq u$).

By a $\Sigma_1^{(1)}$ formula we then mean a formula of the form $\bigvee v\varphi$, where φ is $\Sigma_0^{(1)}$.

How does this hierarchy compare with the Levy hierarchy? If no projectum drops, it turns out to be a useful refinement of the Levy hierarchy:
If $\rho_M^n = \alpha$, then $\Sigma_0^{(n)} \subset \Delta_{n+1}$ and $\Sigma_1^{(n)} = \Sigma_{n+1}$. If, however, a projectum drops, it trivializes and becomes useless. Suppose e.g. that $M = J_\alpha$ and $\rho = \rho_M^1 < \alpha$. Then *every* M–definable relation becomes $\underline{\Sigma}_0^{(1)}(M)$. To see this let $R(\vec{x})$ be defined by the formula $\varphi(\vec{v})$, which we may suppose to be in prenex normal form:

$$\varphi(\vec{v}) = Q_1u_1 \ldots Q_mu_m\varphi'(\vec{v}, \vec{u}),$$

where φ' is quantifier free (hence Σ_0). Then:

$$R(\vec{x}) \leftrightarrow Q_1y_1 \in M \ldots Q_my_m \in MR'(\vec{x}, \vec{y})$$

where R' is Σ_0. By soundness we know that there is a $\underline{\Sigma}_1(M)$ partial map f of ρ onto M. But then:

$$R(\vec{x}) \leftrightarrow Q_1\xi_\xi \in \text{dom}(f)\ldots Q_m\xi_m \in \text{dom}(f)R'(\vec{x}, f(\vec{\xi})).$$

Since f is $\underline{\Sigma}_1$, the relation $R'(\vec{x}, f(\vec{\xi}))$ is $\underline{\Sigma}_1$. But $\text{dom}(f)$ is $\underline{\Sigma}_1$ and $\text{dom}(f) \subset \rho$, hence by induction on m:

$$R(\vec{x}) \leftrightarrow Q_1\xi_1 \in \rho\ldots Q_m\xi_m \in \rho R''(\vec{x}, \vec{\xi}),$$

where R'' is a sentential combination of $\underline{\Sigma}_1$ relations. Hence R'' is $\underline{\Sigma}_0^{(1)}(M)$ and so is R.

The problem is that, in passing from $\Sigma_1^{(0)}$ to $\Sigma_0^{(1)}$ our variables continued to range over the whole of M, despite the fact that M had grown "soft" with respect to $\underline{\Sigma}_1$ sets. Thus we were able to reduce unbounded quantification over M to quantification bounded by ρ, which lies in the "soft" part of M. in section 2.5 we acknowledged softness by reducing to the part $H = H_\rho^M$ which remained "hard" wrt $\underline{\Sigma}_1$ sets. We then formed a reduct M^p containing just the sets in H. If M is sound, we can choose p such that M^p contains complete information about M. In the general case, however, this may not be possible. It can happen that *every* reduct entails a loss of information. Thus we want to hold on to the original structure M. In passing to $\Sigma_0^{(1)}$, however, we want to restrict our variables to H. We resolve this conundrum by introducing *new* varibles which range only over H. We call these variables of *Type 1*, the old ones being of *Type 0*. Using $u^h, v^h (h = 0, 1)$ as metavariables for variables of Type h, we can then reformulate the definition of $\Sigma_0^{(1)}$ formula, replacing the last clause by:

- If φ is in Σ, then so are $\bigwedge v^i \in u^1\varphi$, $\bigvee v^i \in u^1\varphi$ where $i = v, 1$ and $v^i \neq u^1$.

A $\Sigma_1^{(1)}$ formula is then a formula of the form $\bigvee v^1\varphi$, where φ is $\Sigma_0^{(1)}$. We call $A \subset M$ a $\underline{\Sigma}_1^{(1)}$ set if it is definable in parameters by a $\Sigma_1^{(1)}$ formula. The *second projectum* ρ^2 is then the least ρ such that $\rho \cap B \notin M$ for some $\underline{\Sigma}_1^{(1)}$ set B. We then introduce type 2 variables $v^2, u^2, \ldots$ ranging over $|J_{\rho^2}^A|$ ($|J_\gamma^A|$ being the set of elements of the structure J_γ^A, where e.g. $M = \langle J_\alpha^A, B\rangle$.) Proceeding in this way, we arrive at a many sorted language with variables of type n for each $n < \omega$. The resulting hierarchy of $\Sigma_h^{(n)}$ formulae $(h = 0, 1)$ offers a much finer analysis of M–definabilty than was possible with the Levy hierarchy alone. This analysis is known as Σ^* theory. In this section we shall develop Σ^* theory systematically and *ab ovo*.

Before beginning, however, we address a remark to the reader: Most people react negatively on their first encounter with Σ^* theory. The introduction of a many sorted language seems awkward and cumbersome. It is especially annoying that the variable domains diminish as the types increase. The author confesses to having felt these doubts himself. After developing Σ^*–theory and making its first applictions, we spent a couple of months trying vainly to redo the proofs without it. The result was messier proofs and a pronounced loss of perspicuity. It has, in fact, been our consistent experience that Σ^* theory facilitates the fine structural analysis which lies at the heart of inner model theory. We therefore urge the reader to bear with us.

Definition 2.6.1. Let $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$ be acceptable.

The Σ^* *M–language* $\mathbb{L}^* = \mathbb{L}^*_M$ has

- a binary predicate $\dot{\in}$
- unary predicates $\dot{A}_1, \ldots, \dot{A}_n, \dot{B}_1, \ldots, \dot{B}_m$
- variables $v_i^j (i, j < \omega)$

Definition 2.6.2. By induction on $n < \omega$ we define sets $\Sigma_h^{(n)} (h = 0, 1)$ of formulae

$\Sigma_0^{(n)}$ = the smallest set of formulae such that

- all primitive formulae are in Σ.
- $\Sigma_0^{(m)} \cup \Sigma_1^{(m)} \subset \Sigma$ for $m < n$.
- Σ is closed under sentential operations $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$.
- If φ is in $\Sigma, j \leq n$, and $v^j \neq u^n$, then $\bigwedge v^j \in u^n \varphi$, $\bigvee v^j \in u^n \varphi$ are in Σ.

We then set:

$$\Sigma_1^{(n)} =: \text{ The set of formulae } \bigvee v^n \varphi, \text{ where } \varphi \in \Sigma_0^{(n)}.$$

We also generalize the last part of this definition by setting:

Definition 2.6.3. Let $n < \omega$, $1 \leq h < \omega$. $\Sigma_h^{(n)}$ is the set of formulae

$$\bigvee v_1^n \bigwedge v_2^n \ldots Q v_h^n \varphi,$$

where φ is $\Sigma_0^{(n)}$ (and Q is $\bigvee$ if h is odd and $\bigwedge$ if h is even).

We now turn to the interpretation of the formualae in M.

Definition 2.6.4. Let Fml^n be the set of formulae in which only variables of type $\leq n$ occur.

By recursion on n we define:

- The n–th projectum $\rho^n \rho^n_M$.
- The n–th variable domain $H^n = H^n_M$.
- The satisfaction relation $\models^n$ for formulae in Fml^n.

$\models^n$ is defined by interpreting variables of type i as ranging over H^i for $i \leq n$. We set: $\rho^0 = \alpha$, $H^0 = |M| = |J^{\vec{A}}_\alpha|$, when $M = \langle J^{\vec{A}}_\alpha, \vec{B}\rangle$.

Now let ρ^n, H^n be given (hence $\models^n$ is given). Call a set $D \in H^n$ a $\underline{\Sigma}^{(n)}_1$ *set.* if it is definable from parameters by a $\Sigma^{(n)}_1$ formula φ:

$$Dx \leftrightarrow M \models^n \varphi[x, a_1, \ldots, a_p],$$

where $\varphi = \varphi(v^n, u^{i_1}, \ldots, u^{i_m})$ is $\Sigma^{(n)}_1$. ρ^{n+1} is then the least ρ such that there is a $\underline{\Sigma}^{(n)}_1$ set $D \subset \rho$ with $D \notin M$. We then set:

$$H^{n+1} = |J^{\vec{A}}_\rho|.$$

This then defines $\models^{n+1}$.

It is obvious that $\models^i$ is contained in $\models^j$ for $i \leq j$, so we can define the full Σ^* satisfaction relation for M by:

$$\models = \bigcup_{n<\omega} \models^n .$$

Satisfaction is defined in the usual way. We employ v^i, u^i, ω^i etc. as metavariables for variables of type i. We also employ x^i, y^i, z^i etc. as metavariables for elements of H^i. We call $v^{i_1}_1, \ldots, v^{i_n}_n$ a *good sequence* for the formula φ iff it is a sequence of distinct variables containing all the variables which occur free in φ. If $v^{i_1}_1, \ldots, v^{i_n}_n$ is good we write:

$$\models_M \varphi[v^{i_1}_1, \ldots, v^{i_n}_n \setminus x^{i_1}_1, \ldots, x^{i_n}_n]$$

to mean that φ becomes true if $v^{i_n}_h$ is interpreted by $x^{i_n}_h (h = s, \ldots, n)$. We shall follow normal usage in suppressing the sequence $v^{i_1}_1, \ldots, v^{i_n}_n$ writing only:

$$\models_M \varphi[x^{i_1}_1, \ldots, x^{i_n}_n].$$

(However, it is often important for our understanding to retain the upper indices $i_1, \ldots, i_n$.) We often write $\varphi = \varphi(v_1^{i_1}, \ldots, v_n^{i_n}))$ to indicate that these are the suppressed variables. φ (together with $(v_1^{i_1}, \ldots, v_n^{i_n})$ defines a relation:

$$R(x_1^{i_1}, \ldots, x_n^{i_n}) \leftrightarrow \models_M \varphi[x_1^{i_1}, \ldots, x_n^{i_n}].$$

Since we are using a many sorted language, however, we must also employ *many sorted relations.*

The number of argument places of an ordinary one sorted relation is often called its "arity". In the case of a many sorted relation, however, we must know not only the number of argument places, but also the type of each argument place. We refer to this information as its "arity". Thus the arity of the above relation is not n but $\langle i_1, \ldots, i_n \rangle$. An ordinary 1–sorted relation is usually identified with its field. We shall identify a many sorted relation with the pair consisting of its field and its arity:

Definition 2.6.5. A *many sorted relation* R on M is a pair $\langle |R|, r \rangle$ such that for some n:

(a) $|R| \subset M^n$

(b) $r = \langle r_1, \ldots, r_n \rangle$ where $r_i < \omega$

(c) $R(x_1, \ldots, x_n) \rightarrow x_i \subset H^{\nu_i}$ for $i = 1, \ldots, n$.

$|R|$ is called the *field* of R and r is called the *arity* of R.

In practice we adopt a rough and ready notation, writing $R(x_1^{i_1}, \ldots, x_n^{i_n})$ to indicate that R is a many sorted relation of arity $\langle i_1, \ldots, i_n \rangle$.

(**Note** Let $\mathbb{L} = \mathbb{L}_M$ be the ordinary first order language of M (i.e. it has only variables of type 0). Since $H^n \in M$ or $H^n = M$ for all $n < \omega$, it follows that every $\mathbb{L}^*$–definable many sorted relation $R(R(x_1^{i_1}, \ldots, x_n^{i_n})$ has a field which is $\mathbb{L}$–definable in parameters from M.)

(**Note** If R is a relation of arity $\langle i_1, \ldots, i_n \rangle$, then its *complement* is $\Gamma \setminus R$, where:

$$\Gamma = \{\langle x_1, \ldots, x_n \rangle | x_h \in H^{i_n} \text{ for } h = 1, \ldots, n\},$$

the arity remaining unchanged.)

Definition 2.6.6. $R(x_1^{i_1}, \ldots, x_m^{i_m})$ is a $\Sigma_h^{(n)}(M)$ *relation* iff it is defined by a $\Sigma_h^{(n)}$ formula. R is $\Sigma_h^{(n)}(M)$ *in the parameters* $p_1, \ldots, p_r$ iff $R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p})$, where R' is $\Sigma_h^{(n)}(M)$. R is a $\underline{\Sigma}_k^{(n)}(M)$ relation iff it is $\Sigma_h^{(n)}(M)$ in some parameters.

It is easily checked that:

Lemma 2.6.1.

- *If* $R(y^n, \vec{x})$ *is* $\Sigma_1^{(n)}$, *so is* $\bigvee y^n R(y^n, \vec{x})$
- *If* $R(\vec{x}), P(\vec{x})$ *are* $\Sigma_1^{(n)}$, *then so are* $R(\vec{x}) \vee P(\vec{x}),\ R(\vec{x}) \wedge P(\vec{x})$.

Moreover, if $R(x_0^{i_0}, \ldots, x_{m-1}^{i_{m-1}})$ is $\Sigma_1^{(n)}$, so is any relation $R'(y_0^{j_0}, \ldots, y_{r-1}^{j_{r-1}})$ obtained from R by permutation of arguments, insertion of dummy arguments and fusion of arguments having the same type — i.e.

$$R'(y_0^{j_0}, \ldots, y_{r-1}^{j_{r-1}}) \leftrightarrow R(y_{\sigma(0)}^{j_{\sigma(0)}}, \ldots y_{\sigma(m-1)}^{j_{\sigma(m-1)}})$$

where $\sigma : m \to r$ such that $j_{\sigma(l)} = i_l$ for $l < m$.

Using this we get the analogue of Lemma 2.5.6

Lemma 2.6.2. *Let* $M = \langle J_\alpha^A, B\rangle$ *be acceptable. Let* $\rho = \rho^n$, $H = H^n$. *Then*

(a) If $\rho \in M$, *then* ρ *is a cardinal in* M. *(Hence* $H = H_\rho^M$*)*

(b) If D *is* $\underline{\Sigma}_1^{(n)}(M)$ *and* $D \subset H$, *then* $\langle H, D\rangle$ *is amenable.*

(c) If $u \in H$, *there is no* $\Sigma_1^{(n)}(M)$ *partial map of* u *onto* H.

(d) $\rho \in \mathrm{Lm}^*$ *if* $n > 0$.

Proof: By induction on n. The induction step is a virtual repetition of the proof of Lemma 2.5.6.

Definition 2.6.7. Let $R(x_1^{i_1}, \ldots, x_m^{i_m})$ be a many sorted relation. By an *n–specialization* of R we mean a relation $R(x_1^{j_1}, \ldots, x_m^{j_m})$ such that

- $j_l \geq i_l$ for $l = 1, \ldots, m$
- $j_l = i_l$ if $l < n$
- If $z_1, \ldots, z_m$ are such that $z_l \in H^{j_l}$ for $l = 1, \ldots, m$, then: $R(\vec{z}) \leftrightarrow R'(\vec{z})$.

Given a formula φ in which all bound quantifiers are of type $\leq n$, we can easily devise a formula φ' which defines a specialization of the relation defined by φ:

Fact Let $\varphi = \varphi(v_1^{i_1}, \ldots, v_m^{i_m})$ be a formula in which all bound variables are of type $\leq n$. Let $u_1^{j_1}, \ldots, u_m^{j_m}$ be a sequence of district variables such that $j_l \geq i_l$ and $j_l = i_l$ if $i_l < n(l = 1, \ldots, m)$. Suppose that $\varphi' = \varphi'(\vec{u})$ is obtained by replacing each free occurence of $v_l^{i_l}$ by a free occurence of $u_l^{j_l}$ for $l = 1, \ldots, m$. Then for all $x_1, \ldots, x_m$ such that $x_l \in H^{j_l}$ for $l = 1, \ldots, m$ we have:

$$\models_M \varphi(\vec{v})[\vec{x}] \leftrightarrow \models_M \varphi'(\vec{u})[\vec{x}].$$

The proof is by induction on φ. We leave it to the reader. Using this, we get:

Lemma 2.6.3. *Let $R(x_1^{i_1}, \ldots, x_m^{i_m})$ be $\Sigma_l^{(n)}$. Then every n–specialization of R is $\Sigma_l^{(n)}$.*

Proof: $R'(x_1^{i_1}, \ldots, x_m^{i_m})$ be an n–spezialization. Let R be defined by $\varphi(v_1^{i_1}, \ldots, v_m^{i_m})$. Suppose $(u_1^{j_1}, \ldots, v_m^{j_m})$ is a sequence of distinct variables which are new — i.e. none of them occur free or bound in φ. Let φ' be obtained by replacing every free occurence of $v_l^{i_l}$ by $u_l^{j_l}(l = 1, \ldots, m)$. Then $\varphi'(u_1^{j_1}, \ldots, v_m^{j_m})$ defines R' by the above fact. QED (Lemma 2.6.3)

Corollary 2.6.4. *Let R be $\Sigma_1^{(n)}$ in the parameter p. Then every n–spezialization of R is $\Sigma_1^{(n)}$ in p.*

Lemma 2.6.5. *Let $R'(x_1^{j_1}, \ldots, x_m^{j_m})$ be $\Sigma_1^{(n)}$. Then R' is an n–specialization of a $\Sigma_1^{(n)}$ relation $R(x_1^{i_1}, \ldots, x_m^{i_m})$ such that $i_l \leq n$ for $l = 1, \ldots, m$.*

Proof: Let R' be defined by $\varphi'(u_1^{j_1}, \ldots, v_m^{j_m})$, when φ' is $\Sigma_1^{(n)}$. Let $v_1^{i_n}, \ldots, v_m^{i_m}$ be a sequence of distinct new variables, where $i_l = \min(n, j_l)$ for $l = 1, \ldots, m$. Replace each free occurence of $u_l^{j_l}$ by $v_l^{i_l}$ for $l = 1, \ldots, m$ to get $\varphi(u_1^{i_1}, \ldots, v_m^{i_m})$. Let R be defined by φ. Then R' is a specialization of R by the above fact. QED (Lemma 2.6.5)

Corollary 2.6.6. *Let $R'(x_1^{j_1}, \ldots, x_m^{j_m})$ be $\Sigma_1^{(n)}$ in p. Then R' is a specialization of a relation $R(x_1^{i_1}, \ldots, x_m^{i_m})$ which is $\Sigma_1^{(n)}$ in p with $i_l \leq n$ for $l = 1, \ldots, m$.*

Every $\Sigma_1^{(m)}$ formula can appear as a "primitive" component of a $\Sigma_1^{(m+1)}$ formula. We utilize this fact in proving:

Lemma 2.6.7. *Let $n = m+1$. Let $Q_j(z_{j,1}^n, \ldots, z_{j,p_j}^n, x_1^{i_1}, \ldots, x^{i_p})$ be $\Sigma_1^{(m)}(j = 1, \ldots, r)$.*
Set: $Q_{j,\vec{x}} =: \{\langle \vec{z}_j^n \rangle | Q_j(\vec{z}_j^n, \vec{x})\}$.

Set: $H_{\vec{x}} =: \langle H^n, Q_{1,\vec{x}}, \ldots, Q_{r,\vec{x}}\rangle$.
Let $\varphi = \varphi(v_1, \ldots, v_q)$ *be* Σ_l *in the language of* $H_{\vec{x}}$. *Then*

$$\{\langle \vec{x}^n, \vec{x}\rangle | H_{\vec{x}} \models \varphi[\vec{x}^n]\} \textit{ is } \Sigma_l^{(n)}.$$

Proof: We first prove it for $l = 0$, showing by induction on φ that the conclusion holds for any sequence $v_1, \ldots, v_l$ of variables which is good for φ.

We describe some typical cases of the induction.

Case 1 φ is primitive.
Let e.g. $\varphi = \dot{Q}_j(v_{h_1}, \ldots, v_{h_{p_i}})$, where $\dot{Q}_j$ is the predicate for $Q_{j\vec{x}}$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to: $Q_j(x^n_{h_1}, \ldots, x^n_{h_{p_j}}, \vec{x})$, which is $\Sigma_1^{(m)}$ (hence $\Sigma_0^{(n)}$). QED (Case 1)

Case 2 φ arises from a sentential operation.
Let e.g. $\varphi = (\varphi_0 \wedge \varphi_1)$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to:

$$H_{\vec{x}} \models \varphi_0[\vec{x}^n] \wedge H_{\vec{x}} \models \varphi_1[\vec{x}^n]$$

which, by the induction hypothesis is $\Sigma_0^{(n)}$. QED (Case 2)

Case 3 φ arises from a quantification.
Let e.g. $\varphi = \bigwedge w \in v_i \Psi$. By bound relettering we can assume *w.l.o.g.* that w is not among $v_1, \ldots, v_p$. We apply the induction hypothesis to $\Psi(w, v_1, \ldots, v_p)$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to:

$$\bigwedge z \in x_i^n H_{\vec{x}} \models \Psi[w, \vec{x}^n]$$

which is $\Sigma_0^{(n)}$ by the induction hypothesis. QED (Case 3)

This proves the case $l = 0$. We then prove it for $l > 0$ by induction on l, essentially repeating the proof in case 3. QED (Lemma 2.6.7)

Note It is clear from the proof that the set $\{\langle \vec{x}^n, \vec{x}\rangle | H_{\vec{x}} \models \varphi[\vec{x}^n]\}$ is *uniformly* $\Sigma_l^{(n)}$ — i.e. its defining formula χ depends only on φ and the defining formula Ψ_i for $Q_i (i = 1, \ldots, p)$. In fact, the proof implicitly describes an algorithm for the function $\varphi, \Psi_1, \ldots, \Psi_p \mapsto \chi$.

We can invert the argument of Lemma 2.6.7 to get a weak converse:

Lemma 2.6.8. *Let* $n = m + 1$. *Let* $R(\vec{x}^n, x_1^{i_1}, \ldots, x_g^{i_g})$ *be* $\Sigma_l^{(n)}$ *where* $i_l \leq m$ *for* $l = 1, \ldots, g$. *Then there are* $\Sigma_1^{(n)}$ *relations* $Q_i(\vec{z}_i^n, \vec{x})(i = 1, \ldots, p)$ *and a* Σ_l *formula* φ *such that*

$$R(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n],$$

where $H_{\vec{x}}$ is defined as above.

(**Note** This is weaker, since we now require $i_l \leq m$.)

Proof: We first prove it for $l = 0$. By induction on χ we prove:

Claim Let χ be $\Sigma_0^{(n)}$. Let $\vec{v}^n, v_1^{i_1}, \ldots, v_q^{i_q}$ be good for χ, where $i_1, \ldots, i_q \leq m$. Let $\chi(\vec{v}^n, \vec{v})$ define the relation $R(\vec{x}^n, \vec{x})$. Then the conclusion of Lemma 2.6.8 holds for this R (with $l = 0$).

Case 1 χ is $\Sigma_1^{(m)}$.
Let $\chi(\vec{x}^n, \vec{x})$ define $Q(\vec{x}^n, \vec{x})$. Then $R(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \dot{Q}\vec{v}^n[\vec{x}^n]$.

QED (Case 1)

Case 2 χ arises from a sentential operation.
Let e.g. $\chi = (\Psi \wedge \Psi')$. Appliyng the induction hypothesis we get $Q_i(\vec{x}_i^n, \vec{x})(i = 1, \ldots, p)$ and φ such that

$$M \models \Psi[\vec{x}^n, \vec{x}] \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n]$$

where $H_{\vec{x}} = \langle H^n, Q_{1\vec{x}}, \ldots, Q_{p\vec{x}} \rangle$. Similarly we get $Q'_i(\vec{y}_i^n, \vec{x})(i = 1, \ldots, q')$ and φ'

$$M \models \Psi'[\vec{x}^n, \vec{x}] \leftrightarrow H'_{\vec{x}} \models \varphi'[\vec{x}^n].$$

Let $\dot{Q}_i$ be the predicate for $Q_{i\vec{x}}$ in the language of $H_{\vec{x}}$. Let $\dot{Q}'_i$ be the predicate for $Q'_{i\vec{x}}$ in the language of $H'_{\vec{x}}$. Assume *w.l.o.g.* that $\dot{Q}_i \neq \dot{Q}'_j$ for all i, j. Putting the two languages together we get a language for

$$H^*_{\vec{x}} = \langle H^n, \vec{Q}_{\vec{x}}, \vec{Q}'_{\vec{x}} \rangle.$$

Clearly:

$$M \models (\chi \wedge \chi')[\vec{x}^n, \vec{x}] \leftrightarrow H^*_{\vec{x}} \models (\varphi \wedge \varphi')[\vec{x}^n].$$

QED (Case 2)

Case 3 χ arises from the application of a bounded quantifier.
Let e.g. $\chi = \bigwedge w^n \in v_j^n \chi'$. By bound relettering we can assume *w.l.o.g.* that w^n is not among $\vec{v}^n$. Then $w^n\vec{v}^n, \vec{v}$ is a good sequence for χ' and by the induction hypothesis we have for $\chi' = \chi'(w^n, \vec{v}^n, \vec{v})$:

$$M \models \chi'[z^n, \vec{x}^n, x] \leftrightarrow H_{\vec{x}} \models \varphi[z^n, \vec{x}^n, \vec{x}].$$

But then:

$$\begin{aligned} M \models \chi[\vec{x}^n, \vec{x}] &\leftrightarrow \bigwedge z^n \in x_j^n M \models \chi'[z^n, \vec{x}^n, \vec{x}] \\ &\leftrightarrow \bigwedge z^n \in x_j^n H_{\vec{x}} \models \varphi[z^n, \vec{x}^n] \\ &\leftrightarrow H_{\vec{x}} \models \bigwedge w \in v_j \varphi[\vec{x}^n]. \end{aligned}$$

QED (Lemma 2.6.8)

Note Our proof again establishes uniformity. In fact, if χ is the $\Sigma_l^{(n)}$–definition of R, the proof implicitely describes an algorithm for the function

$$\chi \mapsto \varphi, \Psi_1, \ldots, \Psi_p$$

where Ψ_i is a $\Sigma_1^{(m)}$ definition of Q_i.

Remark Lemma 2.6.7 and 2.6.8 taken together give an inductive definition of "$\Sigma_l^{(n)}$ relation" which avoids the many sorted language. It would, however, be difficult to work directly from this definition.

By a function of *arity* $\langle i_1, \ldots, i_n \rangle$ *to* H^j we mean a relation $F(y^j, x^{i_1}, \ldots, x^{i_n})$ such that for all $x^{i_1}, \ldots, x^{i_n}$ there is at most one such y^j. If this y exists, we denote it by $F(x^{i_1}, \ldots, x^{i_n})$. Of particular interest are the $\Sigma_1^{(i)}$ functions to H^i.

Lemma 2.6.9. *$R(y^n, \vec{x})$ be a $\Sigma_1^{(n)}$ relation. Then R has a $\Sigma_1^{(n)}$ uniformizing function $F(\vec{x})$.*

Proof: We can assume *w.l.o.g* that the arguments of R are all of type $\leq n$. (Otherwise let R be a specialization of R', where the arguments of R' are of type $\leq n$. Let F' uniformize R'. Then the appropriate specialization F of F' uniformizes R.)

Case 1 $n = 0$.
Set:

$$F(\vec{x}) \simeq: y \text{ where } \langle z, y \rangle \text{ is } <_M \text{–least such that } R'(z, y, \vec{x}).$$

By section 2.3 we know that $u_M(x)$ is Σ_1, where $u_M(x) = \{y | y <_M x\}$. Thus for sufficient r we have:

$$\begin{gathered} y = F(\vec{x}) \leftrightarrow \bigvee z (R'(z, y, \vec{x}) \wedge \\ \wedge w \in u_M(\langle z, y \rangle) \bigwedge z', y' \in C_r(w) \\ (w = \langle z', y' \rangle \rightarrow \neg R(z', y', \vec{x})), \end{gathered}$$

which is uniformly $\Sigma_1(M)$.

Case 2 $n > 0$. Let $n = m + 1$.
Rearranging the arguments of R if necessary, we can assume that R has the form $R(y^n, \vec{x}^n, \vec{x})$, where the $\vec{x}$ are of type $\leq m$. Then there are $Q_i(\vec{z}_i^n, \vec{x}^n, \vec{x})(i = 1, \ldots, p)$ such that Q_i is $\Sigma_1^{(m)}$ and

$$R(y^n, \vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[y^n, \vec{x}^n],$$

where φ is Σ_1 and

$$H_{\vec{x}} = \langle H^n, Q_{1\vec{x}}, \ldots, Q_{n\vec{x}} \rangle.$$

If e.g. $M = \langle J^A, B \rangle$, we can assume *w.l.o.g.* that $Q_1(z^n, \vec{x}) \leftrightarrow A(z^n)$. Then $<_{H\vec{x}}, u_{H\vec{x}}$ are uniformly $\Sigma_1(H_{\vec{x}})$ and by the argument of Case 1 there is a Σ_1 formula φ' such that F uniformies R where

$$y = F(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi'[\vec{x}^n, \vec{x}].$$

QED (2.6.9)

Note The proof shows that $F(\vec{x})$ is *uniformly* $\Sigma_1^{(n)}$ — i.e. its $\Sigma_1^{(n)}$ definition depends only on the $\Sigma_1^{(n)}$ definition of $R(y^n, \vec{x})$, regardless of M.

Note It is clear from the proof that the $\Sigma_1^{(n)}$ definition of F is *functionally absolute* — i.e. it defines a function over every acceptable M of the same type. Thus:

Corollary 2.6.10. *Every $\Sigma_1^{(n)}$ function $F(\vec{x})$ to H^n has a functionally absolute $\Sigma_1^{(n)}$ definition.*

Note The $\Sigma_1^{(n)}$ functions are closed under permutation of arguments, insertion of dummy arguments, and fusion of arguments of same type. Thus if $F(x_1^{i_1}, \ldots x_n^{i_n})$ is $\Sigma_1^{(n)}$, so is $F'(y_1^{j_1}, \ldots, y_m^{j_m})$ where

$$F'(y_1^{j_1}, \ldots, y_m^{j_m}) \simeq F(y_{\sigma(1)}^{j_{\sigma(1)}}, \ldots, y_{\sigma(n)}^{j_{\sigma(n)}})$$

and $\sigma : n \to m$ such that $j_{\sigma(l)} = i_l$ for $l < n$.

If $R(x_1^{j_1}, \ldots, x_p^{j_p})$ is a relation and $F_i(\vec{z})$ is a function to H^{j_i} for $i = 1, \ldots, n$, we sometimes use the abbreviation:

$$R(\vec{F}(\vec{z})) \leftrightarrow: \bigvee x_1^{j_1}, \ldots x_p^{j_p} (\bigwedge_{i=1}^{p} x_i^{j_i} = F_i(\vec{z}) \wedge R(\vec{x})).$$

Note that $R(\vec{F}(\vec{z}))$ is then false if some $F_i(\vec{z})$ does not exist. $\Sigma_1^{(n)}$ relations are not, in general, closed under substitution of $\Sigma_1^{(n)}$ functions, but we do get:

Lemma 2.6.11. *Let $R(x_1^{j_1}, \ldots, x_p^{j_p})$ be $\Sigma_1^{(n)}$ such that $j_i \leq n$ for $i = 1, \ldots, p$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(j_i)}$ map to H^{j_i} for $i = 1, \ldots, p$. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$ (uniformly in the $\Sigma_1^{(n)}$ definitions of $R, F_1, \ldots, F_8$)*

Before proving Lemma 2.6.11 we show that it has the following corollary:

Corollary 2.6.12. *Let* $R(\vec{x}, y_1^{j_1}, \ldots, y_p^{j_p})$ *be* $\Sigma_1^{(n)}$ *where* $j_i \leq n$ *for* $i = 1, \ldots, p$. *Let* $F_i(\vec{z})$ *be a* $\Sigma_1^{(j_i)}$ *map to* H^{j_i} *for* $i = 1, \ldots, p$. *Then* $R(\vec{x}, \vec{F}(\vec{z}))$ *is (uniformly)* $\Sigma_1^{(n)}$.

Proof: We can assume *w.l.o.g.* that each of $\vec{x}$ has type $\leq n$, since otherwise R is a specialization of an R' with this property. But then $R(\vec{x}, \vec{F}(z))$ is a specialization of $R'(\vec{x}, \vec{F}(z))$. Let $\vec{x} = x_1^{h_1}, \ldots, x_q^{h_q}$ with $h_i \leq n$ for $i = 1, \ldots, q$. For $i = 1, \ldots, p$ set:

$$F'(\vec{x}, \vec{z}) \simeq F(\vec{z}).$$

For $i = 1, \ldots, q$ set:

$$G_h(\vec{x}, \vec{z}) \simeq x_i^{h_i}.$$

By Lemma 2.6.11, $R(\vec{G}(\vec{x}, \vec{z}), F'(\vec{x}, \vec{z}))$ is $\Sigma_1^{(n)}$. But

$$R(\vec{G}(\vec{x}, \vec{z}), F'(\vec{x}, \vec{z})) \leftrightarrow R(\vec{x}, \vec{F}(\vec{z})).$$

QED (Corollary 2.6.12)

We now prove Lemma 2.6.11 by induction on n.

Case 1 $n = 0$.

The conclusion is immediate by the definition of $R(\vec{F}(\vec{z}))$:

$$R(\vec{F}(\vec{z})) \leftrightarrow \bigvee x_1^0 \ldots x_p^0 (\bigwedge_{i=1}^{p} x_1^0 = F_i(\vec{z}) \wedge R(\vec{x})).$$

Case 2 $n = m + 1$.

Then Lemma 2.6.11 holds at m and it is clear from the above proof that Corollary 2.6.12 does, too.

Rearranging the arguments of R if necessary, we can bring R into the form:

$$R(\vec{x}^n, x_1^{l_1}, \ldots, x_q^{l_q}) \text{ where } l_i \leq m \text{ for } i = 1, \ldots, q.$$

We first show:

Claim $R(\vec{x}^n, \vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$.

Proof: Let $Q_i(\vec{z}_i^n, \vec{x})$ be $\Sigma_1^{(m)}(i = 1, \ldots, r)$ such that

$$R(x^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n]$$

where φ is Σ_1 and:

$$H_{\vec{x}} = \langle H^n, Q_{1,\vec{x}}, \ldots, Q_{r,\vec{x}} \rangle.$$

Set:

$$\begin{aligned} \overline{Q}_i(\vec{z}_i^n, \vec{z}) &\leftrightarrow: Q_i(z_i^n, F(\vec{z})) \\ &\leftrightarrow \bigvee \vec{x}(\textstyle\bigwedge_{i=1}^q x_i^{l_i} = F_i(\vec{z}) \wedge R(\vec{x})) \\ \overline{H}_{\vec{z}} &=: \langle H^n, \overline{Q}_{1,\vec{z}}, \ldots, \overline{Q}_{r,\vec{z}} \rangle. \end{aligned}$$

If $x_i^{l_i} = F_i(\vec{z})$ for $i = 1, \ldots, q$, then $\overline{Q}_i(\vec{z}_i^n, \vec{z}) \leftrightarrow Q_i(\vec{z}^n, \vec{x})$ and $\overline{H}_{\vec{z}} = H_{\vec{x}}$. Hence:

$$\begin{aligned} \overline{H}_{\vec{z}} \models \varphi[\vec{x}^n] &\leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n] \\ &\leftrightarrow R(\vec{x}^n, \vec{x}) \\ &\leftrightarrow R(\vec{x}^n, \vec{F}(\vec{z})). \end{aligned}$$

If, on the other hand, $F_i(\vec{z})$ does not exist for some i, then $R(\vec{x}^n, \vec{F}(\vec{z}))$ is false. Hence:

$$\begin{aligned} R(\vec{x}^n, \vec{F}(\vec{z})) &\leftrightarrow (\textstyle\bigwedge_{i=1}^q \bigvee x_i^{l_i}(x_i^{l_i} = F_i(\vec{z})) \\ &\wedge \overline{H}_{\vec{z}} \models \varphi[\vec{x}^n]). \end{aligned}$$

But $\bigwedge\limits_{i=1}^{q} \bigvee x_i^{l_i}(x_i^{l_i} = F_i(\vec{z}))$ is $\Sigma_0^{(n)}$, so the result follows by applying Lemma 2.6.7 to φ. QED (Claim)

But then, setting: $R'(\vec{x}^n, \vec{z}) \leftrightarrow R(\vec{x}^n, F(\vec{z}))$, we have:

$$R(\vec{F}(\vec{x})) \leftrightarrow \vee \vec{x}^n (\bigwedge_{i=1}^{q} x_i^n = F_i(\vec{z}) \wedge R'(\vec{x}^n, \vec{z})).$$

QED (Lemma 2.6.11)

Note that if, in the last claim, we took $R(\vec{x}^n, x_1^{l_1}, \ldots, x_q^{l_q})$ as being $\Sigma_0^{(n)}$ instead of $\Sigma_1^{(n)}$, then in the proof of the claim we could take φ as being Σ_0 instead of Σ_1. But then the application of Lemma 2.6.7 to $\overline{H}_{\vec{z}} \models \varphi[\vec{x}^n]$ yields a $\Sigma_0^{(n)}$ formula. Then we have, in effect, also proven:

Corollary 2.6.13. *Let $R\vec{x}^n, y_1^{l_1}, \ldots, y_q^{l_q})$ be $\Sigma_0^{(n)}$ where $l_1, \ldots, l_r < n$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(l_i)}$ map to H^{l_i} for $i = 1, \ldots, r$. Then $R(x^n, \vec{F}(\vec{z}))$ is (uniformly) $\Sigma_0^{(n)}$.*

As corollaries of Lemma 2.6.11 we then get:

Corollary 2.6.14. *Let $G(x_1^{j_1}, \ldots, x_p^{j_p})$ be a $\Sigma_1^{(n)}$ map to H^n, where $j_1, \ldots, j_p \leq n$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(n)}$ map to H^{j_i} for $i = 1, \ldots, p$. Then $H(\vec{z}) \simeq G(\vec{F}(\vec{z}))$ is uniformly $\Sigma_1^{(n)}$.*

Proof:

$$y = H(\vec{z}) \leftrightarrow \bigvee \vec{x}(\bigwedge_{i=1}^{p} x_i^{j_i} = F_i(\vec{z}) \wedge y = G(\vec{x})).$$

QED (Corollary 2.6.14)

Corollary 2.6.15. *Let $R(x_1^{j_1}, \ldots, x_p^{j_p})$ be $\Sigma_1^{(n)}$ where $j_i \leq n$ for $i = 1, \ldots, p$. There is a $\Sigma_1^{(n)}$ relation $R'(z_1^0, \ldots, z_p^0)$ with the same field*

Proof: Set:

$$R'(\vec{z}) \leftrightarrow: \bigvee \vec{x}(\bigwedge_{i=1}^{p} x_i^{j_i} = z_i^0 \wedge R(\vec{x})).$$

QED (Corollary 2.6.15)

Thus in theory we can always get by with relations that have only arguments of type 0. (Let one make too much of this, however, we remark that the defining formula of R' will still have bounded many sorted variables.)

Generalizing this, we see that if R is a relation with arguments of type $\leq n$, then the property of being $\Sigma_1^{(n)}$ depends only on the field of R. Let us define:

Definition 2.6.8. $R'(z_1^{j_1}, \ldots, z_r^{j_r})$ is a *reindexing* of the relation $R(x_1^{i_1}, \ldots, x_r^{i_r})$ iff both relations have the same field i.e.

$$R'(\vec{y}) \leftrightarrow R(\vec{y}) \text{ for } y_1, \ldots, y_r \in M.$$

Then:

Corollary 2.6.16. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_1^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $R'(z_1^{j_1}, \ldots, z_r^{j_r})$ be a reindexing of R, where $j_1, \ldots, j_r \leq n$. Then R' is $\Sigma_1^{(n)}$.*

Proof:

$$\begin{aligned} R'\vec{z}) &\leftrightarrow R(F_1(z_1), \ldots, F_r(z_r)) \\ &\leftrightarrow \vee \vec{x}(\bigvee_{l=1}^{r} x_l^{i_l} = z_l^{j_l} \wedge R(\vec{x})) \end{aligned}$$

where

$$x^{i_l} = F_l(z^{j_l}) \leftrightarrow: x^{i_l} = z^{j_l}.$$

QED (Corollary 2.6.16)

We now consider the relationship between Σ^* theory and the theory developed in §2.5. $\Sigma_1^{(0)}$ is of course the same as Σ_1 and ρ_1 is the same as the Σ_1 projectum ρ which we defined in §2.5.2. In §2.5.2 we also defined the set P of good parameters and the set R of very good parameters. We then define the reduct M of M for any $\in [\mathrm{On}_M]^{<\omega}$. We now generalize these notions to $\Sigma_1^{(n)}$. We have already defined the $\Sigma_1^{(n)}$ projectum ρ^n. In analogy with the above we now define the sets P^n, R^n of $\Sigma_1^{(n)}$–*good* parameters. We also define the $\Sigma_1^{(n)}$ *reduct* M^{np} of M by $p \in [\mathrm{On}_M]^{<\omega}$].

Under the special assumption of soundness, there will turn out to be the same as the concepts defined in §2.5.3.

Definition 2.6.9. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. We define sets $M^n_{x^{n-1},\ldots,x^0}$ and predicates $T^n(x^n,\ldots,x^0)$ as follows:

$$\begin{aligned} M^0 =&: M, T^0 =: B \text{ (i.e. } M^n_{\vec{x}} = M \text{ for } n = 0)\\ M^{n+1}_{\vec{x}} =&:\langle J^A_{\rho^{n+1}}, T^{n+1}_{\vec{x}}\rangle \text{ for } \vec{x} = x^n,\ldots,x^0\\ T^{n+1}(x^{n+1},\vec{x}) &\leftrightarrow \bigvee z^{n+1} \bigvee i<\omega (x^{n+1} = \langle i, z^{n+1}\rangle\\ &\wedge M^n_{x^{n+1},\ldots,x^0} \models \varphi_i[z^{n+1},x^n]) \end{aligned}$$

(where $\langle \varphi_i | i<\omega\rangle$ is our fixed canonical enumeration of Σ_1 formulae.)

(Then $T^{n+1}(\langle i, x^{n+1}\rangle, x^n,\ldots,x^0) \leftrightarrow M^n_{x^{n-1},\ldots,x^0} \models \varphi_i[x^{n+1},x^n]$).

Clearly T^{n+1} is uniformly $\Sigma_1^{(n)}(M)$.

Lemma 2.6.17.

(a) T^{n+1} is $\Sigma_1^{(n)}$

(b) Let φ be Σ_j. Then $\{\langle \vec{x}^{n+1},\vec{x}\rangle | M^{n+1}_{\vec{x}} \models \varphi[\vec{x}^{n+1}]\}$ is $\Sigma_j^{(n+1)}$.

Proof: We first note that $M^{n+1}_{\vec{x}}$ can be written as $H_{\vec{x}} = \langle H^{n+1}, A^{n+1}_{\vec{x}}, T^{n+1}_{\vec{x}}\rangle$, where $A^{n+1}(x^{n+1},\vec{x}) \leftrightarrow: A(x^{n+1})$. Hence by Lemma 2.6.7:

(1) If (a) holds at n, so does (b). But (a) then follows by induction on n:

Case 1 $n = 0$ is trivial since $\Vdash^{\Sigma_1}_N$ is $\Sigma_1(N)$ for all rud closed N.

Case 2 $n = m+1$. Then $T^{(n+1)}$ is $\Sigma_1^{(n)}$ by (1) applied to m.

QED (Lemma 2.6.17)

We now prove *a* converse to Lemma 2.6.17.

Lemma 2.6.18. *(a) Let $R(x^{n+1},\dots,x^0)$ be $\Sigma_1^{(n)}$. Then there is $i<\omega$ such that*

$$R(x^{n+1},\vec{x}) \leftrightarrow T^{n+1}(\langle i, x^{n+1}\rangle, \vec{x}).$$

(b) Let $R(\vec{x}^{n+1},\dots,x^0)$ be $\Sigma_1^{(n+1)}$. Then there is a Σ_1 formula φ such that

$$R(\vec{x}^{n+1},\vec{x}) \leftrightarrow M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}].$$

Proof:

(1) Let (a) hold at n. Then so does (b).

Proof: We know that

$$R(\vec{x}^{n+1},\vec{x}) \leftrightarrow \bigvee z^{n+1} P(z^{n+1},x^{n+1},\vec{x})$$

for a $\Sigma_0^{(n+1)}$ formula P. Hence it suffices to show:

Claim Let $P(\vec{x}^{n+1},\vec{x})$ be $\Sigma_0^{(n+1)}$. Then there is a Σ_1 formula φ such that

$$P(\vec{x}^{n+1},\vec{x}) \leftrightarrow M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}].$$

Proof: We know that there are $Q_i(\vec{z}_i^{n+1},\vec{x})(i=1,\dots,p)$ such that Q_i is $\Sigma_1^{(n)}$ and

(2) $P(\vec{x}^{n+1},\vec{x}) \leftrightarrow H_{\vec{x}}^{n+1} \models \Psi[\vec{x}^{n+1}]$ where Ψ is Σ_0 and

$$H_{\vec{x}}^{n} = \langle H^{n+1}, \vec{Q}_{\vec{x}}\rangle.$$

Applying (a) to the relation:

$$\bigvee u^{n+1}(u^{n+1} = \langle \vec{z}_i^{n+1}\rangle \wedge Q_i(\vec{z}_i^{n+1},\vec{x}))$$

we see that for each i there is $j_i<\omega$ such that

$$Q_i(\vec{z}_i^{n+1},\vec{x}) \leftrightarrow \langle j_i, \langle \vec{z}^{n+1}\rangle\rangle \in T_{vecx}^{n+1}.$$

Thus $Q_i, \vec{x}$ is uniformly rud in $T_{\vec{x}}^{n+1}$ for $i=1,\dots,p$. $P_{\vec{x}}$ is the restriction of a relation rud in $Q_{i,\vec{x}}(i=1,\dots,p)$ to H^{n+1}, by (2). By §2 Corollary 2.2.8 it follows that $P_{\vec{x}}$ is the restriction of a relation rud in $T_{\vec{x}}^{n+1}$ to H^{n+1} uniformly. Since $M_{\vec{x}}^{n+1} = \langle J_{\rho n+1}^{A}, T_{\vec{x}}^{n+1}\rangle$ is rud closed, it follows by §2 Corollary 2.2.8 that:

$$P(\vec{x}^{n+1},\vec{x}) \leftrightarrow M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}]$$

for a Σ_1 formula φ. QED (1)

Given (1) we can now prove (a) by induction on n.

Case 1 $n = 0$.

Since $\Sigma_1 = \Sigma_1^{(0)}$, there is φ_i such that

$$\begin{aligned} R(x^1, x^0) &\leftrightarrow M \models \varphi_i[x^1, x^0] \\ &\leftrightarrow T^1(\langle i, x^1 \rangle, x^0). \end{aligned}$$

Case 2 $n = m + 1$.

Let $R(x^{n+1}, \ldots, x^0)$ be $\Sigma_1^{(n)}$. By the induction hypothesis and (1) we know that (b) holds at n. Hence:

$$\begin{aligned} &R(x^{n+1}, x^{m+1}, x^m, \ldots, x^0) \leftrightarrow \\ &\leftrightarrow M^n_{x^m, \ldots, x^0} \models \varphi_i[x^{n+1}, x^{m+1}] \end{aligned}$$

for some i. But then

$$R(x^{n+1}, \ldots, x^0) \leftrightarrow T^{n+1}(\langle i, x^{n+1} \rangle, x^{m+1}, \ldots, x^0).$$

QED (Lemma 2.6.18)

Note The reductions in (a) and (b) are both uniform. We have in fact implicitly defined algorithms which in case (a) takes us from the $\Sigma_1^{(n)}$ definition of R to the integer i, and in case (b) takes us from the $\Sigma_1^{(n+1)}$ definition of R to the Σ_1 formula φ.

We now generalize the definition of *reduct* given in §2.5.2 as follows:

Definition 2.6.10. Let $a \in [\mathrm{On}_M]^{<\omega}$. $M^{0,a} =: M$; $M^{n+1,a} = M^{n+1}_{a^{(0)}, \ldots, a^{(n)}}$ where $a^{(i)} = a \cap \rho^i_M$.

Thus $M^{n+1,a} = \langle J^A_{\rho^{n+1}}, T^{n+1,a} \rangle$ where $T^{n+1,a}_{a^{(0)}, \ldots, a^{(n)}}$.

Thus by Lemma 2.6.18

Corollary 2.6.19. *Set* $a^{(i)} = a \cap \rho^i$ *for* $a \in [\mathrm{On}_M]^{<\omega}$.

(a) If $D \subset H^{n+1}$ is $\Sigma_1^{(n)}$ in $a^{(0)}, \ldots, a^{(n)}$, there is (uniformly) an $i < \omega$ such that

$$D(x^{n+1}) \leftrightarrow \langle i, x^{n+1} \rangle \in T^{n+1,a}$$

(b) If $D(\vec{x}^{n+1})$ is $\Sigma_1^{(n+1)}$ in $a^{(0)}, \ldots, a^{(n)}$ there is (uniformly) a Σ_1 formula φ such that $D(\vec{x}^{n+1}) \leftrightarrow M^{n+1,a} \models \varphi \vec{x}^{n+1}$.

(**Note** Being $\Sigma_1^{(n)}$ in a is the same as being $\Sigma_1^{(n)}$ in $a^{(0)}, \ldots, a^{(n)}$, but I do not see how this is uniformly so. To see that a $\Sigma_1^{(n)}$ relation R in $a^{(0)}, \ldots, a^{(n)}$ is $\Sigma_1^{(n)}$ in a we note that for each n there is k such that $y = a \cap \rho^n \leftrightarrow \bigvee f$ (f is the monotone enumeration of a and $y = f''k$), which is Σ_1 in a. However, k cannot be inferred from the $\Sigma_1^{(n)}$ definition of R, so the reduction is not uniform.)

We can generalize the good parameter sets P, R of §2.5.2 as follows:

Definition 2.6.11. $P_M^0 =: [\mathrm{On}]^{<\omega}$.

P_M^{n+1} =: the set of $a \in P_M^n$ such that there is D which is $\Sigma_1^{(n)}(M)$ in a with $D \cap H_M^n \notin M$.

(Thus we obviously have $P^1 = P$.)

Similarly:

Definition 2.6.12. $R_M^0 =: P_M^0$.

R_M^{n+1} =: The set of $a \in R_M^n$ such that

$$M^{n,a} = h_{M^{n,a}}(\rho^{n+1} \cup (a \cap \rho^n)).$$

Comparing these definitions with those in §2.5.6 it is apparent that R_M^n has the same meaning and that, whenever $a \in R_M^n$, then $M^{n,a}$ is the same structure.

By a virtual repetition of the proof of Lemma 2.5.8 we get:

Lemma 2.6.20. *$a \in P^n \leftrightarrow T^{na} \notin M$.*

We also note the following fact:

Lemma 2.6.21. *Let $a \in R^n$. Let D be $\underline{\Sigma}_1^{(n)}$. Then D is $\Sigma_1^{(n)}$ in parameters from $\rho^{n+1} \cup \{a^{(0)}, \ldots, a^{(n)}\}$, where $a^{(i)} =: a \cap \rho^i$. (Hence D is $\Sigma_1^{(n)}(M)$ in parameters from $\rho^{n+1} \cup \{a\}$.)*

Proof: We use induction on n. Let it hold below n. Then:

$$D(\vec{x}) \leftrightarrow D'(\vec{x}; a^{(0)}, \ldots, a^{(n-1)}, \vec{\xi}),$$

where $\xi_1, \ldots, \xi_r < \rho^n$. (If $n = 0$ the sequence $a^{(0)}, \ldots, a^{(n-1)}$ is vacuous and $\rho^n = \mathrm{On}_M$.)

Let $\xi_i = h_{M^{n+1}}(j_i, \langle \mu_i, a^{(n)} \rangle)$, where $\mu_1, \ldots, \mu_r < \rho^{n+1}$. The functions:

$$F_i(x) \simeq h_{M^{na}}(j_i, \langle x, a^{(n)} \rangle)$$

are $\Sigma_1^{(n)}$ to H^n in the parameters $a^{(0)}, \ldots, a^{(n)}$. But $D(\vec{x})$ then has the form:

$$D'(\vec{x}, a^{(0)}, \ldots, a^{(n-1)}, F_1(\mu_1), \ldots, F_r(\mu_r)),$$

which is $\Sigma_1^{(n)}$ in $a^{(0)}, \ldots, a^{(n)}, \mu_1, \ldots, \mu_k$ by Corollary 2.6.12.

QED (Lemma 2.6.21)

Definition 2.6.13. π is a $\Sigma_h^{(n)}$ *preserving map of* $\overline{M}$ to M (in symbols $\pi : \overline{M} \to_{\Sigma_h^{(n)}} M$) iff the following hold:

- $\overline{M}, M$ are acceptable structures of the same type.
- $\pi'' H^i_{\overline{M}} \subset H^i_M$ for $i \leq n$.
- Let $\varphi = \varphi(v_1^{j_1}, \ldots, v_m^{j_m})$ be a $\Sigma_h^{(n)}$ formula with a good sequence $\vec{v}$ of variables such that $j_1, \ldots, j_m \leq n$. Let $x_i \in H^{j_i}_{\overline{M}}$ for $i = 1, \ldots, m$. Then:
$$\overline{M} \models \varphi[\vec{x}] \leftrightarrow M \models \varphi[\pi(\vec{x})].$$

π is then a structure preserving injection. If it is $\Sigma_h^{(n)}$–preserving, it is $\Sigma_1^{(m)}$–preserving for $m < n$ and $\Sigma_i^{(n)}$–preserving for $i < h$. If $h \geq 1$ then $\pi^{-1}{}'' H^n_M \subset H^n_{\overline{M}}$, as can be seen using:

$$x \in H^n_M \leftrightarrow M \models V u^n u^n = v^0[x].$$

We say that π is *strictly* $\Sigma_h^{(n)}$ *preserving* (in symbols $\pi : \overline{M} \to_{\Sigma_h^{(n)}} M$ strictly) iff it is $\Sigma_h^{(n)}$ preserving and $\pi^{-1}{}'' H^n \subset \overline{H}^n$. (Only if $h = 0$ can the embedding fail to be strict.)

We say that π is Σ^* *preserving* ($\pi : \overline{M} \to_{\Sigma^*} M$) iff it is $\Sigma_1^{(n)}$ preserving for all $n < \omega$. We call π $\Sigma_\omega^{(n)}$ *preserving* iff it is $\Sigma_h^{(n)}$ preserving for all $h < \omega$.

Good functions

Let $n < \omega$. Consider the class $\mathbb{F}$ of all $\Sigma_1^{(n)}$ functions $F(x^{i_1}, \ldots, x^{i_m})$ to H^j, where $j, i_1, \ldots, i_m \leq n$. This class is not necessarily closed under composition. If, however, $\mathbb{G}^0$ is the class of $\Sigma_1^{(j)}$ functions $G(z^{i_1}, \ldots, z^{i_m})$ to H^j where $j, i_1, \ldots, i_m \leq n$, then $\mathbb{G}^0 \subset \mathbb{F}$ and, as we have seen, elements of $\mathbb{G}^0$

can be composed into elements of $\mathbb{F}$ — i.e. if $F(z^{i_1}, \dots, z^{i_m})$ is in $\mathbb{F}$ and $G_l(\vec{x})$ is in $\mathbb{G}^0$ for $l = 1, \dots, n$, then $F(\vec{G}(\vec{x}))$ lies in $\mathbb{F}$. The class $\mathbb{G}$ of *good* $\Sigma_1^{(n)}$ functions is the result of closing $\mathbb{G}^0$ under composition. The elements of $\mathbb{G}$ are all $\Sigma_1^{(n)}$ functions and $\mathbb{G}$ is closed under composition. The precise definition is:

Definition 2.6.14. Fix acceptable M. We define sets $\mathbb{G}^k = \mathbb{G}_n^k$ of $\Sigma_1^{(n)}$ functions by:

$\mathbb{G}^0 =$ The set of partial $\Sigma_1^{(i)}$ maps $F(x_1^{j_1}, \dots, x_m^{j_m})$ to H^i, where $i \leq n$ and $j_1, \dots, j_m \leq n$.

$\mathbb{G}^{k+1} =$ The set of $H(\vec{x}) \simeq G(\vec{F}(\vec{x}))$, such that $G(y^{j_1}, \dots, y_m^{j_m})$ is in G^k and $F_l \in \mathbb{G}^0$ is a map to j_l for $l = 1, \dots, m$.

It follows easily that $\mathbb{G}^k \subset \mathbb{G}_{k+1}^k$ (since $G(\vec{y}) \simeq G(\vec{h}(\vec{y}))$ where $h(y_1^{j_1}, \dots, y_m^{j_m}) = y_i^{j_i}$ for $i = 1, \dots, m$). $\mathbb{G} = \mathbb{G}_n =: \bigcup_k \mathbb{G}^k$ is then the set of all *good* $\Sigma_1^{(n)}$ *functions* $\mathbb{G}^* = \bigcup_n \mathbb{G}_n$ is the set of all *good* Σ^* *functions.* All good $\Sigma_1^{(n)}$ functions have a functionally absolute $\Sigma_1^{(n)}$ definition. Moreover, the good $\Sigma_1^{(n)}$ functions are closed under permutation of arguments, insertion of dummy arguments, and fusion of arguments of same type (i.e. if $F(x_0^{i_1}, \dots, x_{m-1}^{j_p})$ is good, then so is $F'(\vec{y}) \simeq F(y_{\sigma(1)}^{j_{\sigma(1)}}, \dots, y_{\sigma(m)}^{j_{\sigma(m)}})$ and $\sigma : m \to p$ such that $j_{\sigma(l)} = i_l$ for $l < m$.

To see this, one proves by a simple induction on k that:

Lemma 2.6.22. *Each $\mathbb{G}_n^k$ has the above properties.*

The proof is quite straightforward. We then get:

Lemma 2.6.23. *The good $\Sigma_1^{(n)}$ functions are closed under composition: Let $G(y_1^{j_1}, \dots, y_m^{j_m})$ be good and let $F_l(\vec{x})$ be a good function to H^{j_l} for $l = \dots, m$. Then the function $G(\vec{F}(\vec{x}))$ is good.*

Proof: By induction in $k < \omega$ we prove:

Claim The above holds for $F_l \in \mathbb{G}^k (l = 1, \dots, m)$.

Case 1 $k = 0$.

This is trivial by the definition of "good function".

Case 2 $k = h + 1$.

Let:

$$F_l(\vec{x}) \simeq H_l(F_{l,1}(\vec{x}), \ldots, F_{l,p_l}(\vec{x}))$$

for $l = 1, \ldots, m$, where $H_l(z_{l,1}, \ldots, z_{l,p_l})$ is in $\mathbb{G}^h$ and $F_{l,i} \in G^0$ is a map to $H^{j_{l,i}}$ for $l = 1, \ldots, m, i = 1, \ldots, p_l$.

Let $\langle\langle l_\xi, i_\xi\rangle|\xi = 1, \ldots, p\rangle$ enumerate

$$\{\langle l, i\rangle | l = 1, \ldots, m;\ i = 1, \ldots, p_l\}.$$

Define $\sigma_l : \{1, \ldots, p_l\} \to \{1, \ldots, p\}$ by:

$$\sigma_l(i) = \text{ that } \xi \text{ such that } \langle l, i\rangle = \langle l_\xi, i_\xi\rangle.$$

Set:

$$H'_l(z_1, \ldots, z_p) \simeq H_l(z_{\sigma_l(1)}, \ldots, z_{\sigma_l(p_l)})$$

for $l = 1, \ldots, m$. $F'_\xi = F_{l_\xi, i_\xi}$ for $\xi = 1, \ldots, p$.

Clearly we have:

$$F_l(\vec{x}) = H'_l(F'_1(\vec{x}), \ldots, F'_p(\vec{x}))$$

where $H'_l \in \mathbb{G}^h$ for $l = 1, \ldots, m$. Set:

$$G'(z_1, \ldots, z_p| \simeq G(H_1(\vec{z}), \ldots, H_m(\vec{z})).$$

Then G' is a good $\Sigma_1^{(n)}$ function by the induction hypothesis. But:

$$G(\vec{F}(\vec{x})) \simeq G'(F'_1(\vec{x}), \ldots, F'_p(\vec{x})).$$

The conclusion then follows by Case 1, since $F'_i \in \mathbb{G}^0$ for $i = 1, \ldots, p$.

QED (Lemma 2.6.23)

An entirely similar proof yields:

Lemma 2.6.24. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_1^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $F_l(\vec{z})$ be a good $\Sigma_1^{(n)}$ map to $H^{i_l}(L = 1, \ldots, m)$. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$.*

(Recall that $R(\vec{F}, \vec{z}))$ means:

$$\bigvee y_1, \ldots, y_r(\bigwedge_{l=1}^{r} y_l = F(\vec{z}) \wedge R(\vec{y})).)$$

Applying Corollary 2.6.13 we also get:

Lemma 2.6.25. *Let $n = m+1$. Let $R(\vec{x}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_r \leq m$. Let $F_l(\vec{z})$ be a good $\Sigma_1^{(n)}$ map to H^{i_l} for $l = 1, \ldots, r$. Then $R(\vec{x}^n, F(\vec{z}))$ in $\Sigma_0^{(n)}$.*

By a *reindexing* of a function $G(x_1^{i_1}, \ldots, x_r^{i_r})$ we mean any function G' which is a reindexing of G as a relation. (In other words G, G' have the same field, i.e.

$$G(\vec{x}) \simeq G'(\vec{x}) \text{ for all } x_1, \ldots, x_r \in M.)$$

Then:

Corollary 2.6.26. *Let $G(x_1^{i_1}, \ldots, x_r^{i_r})$ be a good $\Sigma_1^{(m)}$ map to H^i. Let $G'(y_1^{j_1}, \ldots, y_r^{j_r})$ be a map to H^j, where $j, j_1, \ldots, j_r \leq n$. If G' is a reindexing of G, then G' is a good $\Sigma_1^{(m)}$ function.*

Proof: $G'(y) \simeq F(G(F_1(y_1^{j_1}), \ldots, F(y_r^{j_r})))$ where F is defined by $x^i = y^i$ and F_l is defined by $x_l^{i_l} = y_l^{j_l}$. (Then e.g.

$$F(y) = \begin{cases} y \text{ if } y \in H_M^{\min\{i,j\}}, \\ \text{undefined if not.} \end{cases}$$

where F is a map to i with arity j.)
But $F, F_1 \ldots, F_r$ are $\Sigma_1^{(n)}$ good. QED (Corollary 2.6.26)

The statement made earlier that every good $\Sigma_1^{(n)}$ function has a functionally absolute $\Sigma_1^{(n)}$ definition can be improved. We define:

Definition 2.6.15. φ is a *good* $\Sigma_1^{(n)}$ *definition* iff φ is a $\Sigma_1^{(n)}$ formula which defines a good $\Sigma_1^{(n)}$ function over any acceptable M of the given type.

Lemma 2.6.27. *Every good $\Sigma_1^{(n)}$ function has a good $\Sigma_1^{(n)}$ definition.*

Proof: By induction on k we show that it is true for all elements of $\mathbb{G}^k$. If $F \in \mathbb{G}^0$, then F is a $\Sigma_1^{(i)}$ map to H^i for an $i \leq n$. Hence any functionally absolute $\Sigma_1^{(i)}$ definition will do. Now let $F \in \mathbb{G}^{k+1}$. Then $F(\vec{x}) \simeq G(H_1(\vec{x}), \ldots, H_p(\vec{x}))$ where $G \in \mathbb{G}^k$ and $H_i \in \mathbb{G}^0$ for $i = 1, \ldots, p$. Then G has a good definition φ and every H_i has a good definition Ψ_i. By the uniformity expressed in Corollary 2.6.14 there is a $\Sigma_1^{(n)}$ formula χ such that, given *any* acceptable M of the given type, if φ defines G' and Ψ_i defines $H_i'(i = 1, \ldots, p)$, then χ defines $F'(\vec{x}) \simeq G'(\vec{H}'(\vec{x}))$. Thus χ is a good $\Sigma_1^{(n)}$ definition of F. QED (Lemma 2.6.27)

Definition 2.6.16. Let $a \in [\mathrm{On}_m]^{<\omega}$. We define partial maps h_a from $\omega \times H^n$ to H^n by:

$$h^n_a(i,x) \simeq: h_{M^{n,a}}(i, \langle x, a^{(n)}\rangle).$$

Then h^n_a is uniformly $\Sigma^{(n)}_1$ in $a^{(n)}, \ldots, a^{(0)}$ by Corollary **??**. We then define maps $\tilde{h}^n_a$ from $\omega \times H^n$ to H^0 by:

$$\begin{aligned}\tilde{h}^0_a(i,x) &\simeq h^o_a(i,x)\\ \tilde{h}^{n+1}_a(i,x) &\simeq \tilde{h}^n_a((i)_0, h^{n+1}_a((i)_1, x)).\end{aligned}$$

Then $\tilde{h}^n_a$ is a good $\Sigma^{(n)}_1$ function uniformly in $a^{(n)}, \ldots, a^{(0)}$.

Clearly, if $a \in R^{n+1}$, then

$$h^{n\prime\prime}_a(\omega \times \rho^{n+1}) = H^n.$$

Hence:

Lemma 2.6.28. *If $a \in R^{n+1}$, then $\tilde{h}^{n\prime\prime}_a(\omega \times \rho^{n+1}) = M$.*

Corollary 2.6.29. *If $R^n \neq \emptyset$, then $\underline{\Sigma}_l \subset \underline{\Sigma}^{(n)}_l$ for $l \geq 1$.*

Proof: Trivial for $n = 0$, since $\Sigma^{(0)}_l = \Sigma_l$. Now let $n = m+1$. Set: $D = H^n \cap \mathrm{dom}(h^n_a)$, where $a \in R^n$. Then D is $\underline{\Sigma}^{(n)}_1$ by Lemma 2.6.24, since:

$$\begin{aligned}x^n \in D &\leftrightarrow h^n_a(x^n) = h^n_a(x^n)\\ &\leftrightarrow \bigvee z^0(z^0 = h^n_a(x^n) \wedge z^0 = z^0).\end{aligned}$$

Let $R(\vec{x})$ be $\Sigma_l(M)$. Let

$$R(\vec{x}) \leftrightarrow Q_1 z_1 \ldots Q z_l P(\vec{z}, \vec{x})$$

where P is Σ_0. Set:

$$P'(\vec{u}^n, \vec{x}) \leftrightarrow: P(\vec{h}^n(\vec{u}^n), \vec{x}).$$

Then P' is $\Sigma^{(n)}_1$ in a. But for $u^n_1, \ldots, u^n_l \in D$, $\neg P'(\vec{u}^n, \vec{x})$ can also be written as a $\Sigma^{(n)}_1$ formula. Hence

$$R(\vec{x}) \leftrightarrow Q u^n_1 \in D \ldots Q u^n_l \in D P'(\vec{u}^n, \vec{x})$$

is $\Sigma^{(n)}_l$ in a. QED (Corollary 2.6.29)

We have seen that every $\underline{\Sigma}^{(n)}_\omega$ relation is $\underline{\Sigma}_\omega$. Hence:

Corollary 2.6.30. *Let $R^n \neq \emptyset$. Then $\underline{\Sigma}^{(n)}_\omega = \underline{\Sigma}_\omega$.*

An obvious corollary of Lemma 2.6.28 is:

Corollary 2.6.31. *Let $a \in R^n_M$. Then every element of M has the form $F(\xi, a^{(0)}, \dots, a^{(n)})$ where F is a good $\Sigma_1^{(n)}$ function and $\xi < \rho^{n+1}$.*

Using this we now prove a downward extension of embeddings lemma which strengthens and generalizes Lemma 2.5.12

Lemma 2.6.32. *Let $n = m + 1$. Let $a \in [\mathrm{On}_M]^{<\omega}$ and let $N = M^{na}$. Let $\overline{\pi} : \overline{N} \to_{\Sigma_j} N$, where $\overline{N}$ is a J–model. Then:*

(a) There are unique $\overline{M}, \overline{a}$ such that $\overline{u} \in R^n_{\overline{M}}$ and $\overline{M}^{n\overline{a}} = \overline{N}$.

(b) There is a unique $\pi \supset \overline{\pi}$ such that $\pi : \overline{M} \to_{\Sigma_0^{(m)}} M$ strictly and $\pi(\overline{a}) = a$.

(c) $\pi : M \to_{\Sigma_j^{(n)}} M$.

Proof: We first prove existence, then uniquenes. The existence assertion in (a) follows by:

Claim 1 There are $\overline{M}, \overline{a}, \hat{\pi} \supset \overline{\pi}$ such that $\overline{M}^{na} = \overline{N}$, $a \in R^n_{\overline{M}}$, $\hat{\pi} : \overline{M} \to_{\Sigma_1} M$, $\hat{\pi}(\overline{a}) = a$.

Proof: We proceed by induction on m. For $m = 0$ this immediate by Lemma 2.5.12. Now let $m = h + 1$. We first apply Lemma 2.5.12 to M^{ma}. It is clear from our definition that $\rho_{M^{m,a}} \geq \rho^n_M$. Set $N' = (M^{m,a})^{a \cap \rho^m_M}$. Then $N' = \langle J^A_{\rho'}, T' \rangle$, where $\rho' = \rho_{M^{ma}}$. But it is clear from our definition that $T^{na} = T' \cap J^A_{\rho^n_M}$. Hence:

(1) $\overline{\pi} : \overline{N} \to_{\Sigma_0} N'$.
By Lemma 2.5.12 there are then $\tilde{M}, \tilde{a}, \tilde{\pi} \supset \overline{\pi}$ such that $\tilde{M}^{\tilde{a}} = N'$, $\tilde{a} \in R_{\tilde{M}}$, $\tilde{\pi} : \tilde{M} \to_{\Sigma_1} M^{m,a}$ and $\tilde{\pi}(\tilde{a}) = a \cap \rho^m_M = a^{(m)}$.
(**Note:** Throughout this proof we use the notation:

$$a^{(i)} =: a \cap \rho^i \text{ for } i = 0, \dots, m.)$$

By the induction hypothesis there are then $\overline{M}, \overline{a}, \hat{\pi} \supset \tilde{\pi}$ such that $\overline{M}^{m\overline{a}} = \tilde{M}$, $\hat{\pi} : \overline{M} \to_{\Sigma_1} M$, and $\hat{\pi}(\overline{a}) = a$.
We observe that:

(2) $\tilde{a} = \overline{a} \cap \rho^m_{\overline{M}}$.
Proof:

($\subset$) Let $\tilde{\rho} =: \rho^m_{\overline{M}} = \mathrm{On} \cap \tilde{M}$. Then $\tilde{a} \subset \tilde{\rho}$. But $\hat{\pi}(\tilde{a}) = \tilde{\pi}(\tilde{a}) = a \cap \rho^m_M \subset a = \hat{\pi}(\overline{a})$. Hence $\tilde{a} \subset a$.

($\supset$) $\hat{\pi}(\overline{a} \cap \tilde{\rho}) = \hat{\pi}''(\overline{a} \cap \tilde{\rho}) \subset \rho^m_M \cap a = \hat{\pi}(\tilde{a})$, since $\hat{\pi}''\tilde{\rho} \subset \rho^m_M$. Hence $\overline{a} \cap \tilde{\rho} = \tilde{a}$. QED (2)

Since $\tilde{a} \in R^{m\overline{a}}_{\overline{M}}$ we conclude that $a \in R^n_{\overline{M}}$ and $\overline{N} = (M^{m\overline{a}})^{a \cap \tilde{\rho}} = \overline{M}^{n,\overline{a}}$. QED (Claim 1)

We now turn to the existence assertion in (b).

Claim 2 Let $\overline{M}^{\overline{a}} = N$ and $\overline{a} \in R^n_{\overline{M}}$. There is $\pi \supset \overline{\pi}$ sucht that $\pi : \overline{M} \to_{\Sigma_1^{(m)}} M$ and $\pi(\overline{a}) = a$.

Proof: Let $x_1, \ldots, x_n \in \overline{M}$ with $x_i = \overline{F}_i(z_i)(i = 1, \ldots, r)$, where $\overline{F}_i$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameters $\overline{a}^{(0)}, \ldots, \overline{a}^{(n)}$ and $z_i \in \overline{N}$. Let F_i have the same $\Sigma_1^{(m)}(M)$–good definition in $a^{(0)}, \ldots, a^{(m)}$. Let $\overline{R}(u_1, \ldots, u_r)$ be a $\Sigma_1^{(n)}(\overline{M})$ relation and let R be $\Sigma_1^{(n)}(M)$ by the same definition.

Then $\overline{R}(\overline{F}_1(z_1), \ldots, \overline{F}_r(z_r))$ is $\Sigma_1^{(m)}(\overline{M})$ in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$ and $R(F_1(z_1), \ldots, F_r(z_r))$ is $\Sigma_1^{(m)}(M)$ in $a^{(0)}, \ldots, a^{(m)}$ by the same definition. Hence there is $i < \omega$ such that

$$\overline{R}(\overline{F}(\vec{z}) \leftrightarrow \langle i, \langle \vec{z} \rangle \rangle \in \overline{T}$$
$$R(F(\vec{z})) \leftrightarrow \langle i, \langle \vec{z} \rangle \rangle \in T$$

where $\overline{N} = \langle J^{\overline{A}}_{\overline{\rho}}, \overline{T} \rangle, N = \langle J^A_\rho, T \rangle$. Thus $\overline{R}(\overline{F}(\vec{z}))$ is rud in $\overline{N}$ and $R(F(\vec{z}))$ is rud in N by the same rud definition. But $\overline{\pi} : \overline{N} \to_{\Sigma_0} N$.

Hence:

$$\overline{R}(\overline{F}_1(z_i), \ldots, \overline{F}_r(z_r)) \leftrightarrow R(F_1(\overline{\pi}(z_1)), \ldots, F_r(\overline{\pi}(z_r))).$$

Thus there is $\pi : \overline{M} \to_{\Sigma_1^{(n)}} M$ defined by $\pi(\overline{F}(\xi)) =: F(\overline{\pi}(\xi))$ whenever $\xi \in \mathrm{On} \cap \overline{N}$, $\overline{F}$ is $\Sigma_1^{(m)}(\overline{M})$– good in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$ and F is $\Sigma_1^{(m)}(M)$–good in $a^{(0)}, \ldots, a^{(m)}$ by the same definition. But then

$$\pi(z) = \pi(\mathrm{id}(z)) = \overline{\pi}(z) \text{ for } z \in \overline{N}.$$

Hence $\pi \supset \overline{\pi}$. But clearly

$$\begin{aligned} \pi(\overline{a}) &= \pi(\overline{a}^{(0)} \cup \ldots \cup \overline{a}^{(m)}) \\ &= a^{(0)} \cup \ldots \cup a^{(m)} = a. \end{aligned}$$

QED (Claim 2)

We now verify (c):

Claim 3 Let $\overline{M}, \overline{a}, \pi$ be as in Claim 2. Then $\pi : \overline{M} \to_{\Sigma_j^{(n)}} M$.

Proof: We first note that π, being $\Sigma_1^{(n)}$–preserving, is *strictly* so — i.e. $\rho^i_{\overline{M}} = \pi^{-1}{}''\rho^i_M$ for $i = 0, \ldots, m$. It follows easily that:

$$\pi(\overline{a}^{(i)}) = \pi''\overline{a}^{(i)} = a^{(i)} \text{ for } i = 0, \ldots, m.$$

We now proceed the cases.

Case 1 $j = 0$.

It suffices to show that if φ is $\Sigma_1^{(n)}$ and $x_1, \ldots, x_r \in \overline{N}$, then

$$\overline{M} \models \varphi[x_1, \ldots, x_r] \to M \models \varphi[\pi(x_1), \ldots, \pi(x_r)].$$

Let $x_1, \ldots, x_r \in \overline{M}$. Then $x_i = \overline{F}_i(z_i)(i = 1, \ldots, r)$ where $z_i \in \overline{N}$ and $\overline{F}_i$ is $\Sigma_1^{(m)}(\overline{M})$–good in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$. Let F_i be $\Sigma_1^{(m)}(M)$–good in $a^{(0)}, \ldots, a^{(m)}$ by the same good definition.

By Corollary 2.6.19, we know that $\overline{M} \models \varphi[\overline{F}_1(z_1), \ldots, \overline{F}_r(z_r)]$ is equivalent to

$$\overline{N} \models \Psi[z_1, \ldots, z_r]$$

for a certain Σ_1 formula Ψ. The same reduction on the M side shows that $M \models \varphi[F_1(z_1), \ldots, F_r(z_r)]$ is equivalent to: $N \models \Psi[z_1, \ldots, z_r]$ for $z_1, \ldots, z_r \in N$, where Ψ is the same formula.

Since π is Σ_0–preserving we then get:

$$\begin{aligned} \overline{M} \models \varphi[\vec{x}] &\leftrightarrow \overline{M} \models \varphi[\overline{F}(\vec{z})] \\ &\leftrightarrow \overline{N} \models \Psi[\vec{z}] \\ &\to N \models \Psi[\pi(\vec{z})] \\ &\leftrightarrow M \models \varphi[F(\pi(\vec{z}))] \\ &\leftrightarrow M \models \varphi[\pi(\vec{x})]. \end{aligned}$$

QED (Case 1)

Case 2 $j > 0$.

This is entirely similar. Let φ be $\Sigma_j^{(n)}$. By Corollary 2.6.19 it follows easily that there is a Σ_j formula Ψ such that: $\overline{M} \models \varphi[\overline{F}_1(z_1), \ldots, \overline{F}_r(z_r)]$ is equivalent to:

$$\overline{N} \models \Psi[z_1, \ldots, z_r].$$

Since the corresponding reduction holds on the M–side, we get

$$\overline{M} \models \varphi[\vec{x}] \leftrightarrow M \models \varphi[\pi(\vec{x})],$$

since $\pi(x_i) = \pi(\overline{F}_i(z_i)) = F_i(\overline{\pi}(z_i))$. QED (Claim 3)

This proves existence. We now prove uniqueness.

Claim 4 The uniqueness assertion of (a) holds.

Proof: Let $\hat{M}, \hat{a}$ be such that $\hat{M}^{n,\hat{a}} = \overline{N}$ and $\hat{a} \in R^N_{\hat{M}}$.

Claim $\hat{M} = \overline{M}, \hat{a} = \overline{a}$.

Proof: By a virtual repetition of the proof in Claim 2 there is a $\pi : \hat{M} \to_{\Sigma_1^{(m)}} \overline{M}$ defined by:

(3) $\pi(\hat{F}(z)) = \overline{F}(z)$ whenever $z \in \overline{N}$, $\hat{F}$ is a good $\Sigma_1^{(m)}(\hat{M})$ function in $\hat{a}^{(0)}, \ldots, \hat{a}^{(m)}$ and $\overline{F}$ is the $\Sigma_1^{(m)}(\overline{M})$ function in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$ with the same good definition.

But π is then onto. Hence π is an isomorphism of $\hat{M}$ with $\overline{M}$. Since $\hat{M}, \overline{M}$ are transitive, we conclude that $\overline{M} = \hat{M}, \overline{a} = \hat{a}$.

QED (Claim 4)

Finally we prove the uniqueness assertion of (b):

Claim 5 Let $\pi' : \overline{M} \to_{\Sigma_0^{(m)}} M$ strictly, such that $\pi'(\overline{a}) = a$. Then $\pi' = \pi$.

Proof: By strictness we can again conclude that $\pi'(\overline{a}^{(i)}) = a^{(i)}$ for $i = 0, \ldots, m$. Let $x \in \overline{M}$, $x = \overline{F}(z)$, where $z \in \overline{N}$ and $\overline{F}$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameters $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$. Let F be $\Sigma_1^{(m)}(M)$ in $a^{(0)}, \ldots, a^{(m)}$ by the same good definition.

The statement: $x = \overline{F}(z)$ is $\Sigma_2^{(m)}(\overline{M})$ in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$. Since π' is $\Sigma_0^{(m)}$–preserving, the corresponding statement must hold in M — i.e. $\pi'(x) = F(\overline{\pi}(z)) = \pi(x)$.

QED (Lemma 2.6.32)

2.7 Liftups

2.7.1 The Σ_0 liftup

A concept which, under a variety of names, is frequently used in set theory is the *liftup* (or as we shall call it here, the Σ_0 *liftup*). We can define it as follows:

Definition 2.7.1. Let M be acceptable. Let $\tau > \omega$ be a cardinal in M. Let $H = H_\tau^M$ and let $\pi : H \to_{\Sigma_0} H'$ cofinally. We say that $\langle M', \pi' \rangle$ is a Σ_0 *liftup* of $\langle M, \pi \rangle$ iff M' is transitive and:

(a) $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0} M'$

(b) Every element of M' has the form $\pi'(f)(x)$ for an $x \in H'$ and an $f \in \Gamma^0$, where $\Gamma^0 = \Gamma^0(\tau, M)$ is the set of functions $f \in M$ such that $\mathrm{dom}(f) \in H$.

(**Note** The condition of acceptability can be relaxed considerably, but that is uninteresting for our purposes.)

If $\langle M', \pi' \rangle$ is a liftup of $\langle M, \pi \rangle$ it follows easily that:

Lemma 2.7.1. *$\pi' : M \to_{\Sigma_0} M'$ cofinally.*

Proof: Let $y \in M'$, $y = \pi'(f)(x)$ where $x \in H'$ and $f \in \Gamma^0$, then $y \in \pi'(\mathrm{rng}(f))$. QED (Lemma 2.7.1)

Lemma 2.7.2. *$\langle M', \pi' \rangle$ is the only liftup of $\langle M, \pi \rangle$.*

Proof: Suppose not. Let $\langle M^*, \pi^* \rangle$ be another liftup. Let $\varphi(v_1, \ldots, v_n)$ be Σ_0. Then

$$M' \models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_n)(x_n)] \leftrightarrow$$
$$\langle x_1, \ldots, x_n \rangle \in \pi(\{\langle \vec{z} \rangle | M | \varphi[\vec{f}(\vec{z})]\}) \leftrightarrow$$
$$M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_n)(x_n)].$$

Hence there is an isomorphism σ of M' onto M^* defined by:

$$\sigma(\pi'(f)(x)) = \pi^*(f)(x)$$
$$\text{for } f \in \Gamma^0,\ x \in \pi(\mathrm{dom}(f)).$$

But M', M^* are transitive. Hence $\sigma = \mathrm{id}$, $M' = M^*$, $\pi' = \pi^*$.
QED (Lemma 2.7.2)

(**Note** $M \models \varphi[\vec{f}(\vec{z})]$ means the same as

$$\bigvee y_1 \ldots y_n (\bigwedge_{i=1}^{n} y_i = f_i(z_i) \wedge M \models \varphi[\vec{y}]).$$

Hence if $e = \{\langle \vec{z} \rangle | M \models \varphi[\vec{f}(\vec{z})]\}$, then $e \subset \underset{i=1}{\overset{n}{\times}} \mathrm{dom}(f_i) \in H$. Hence $e \in M$ by rud closure, since e is $\underline{\Sigma}_0(M)$. But then $e \in H$, since $\mathbb{P}(u) \cap M \subset H$ for $u \in H$.)

But when des the liftup exist? In answering this question it is useful to devise a 'term model' for the putative liftup rather like the ultrapower construction:

Definition 2.7.2. Let $M, \tau, \pi : H \to_{\Sigma_0} H'$ be as above. The term model $\mathbb{D} = \mathbb{D}(M, \pi)$ is defined as follows. Let e.g. $M = \langle J_\alpha^A, B \rangle$. $\mathbb{D} =: \langle D, \cong, \tilde{\in}, \tilde{A}, \tilde{B} \rangle$ where

$D =$ the set of pairs $\langle f, x\rangle$ such that $f \in \Gamma_0$ and $x \in H'$

$$\langle f,x\rangle \cong \langle g,y\rangle \leftrightarrow: \langle x,y\rangle \in \pi(\{\langle z,w\rangle | f(z) = g(y)\})$$
$$\langle f,x\rangle \tilde{\in} \langle g,y\rangle \leftrightarrow: \langle x,y\rangle \in \pi(\{\langle z,w\rangle | f(z) \in g(y)\})$$
$$\tilde{A}\langle f,x\rangle \leftrightarrow: x \in \pi(\{z | Af(z)\})$$
$$\tilde{B}\langle f,x\rangle \leftrightarrow: x \in \pi(\{z | Bf(z)\})$$

(**Note** $\mathbb{D}$ is an 'equality model', since the identity predicate $=$ is interpreted by $\cong$ rather than the identity.)

Łoz theorem for $\mathbb{D}$ then reach:

Lemma 2.7.3. *Let $\varphi = \varphi(v_1, \ldots, v_n)$ be Σ_0. Then*

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle] \leftrightarrow \langle x_1, \ldots, x_n\rangle \in \pi(\{\langle \vec{z}\rangle | M \models \varphi[\vec{f}(\vec{z})]\}).$$

Proof: (Sketch)
We prove this by induction on the formula φ. We display a typical case of the induction. Let $\varphi = \bigvee u \in v_1 \Psi$. By bound relettering we can assume *w.l.o.g.* that u is not among $v_1, \ldots, v_n$. Hence $u, v_1, \ldots, v_n$ is a good sequence for Ψ. We first prove $(\rightarrow)$. Assume:

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle].$$

Claim $\langle x_1, \ldots, x_n\rangle \in \pi(e)$ where

$$e = \{\langle z_1, \ldots, z_n\rangle | M \models \varphi[f_1(z_1) \ldots f_n(z_n)]\}.$$

Proof: By our assumption there is $\langle g, y\rangle \in D$ such that $\langle g, y\rangle \tilde{\in} \langle f_1, ?\rangle$ and:

$$\mathbb{D} \models \Psi[\langle g, y\rangle, \langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle].$$

By the induction hypothesis we conclude that $\langle y, \vec{x}\rangle \in \pi(\tilde{e})$ where:

$$\tilde{e} = \{\langle w, \vec{z}\rangle | g(w) \in f_1(z_1) \wedge M \models \Psi[g(w), \vec{f}(\vec{z})\}.$$

Clearly $e, \tilde{e} \in H$ and

$$H \models \wedge w, \vec{z}(\langle w, \vec{z}\rangle \in \tilde{e} \rightarrow \langle \vec{z}\rangle \in e).$$

Hence

$$H' \models \wedge w, \vec{z}(\langle w, \vec{z}\rangle \in \pi\tilde{e} \rightarrow \langle \vec{z}\rangle \in \pi(e)).$$

Hence $\langle \vec{x}\rangle \in \pi(e)$. QED $(\rightarrow)$

We now prove ($\leftarrow$)
We assume that $\langle x_1, \dots, x_n\rangle \in \pi(e)$ and must prove:

Claim $\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \dots, \langle f_n, x_n\rangle]$.

Proof: Let $r \in M$ be a well ordering of $\text{rng}(f_1)$. For $\langle\vec{z}\rangle \in e$ set:

$$\begin{aligned} g(\langle\vec{z}\rangle) = \ & \text{the } r\text{–least } w \text{ sucht that } M \models \\ & M \models \Psi[w, f_1(z_1), \dots, f_n(z_n)]. \end{aligned}$$

Then $g \in M$ and $\text{dom}(g) = e \in H$. Now let $\tilde{e}$ be defined as above with this g. Then:

$$H \models \bigwedge z_1, \dots, z_n(\langle\vec{z}\rangle \in e \leftrightarrow \langle\langle\vec{z}\rangle, \vec{z}\rangle \in \tilde{e}).$$

But then the corresponding statement holds of $\pi(e), \pi(\tilde{e})$ in H'. Hence

$$\langle\langle\vec{x}\rangle, \vec{x}\rangle \in \pi(\tilde{e}).$$

By the induction hypothesis we conclude:

$$\mathbb{D} \models \Psi[\langle g, \langle\vec{x}\rangle\rangle, \langle f_1, x_1\rangle, \dots, \langle f_n, x_n\rangle].$$

The conclusion is immediate. QED (Lemma 2.7.3)

The liftup of $\langle M, \pi\rangle$ can only exist if the relation $\tilde{e}$ is well founded:

Lemma 2.7.4. *Let $\tilde{\in}$ be ill founded. Then there is no $\langle M', \pi'\rangle$ such that $\pi' : M \to_{\Sigma_0} M'$. M' is transitive, and $\pi' \supset \pi$.*

Proof: Suppose not. Let $\langle f_{i+1}, x_{i+1}\rangle \tilde{\in} \langle f_i, x_i\rangle$ for $i < w$. Then

$$\langle x_{i+1}, x_i\rangle \in \pi\{\langle z, w\rangle | f_{i+1}(z) \in f_i(w)\}.$$

Hence $\pi'(f_{i+1})(x_{i+1}) \in \pi'(f_i)(x_i)(i < w)$.
Contradiction! QED (Lemma 2.7.4)

Conversely we have:

Lemma 2.7.5. *Let $\tilde{\in}$ be well founded. Then the liftup of $\langle M, \pi\rangle$ exists.*

Proof: We shall explicitly construct a liftup from the term model $\mathbb{D}$. The proof will stretch over several subclaims.

Definition 2.7.3. $x^* = \pi^*(x) =: \langle \text{const}_x, 0\rangle$, where $\text{const}_x =: \{\langle x, 0\rangle$ = the constant function x defined on $\{0\}$.

Then:

(1) $\pi^* : M \to_{\Sigma_0} \mathbb{D}$.
Proof: Let $\varphi(v_1, \ldots, v_n)$ be Σ_0. Set:

$$e = \{\langle z_1, \ldots, z_n\rangle | M \models \varphi[\mathrm{const}_{x_1}(z_1), \ldots, \mathrm{const}_{x_n}(z_n)]\}.$$

Obviously:

$$e = \begin{cases} \{\langle 0, \ldots, 0\rangle\} \text{ if } M \models \varphi[x_1, \ldots, x_n] \\ \emptyset \text{ if not.} \end{cases}$$

Hence by Łoz theorem:

$$\begin{aligned} \mathbb{D} \models \varphi[x_1^*, \ldots, x_n^*] \quad &\leftrightarrow \langle 0, \ldots, o\rangle \in \pi(e) \\ &M \models \varphi[x_1, \ldots, x_n] \end{aligned}$$

(2) $\mathbb{D} \models$ Extensionality.
Proof: Let $\varphi(u, v) =: \bigwedge w \in u\, w \in v \ \wedge \bigwedge w \in v\ w \in u$.

Claim $\mathbb{D} \models \varphi[a, b] \to a \cong b$ for $a, b \in \mathbb{D}$. This reduces to the Claim: Let $a = \langle f, x\rangle, b = \langle g, y\rangle$. Then

$$\begin{aligned} \mathbb{D} \models \varphi[\langle f, x\rangle, \langle g, y\rangle] \quad &\leftrightarrow \langle x, y\rangle \in \pi(e) \\ &\leftrightarrow \langle f, x\rangle \cong \langle g, y\rangle \end{aligned}$$

where

$$\begin{aligned} e \quad &= \{\langle z, w\rangle | M \models \varphi[z, w]\} \\ &= \{\langle z, w\rangle | f(z) = g(w)\} \end{aligned}$$

QED (2)

Since $\cong$ is a congruence relation for $\mathbb{D}$ we can factor $\mathbb{D}$ by $\cong$, getting:

$$\hat{\mathbb{D}} = (\mathbb{D} \setminus \cong) = \langle \hat{D}, \hat{\in}, \hat{A}, \hat{B}\rangle$$

where:

$$\begin{aligned} &\hat{D} = \{\hat{s} | s \in D\} \\ &\hat{s} =: \{t | t \cong s\} \text{ for } s \in D \\ &\hat{s} \hat{\in} \hat{t} \leftrightarrow: s \tilde{\in} t \\ &\hat{A}\hat{s} \leftrightarrow: \tilde{A}s, \hat{B}\hat{s} \leftrightarrow: \hat{B}s. \end{aligned}$$

Then $\hat{\mathbb{D}}$ is a well founded identity model satisfying extensionality. By Mostowski's isomorphism theorem there is an isomorphism k of $\hat{\mathbb{D}}$ onto M', where $M' = \langle |M'|, \in, A', B'\rangle$ is transitive.

Set:

$$\begin{aligned} &[s] =: k(\hat{s}) \text{ for } s \in D \\ &\pi'(x) =: [x^*] \text{ for } x \in M. \end{aligned}$$

Then by (1):

(3) $\pi' : M \to_{\Sigma_0} M'$.
Lemma 2.7.5 will then follow by:

Lemma 2.7.6. *$\langle M', \pi' \rangle$ is the liftup of $\langle M, \pi \rangle$.*

We shall often write $[f, x]$ for $[\langle f, x \rangle]$. Clearly every $s \in M'$ has the form $[f, x]$ where $f \in M$; $\mathrm{dom}(f) \in H$, $x \in H'$.

Definition 2.7.4. $\tilde{H}$ =: the set of $[f, x]$ such that $\langle f, x \rangle \in D$ and $f \in H$.

We intend to show that $[f, x] = \pi(f)(x)$ for $x \in \tilde{H}$. As a first step we show:

(4) $\tilde{H}$ is transitive.

Proof: Let $s \in [f, x]$ where $f \in H$.

Claim $s = [g, y]$ for a $g \in H$.

Proof: Let $s = [g', y]$. Then $\langle y, x \rangle \in \pi(e)$ where: $e = \{\langle u, v \rangle | g'(u) \in f(v)\}$ set:

$$e' = \{u | g'(u) \in \mathrm{rng}(f)\}, \ g = g' \upharpoonright e'.$$

Then $g \subset \mathrm{rng}(f) \times \mathrm{dom}(g') \in H$. Hence $g \in H$. Then $[g', y] = [g, y]$ since $\pi(g')(y) = \pi(g)(y)$ and hence
$\langle y, y \rangle \in \pi(\{\langle u, v \rangle | g'(u) = g(v)\})$. But $e = \{\langle u, v \rangle | g(u) \in f(v)\}$. Hence $[g, y] \in [f, x]$. QED (4)

But then:

(5) $[f, x] = \pi(f)(x)$ for $f \in H, \langle f, x \rangle \in D$.

Proof: Let $f, g \in H, \langle f, x \rangle, \langle g, y \rangle \in D$. Then:

$$\begin{aligned} [f, x] \in [g, y] &\leftrightarrow \langle x, y \rangle \in \pi(e) \\ &\leftrightarrow \pi(f)(x) \in \pi(g)(y) \end{aligned}$$

where $e = \{\langle u, v \rangle | f(u) \in g(v)\}$. Hence there is an $\in$–isomorphism σ of H onto $\tilde{H}$ defined by:

$$\sigma(\pi(f)(x)) =: [f, x].$$

But then $\sigma = \mathrm{id}$, since $H, \tilde{H}$ are transitive. (5)

But then:

(6) $\pi' \supset \pi$.

Proof: Let $x \in H$. Then $\pi'(x) = [\mathrm{const}_x, 0] = \pi(\mathrm{const}_x)(0) = \pi(x)$ by (5).

(7) $[f,x] = \pi'(f)(x)$ for $\langle f,x\rangle \in D$.

Proof: Let $a = \text{dom}(f)$. Then $[\text{id}_a, x] = \text{id}_{\pi(a)}(x) = x$ by (5). Hence it suffices to show:

$$[f,x] = [\text{const}_f, 0]([\text{id}_a, x]).$$

But this says that $\langle x, 0\rangle \in \pi(e)$ where:

$$\begin{aligned} e &= \{\langle z,u\rangle | f(z) = \text{const}_f(u)(\text{id}_a(z))\} \\ &= \{\langle z,0\rangle | f(z) = f(z)\} = a \times \{0\}. \end{aligned}$$

QED (7)

Lemma 2.7.6 is then immediate by (3), (6) and (7). QED (Lemma 2.7.5)

Lemma 2.7.7. *Let $\pi^* \supset \pi$ such that $\pi^* : M \to_{\Sigma_0} M^*$. Then the liftup $\langle M', \pi'\rangle$ of $\langle M, \pi\rangle$ exists. Moreover there is a $\sigma : M' \to_{\Sigma_0} M^*$ uniquely defined by the condition:*

$$\sigma \restriction H' = \text{id}, \ \sigma\pi' = \pi^*.$$

Proof: $\langle M', \pi'\rangle$ exists, since $\tilde{\in}$ is well founded, since $\langle f,x\rangle \tilde{\in} \langle g,y\rangle \leftrightarrow \pi^*(f)(x) \in \pi^*(g)(y)$. But then:

$$\begin{aligned} M' &\models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)(x_r)] \leftrightarrow \\ &\leftrightarrow \langle x_1, \ldots, x_r\rangle \in \pi(e) \\ &\leftrightarrow M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_r)(x_r)] \end{aligned}$$

where $e = \{\langle z_1, \ldots, z_r\rangle | M \models \varphi[\vec{f}(\vec{z})]\}$. Hence there is $\sigma : M' \to_{\Sigma_0} M^*$ defined by:

$$\sigma(\pi'(f)(x)) = \pi^*(f)(x) \text{ for } \langle f,x\rangle \in D.$$

Now let $\tilde{\sigma} : M' \to_{\Sigma_0} M^*$ such that $\tilde{\sigma} \restriction H' = \text{id}$ and $\tilde{\sigma}\pi' = \pi^r$.

Claim $\tilde{\sigma} = \sigma$.

Let $s \in M'$, $s = \pi'(f)(x)$. Then $\tilde{\sigma}(\pi'(f)) = \pi^*(f)$, $\tilde{\sigma}(x) = x$. Hence $\tilde{\sigma}(s) = \pi^*(f)(x) = \sigma(s)$. QED (Lemma 2.7.7)

2.7.2 The $\Sigma_0^{(n)}$ liftup

We now attempt to generalize the notion of Σ_0 liftup. We suppose as before that $\tau > w$ is a cardinal in M and $H = H_\tau^M$. As before we suppose that $\pi' : H \to_{\Sigma_0} H'$ cofinally. Now let $\rho^n \geq \tau$. The Σ_0–liftup was the "minimal"

$\langle M', \pi' \rangle$ such that $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0} M'$. We shall now consider pairs $\langle M', \pi' \rangle$ such that $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0} M'$. Among such pairs $\langle M', \pi' \rangle$ we want to define a "minimal" one and show, if possible, that it exists. The minimality of the Σ_0 liftup was expressed by the condition that every element of M' have the $\pi'(f)(x)$, where $x \in H'$ and $f \in \Gamma^0(\tau, M)$. As a first step to generalizing this definition we replace $\Gamma^0(\tau, M)$ by a larger class of functions $\Gamma^n(\tau, M)$.

Definition 2.7.5. Let $n > 0$ such that $\tau \leq \rho^n_M$. $\Gamma^n(\tau, M)$ is the set of maps f such that

(a) $\mathrm{dom}(f) \in H$

(b) For some $i < n$ there is a $\Sigma^{(i)}(M)$ good function G and a parameter $p \in M$ such that $f(x) = G(x, p)$ for all $x \in \mathrm{dom}(f)$.

Note $\Sigma^{(i)}_1$ good functions are many sorted, hence any such function can be identified with a pair consisting of its field and its arity. An element of Γ^n, on the other hand, is 1–sorted in the classical sense, and can be identified with its field.

Note This definition makes sense for the case $n = \omega$, and we will not exclude this case. A $\Sigma^{(\omega)}_0$ formula (or relation) then means any formula (or relation) which is $\Sigma^{(i)}_0$ for an $i < \omega$ — i.e. $\Sigma^{(\omega)}_0 = \Sigma^*$.

We note:

Lemma 2.7.8. *Let $f \in \Gamma^n$ such that $\mathrm{rng}(f) \subset H^i$, where $i < n$. Then $f(x) = G(x, p)$ for $x \in \mathrm{dom}(f)$ where G is a good $\Sigma^{(h)}_1$ function to H^i for some $h < n$.*

Proof: Let $f(x) = G'(x, p)$ for $x \in \mathrm{dom}(f)$ where G' is a good $\Sigma^{(n)}_1$ function to H^j where $h, j < n$. Since every good $\Sigma^{(n)}_1$ function for $k \geq h$, we can assume *w.l.o.g.* that $i, j \leq h$. Let F be the identity function defined by $v^i = u^j$ (i.e. $y^i = F(x^j) \leftrightarrow y^i = x^j$). Set: $G(x, y) \simeq: F(G'(x, y))$. Then F is a good $\Sigma^{(h)}_1$ function and so is G, where $f(x) = G(x, p)$ for $x \in \mathrm{dom}(f)$.

QED (Lemma 2.7.8)

Lemma 2.7.9. $\Gamma^i(\tau, m) \subset \Gamma^n(\tau, M)$ *for $i < n$.*

Proof: For $0 < i$ this is immediat by the definition. Now let $i = 0$. If $f \in \Gamma^0$, then $f(x) = G(x, f)$ for $x \in \mathrm{dom}(f)$ where G is the $\Sigma^{(0)}_0$ function defined by

$$y = G(x, f) \leftrightarrow: \quad (f \text{ is a function } \wedge \\ \wedge \langle y, x \rangle \in f).$$

QED (Lemma 2.7.9)

The "natural" minimality condition for the $\Sigma_0^{(n)}$ liftup would then read: Each element of M has the form $\pi'(f)(x)$ where $x \in H'$ and $f \in \Gamma^n$. But what sequence can we make of the expression "$\pi'(f)(x)$" when f is not an element of M? The following lemma rushes to our aid:

Lemma 2.7.10. *Let $\pi' : M \to_{\Sigma_0^{(n)}} M'$ where $n > 0$ and $\pi' \supset \pi$. There is a unique map π'' of $\Gamma^n(\tau, M)$ to $\Gamma^n(\pi(\tau), M')$ with the following property:*

- $*$ *Let $f \in \Gamma^n(\tau, M)$ such that $f(x) = G(x,p)$ for $x \in \mathrm{dom}(f)$ where G is a good $\Sigma_1^{(i)}$ function for an $i < n$ and χ is a good $\Sigma_1^{(i)}$ definition of G. Let G' be the function defined on M' by χ. Let $f' = \pi''(f)$. Then $\mathrm{dom}(f') = \pi(\mathrm{dom}(f))$ and $f'(x) = G'(x, \pi'(p))$ for $x \in \mathrm{dom}(f')$.*

Proof: As a first approximation, we simply pick G, χ with the above properties. Let G' then be as above. Let $d = \mathrm{dom}(f)$. The statement
$\bigwedge x \in d \bigvee y\, y = G(x,p)$ is $\Sigma_0^{(n)}$ is d, p, so we have:

$$\bigwedge x \in \pi(d) \bigvee y\, y = G'(x, \pi(p)).$$

Define f_0 by $\mathrm{dom}(f_0) = \pi(d)$ and $f_0(x) = G'(x, \pi(p))$ for $x \in \pi(d)$. The problem is, of course, that G, χ where picked arbitrarily. We might also have:

$$f(x) = H(x,q) \text{ for } x \in d,$$

where H is $\Sigma_1^{(j)}(M)$ for a $j < n$ and Ψ is a good $\Sigma_1^{(j)}$ definition of H. Let H' be the good function on M' defined by Ψ. As before we can define f_1 by $\mathrm{dom}(f_1) = \pi(d)$ and $f_1(x) = H'(x, \pi'(q))$ for $x \in \pi(d)$. We must show: $f_0 = f_1$. We note that:

$$\bigwedge x \in d G(x,p) = H(x,q).$$

But this is a $\Sigma_0^{(n)}$ statement. Hence

$$\bigwedge x \in \pi(d) G'(x,p) = H'(x,q).$$

Then $f_0 = f_1$. QED (Lemma 2.7.10)

Moreover, we get:

Lemma 2.7.11. *Let $n, \pi, \tau, \pi', \pi''$ be as above. Then $\pi''(f) = \pi'(f)$ for $f \in \Gamma^0(\tau, M)$.*

Proof: We know $f(x) = G(x, f)$ for $x \in d = \mathrm{dom}(f)$, where:

$$y = G(x, f) \leftrightarrow: (f \text{ is a function } \wedge y = f(x)).$$

Then $\pi''(f)(x) = G'(x, \pi'(f)) = \pi'(f)(x)$ for $x \in \pi(d)$, where G' has the same definition over M'. QED (Lemma 2.7.11)

Thus there is no ambignity in writing $\pi'(f)$ instead of $\pi''(f)$ for $f \in \Gamma^n$. Doing so, we define:

Definition 2.7.6. Let $\omega < \tau < \rho_M^n$ where $n \leq \omega$ and τ is a cardinal in M. Let $H = H_\tau^M$ and let $\pi : H \to_{\Sigma_0} H'$ cofinally. We call $\langle M', \pi' \rangle$ a $\Sigma_0^{(n)}$ *liftup* of $\langle M, \pi \rangle$ iff the following hold:

(a) $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0^{(n)}} M'$.

(b) Each element of M' has the form $\pi'(f)(x)$, where $f \in \Gamma^n(\tau, M)$ and $x \in H'$.

(Thus the old Σ_0 liftup is simply the special case: $n = 0$.)

Definition 2.7.7. $\Gamma_i^n(\tau, M)$ =: the set of $f \in \Gamma^n(\tau, M)$ such that either $i < n$ and $\mathrm{rng}(f) \subset H_M^i$ or $i = n < \omega$ and $f \in H_M^i$.

(Here, as usual, $H^i = J_{\rho_M^i}[A]$ where $M = \langle J_\alpha^A, B \rangle$.)

Lemma 2.7.12. *Let $f \in \Gamma_i^n(\tau, M)$. Let $\pi' : M \to_{\Sigma_0^{(n)}} M'$ where $\pi' \supset \pi$. Then $\pi'(f) \in \Gamma_i^n(\pi'(\tau), M')$.*

Proof:

Case 1 $i = n$. Then $f \in H_{\rho_M^n}^M$. Hence $\pi'(f) \in H_{\rho_M^n}^{M'}$.

Case 2 $i < n$.

By Lemma 2.7.9 for some $h < n$ there is a good $\Sigma_1^{(n)}(M)$ function $G(u, v)$ to H^i and a parameter p such that

$$f(x) = G(x, p) \text{ for } x \in \mathrm{dom}(f).$$

Hence:

$$\pi'(f)(x) = G'(x, \pi'(p)) \text{ for } x \in \mathrm{dom}(\pi(f)),$$

where G' is defined over M' by the same good $\Sigma^(n)$ definition. Hence $\mathrm{rng}(\pi'(f)) \subset H_M^i$. QED (Lemma 2.7.12)

The following lemma will become our main tool in understanding $\Sigma_0^{(n)}$ liftups.

Lemma 2.7.13. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $f_l \in \Gamma_{i_l}^n (l = 1, \ldots, r)$. Then:*

(a) The relation P is $\Sigma_0^{(n)}$ in a parameter where:

$$P(\vec{z}) \leftrightarrow: R(f_1(z_1), \ldots, f_r(z_r)).$$

(b) Let $\pi' \supset \pi$ such that $\pi' : M \to_{\Sigma_0^{(n)}} M'$. Let R' be $\Sigma_0^{(n)}(m')$ by the same definition as R. Then P' is $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition as P in p, where:

$$P'(\vec{z}) \leftrightarrow: R'(\pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)).$$

Before proving this lemma we note some corollaries:

Corollary 2.7.14. *Let $e = \{\langle \vec{z} \rangle | P(\vec{z})\}$. Then $e \in H$ and $\pi(e) = \{\langle \vec{z} \rangle | P'(\vec{z})\}$.*

Proof: Clearly $e \subset d = \underset{l=1}{\overset{r}{\times}} \mathrm{dom}(f_l) \in H$. But then $d \in H_{\rho^n}$ and $e \in H_{\rho^n}$ since $\langle H_{\rho^n}, P \cap H_{\rho^n} \rangle$ is amenable. Hence $e \in H$, since $H = H_\tau^M$ and therefore $\mathbb{P}(u) \cap M \subset H$ for $u \in H$.

Now set $e' = \{\langle \vec{z} \rangle | P'(\vec{z})\}$. Then $e' \subset \pi(d) = \underset{l=1}{\overset{r}{\times}} \mathrm{dom}(\pi(f_l))$ since $\pi' \supset \pi$ and hence $\pi(\mathrm{dom}(f_l)) = \mathrm{dom}(\pi(f_l))$. But

$$\bigwedge \langle \vec{z} \rangle \in d(\langle \vec{z} \rangle \in e \leftrightarrow P(\vec{z}))$$

which is a $\Sigma_0^{(n)}$ statement about e, p. Hence the same statement holds of $\pi(e), \pi(p)$ in M'. Hence

$$\bigwedge \langle \vec{z} \rangle \in \pi(d)(\langle \vec{z} \rangle \in \pi(e) \leftrightarrow P'(\vec{z})).$$

Hence $\pi(e) = e'$. QED (Corollay 2.7.14)

Corollary 2.7.15. *$\langle M, \pi \rangle$ has at most one $\Sigma_0^{(n)}$ liftup $\langle M', \pi' \rangle$.*

Proof: Let $\langle M^*, \pi^* \rangle$ be a second such. Let $\varphi(v_1^{i_1}, \ldots, v_r^{i_r})$ be a $\Sigma_0^{(n)}$ formula. (In fact, we could take it here as being $\Sigma_0^{(0)}$.) Let $e = \{\langle \vec{z} \rangle | M \models \varphi[f_1(z_1), \ldots, f_r(z_r)]\}$ where $f_l \in \Gamma_{i_l}^n (l = 1, \ldots, r)$. Then:

$$\begin{aligned} M' &\models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)(x_r)] \leftrightarrow \\ &\leftrightarrow \langle x_1, \ldots, x_r \rangle \in \pi(e) \\ &\leftrightarrow M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_r)(x_r)] \end{aligned}$$

for $x_l \in \pi(\text{dom}(f_l)(l = 1, \ldots, r)$.

Hence there is an isomorphism $\sigma : M' \tilde{\to} M^*$ defined by:

$$\sigma(\pi'(f)(x)) =: \pi^*(f)(x)$$

for $f \in \Gamma^n,\ x \in \pi(\text{dom}(f))$. But M', M^* are transitive. Hence $\sigma = \text{id}, M' = M^*, \pi' = \pi^*$. QED (Corollary 2.7.15)

We now prove Lemma 2.7.13 by induction on n.

Case 1 $n = 0$.

Then $f_1, \ldots, f_r \in M$ and P is Σ_0 in $p = \langle f_1, \ldots, f_r \rangle$, since f_i is rudimentary in p and for sufficiently large h we have:

$$P(\vec{z}) \leftrightarrow \bigvee y_1, \ldots, y_r \in C_h(p)(\bigwedge_{i=1}^{r} y_i = f_i(\vec{z}_i) \wedge R(\vec{y}))$$

where R is Σ_0. If P' has the same Σ_0 definition over M' in $\pi'(p)$, then

$$\begin{aligned} P'(z) &\leftrightarrow \bigvee\nolimits_{y_1, \ldots, y_r} \in C_h(\pi(p))(\bigwedge_{n=1}^{r} y_i = \pi(f_i)(z_i) \wedge R(\vec{y})) \\ &\leftrightarrow R(\pi(\vec{f})(\vec{z})) \end{aligned}$$

QED

Case 2 $n = w$.

Then $\Sigma_0^w = \bigcup_{h<w} \Sigma_1^{(n)}$. Let $R(x_1^{i_1}, \ldots, x_r^{l_r})$ be $\Sigma_1^{(h)}$. Since every $\Sigma_1^{(h)}$ relation is $\Sigma_1^{(k)}$ for $k \geq h$, we can assume h taken large enough that $i_1, \ldots, i_r \leq h$. We can also choose it large enough that:

$$f_l(z) \simeq G_l(z, p) \text{ for } l = 1, \ldots, v$$

where G_l is a good $\Sigma_1^{(h)}$ map to H^{i_l}. (We assume *w.l.o.g.* that p is the same for $l = 1, \ldots, r$ and that $d_l = \text{dom}(f_l)$ is rudimentary in p.) Set:

$$P(\vec{z}, y) \leftrightarrow: R(G_1 x_1, y), \ldots, G(x_r, y)).$$

By §6 Lemma **??**, P is $\Sigma_1^{(h)}$ (uniformly in the $\Sigma_1^{(h)}$ definition of R and $G_1, \ldots, G_r$). Moreover:

$$P(\vec{z}) \leftrightarrow P(\vec{z}, p).$$

Thus P is uniformly $\Sigma_1^{(h)}$ in p, which proves (a). But letting P' have the same $\Sigma_1^{(h)}$ definition in $\pi'(p)$ over M', we have:

$$\begin{aligned} P'(\vec{z}) &\leftrightarrow P'(\vec{z}, \pi'(p)) \\ &\leftrightarrow R'(\pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)), \end{aligned}$$

which proves (b). QED (Case 2)

Case 3 $0 < n < w$.

Let $n = m+1$. Rearranging arguments as necessary, we can take R as given in the form:

$$R(y_1^n, \ldots, y_s^n, x_1^{i_1}, \ldots, x_r^{i_r})$$

where $i_1, \ldots, i_r \leq m$. Let $f_l \in \Gamma_{i_l}^n$ for $l = 1, \ldots, r$ and let $g_1, \ldots, g_1 \in \Gamma_n^n$.

Claim

(a) P is $\Sigma_0^{(n)}$ in a parameter p where

$$P(\vec{w}, \vec{z}) \leftrightarrow: R(\vec{g}(\vec{w}), \vec{f}(\vec{z})).$$

(b) If π', M' are as above and P' is $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition, then

$$P'(w, \vec{z}) \leftrightarrow R'(\pi'(\vec{g})(\vec{w}), \pi'(\vec{f})(\vec{z}))$$

where R' has the same $\Sigma_0^{(n)}$ definition over M'.

We prove this by first substituting $\vec{f}(\vec{z})$ and then $\vec{g}(\vec{w})$, using two different arguments. The claim then follows from the pair of claims:

Claim 1 Let:

$$P_0(\vec{y}^n, \vec{z}) \leftrightarrow= R(y^n, f_1(z_1), \ldots, f_r(z_r)).$$

Then:

(a) P_0 is $\Sigma_0^{(n)}(M)$ in a parameter p_0.

(b) Let π', M', R' be as above. Let P_0' have the same $\Sigma_0^{(n)}(M')$ definition in $\pi'(p_0)$. Then:

$$P_0'(\vec{y}^n, \vec{z}) \leftrightarrow R'(y^n, \pi'(\vec{f})(\vec{z})).$$

Claim 2 Let

$$P(\vec{w}, \vec{z}) \leftrightarrow: P_0(g_1(w_1), \ldots, g_s(w_s), \vec{z}).$$

Then:

(a) P is $\Sigma_0^{(n)}(M)$ in a parameter p.

(b) Let π', M', P_0' be as above. Let P' have the same $\Sigma_1^{(n)}(M')$ definition in $\pi'(p)$. Then

$$P'(\vec{w}, \vec{z}) \leftrightarrow P_0'(\pi'(\vec{g})(\vec{w}), \vec{z}).$$

We prove Claim 1 by imitating the argument in Case 2, taking $h = m$ and using §6 Lemma 2.6.11. The details are left to the reader. We then prove Claim 2 by imitating the argument in Case 1: We know that $g_1, \ldots, g_s \in H^n$. Set: $p = \langle g_1, \ldots, g_n, p\rangle$. Then P is $\Sigma_0^{(n)}(M)$ in p, since:

$$P(\vec{w}, \vec{z}) \leftrightarrow \bigvee y_1 \ldots y_s \in C_h(p)(\bigwedge_{i=1}^{s} y_i = g_i(w_i) \wedge P_0(\vec{y}, \vec{z}))$$

where g_i, p_0 are rud in P, for a sufficiently large h. But if P' is $\Sigma_0^{(n)}(M')$ in $\Pi'(P)$ by the same definition, we obviously have:

$$\begin{aligned} P'(\vec{w}, \vec{z}) \ & \leftrightarrow \bigvee y_1 \ldots y_r(\bigwedge_{i=1}^{s} y_i = \pi'(g)(w_i) \wedge P'_0(\vec{y}, \vec{z})) \\ & P'_0(\pi'(\vec{g})(\vec{w}), \vec{z}). \end{aligned}$$

QED (Lemma 2.7.13)

We can repeat the proof in Case 3 with "extra" arguments $\vec{u}^n$. Thus, after rearranging arguments we would have $R(\vec{u}^n, \vec{y}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ where $i_1, \ldots, i_r < n$. We would then define

$$P(\vec{u}^n, \vec{w}, \vec{z}) \leftrightarrow: R(\vec{u}^n, \vec{g}(\vec{w}), \vec{f}(\vec{z})).$$

This gives us:

Corollary 2.7.16. *Let $n < w$. Let $R(\vec{u}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_p \leq n$. Let $f_l \in \Gamma_{i_l}^n$ for $l = 1, \ldots, r$. Set:*

$$P(\vec{u}^n, \vec{z}) \leftrightarrow: R(\vec{u}^n, f_1(z_1), \ldots, f_r(z_r)).$$

Then:

(a) $P(\vec{u}^n, \vec{z})$ is $\Sigma_0^{(n)}$ in a parameter p.

(b) Let $\pi' \supset \pi$ such that $\pi' : M \to_{\Sigma_0^{(n)}} M'$. Let R' be $\Sigma_0^{(n)}(M')$ by the same definition. Let P' be $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition. Then

$$P'(\vec{u}^n, \vec{z}) \leftrightarrow R'(\vec{u}^n, \pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)).$$

By Corollary 2.7.15 $\langle M, \pi\rangle$ can have at most one $\Sigma_0^{(n)}$ liftup. But when does it have a liftup? In order to answer this — as before — define a term model $\mathbb{D} = \mathbb{D}^{(n)}$ for the supposed liftup, which will then exist whenever $\mathbb{D}$ is well founded.

Definition 2.7.8. Let M, τ, H, H', π be as above where $\rho_M^n \geq \tau, n \leq w$. The $\Sigma_0^{(n)}$ *term model* $\mathbb{D} = \mathbb{D}^{(n)}$ is defined as follows: (Let e.g. $M = \langle J_\alpha^A, B \rangle$.) We set: $\mathbb{D} = \langle D, \cong, \tilde{\in}, \tilde{A}, \tilde{B} \rangle$ where:

$$D = D^{(n)} =: \text{ the set of pairs } \langle f, x \rangle \text{ such that } f \in \Gamma^n(\tau, M) \text{ and } x \in \pi(\mathrm{dom}(f))$$

$\langle f, x \rangle \cong \langle g, y \rangle \leftrightarrow: \langle x, y \rangle \in \pi(e)$, where

$$e = \{\langle z, w \rangle | f(z) = g(w)\}.$$

$\langle f, x \rangle \tilde{\in} \langle g, y \rangle \leftrightarrow: \langle x, y \rangle \in \pi(e)$, where

$$e = \{\langle z, w \rangle | f(z) \in g(w)\}$$

(similarly for $\tilde{A}, \tilde{B}$).

We shall interpret the model $\mathbb{D}$ in a many sorted language with variables of type $i < \omega$ if $n = \omega$ and otherwise of type $i \leq n$. The variables v^i will range over the domain D_i defined by:

Definition 2.7.9. $D_i = D_i^{(n)} =: \{\langle f, x \rangle \in D | f \in \Gamma_i^n\}$.

Under this interpretation we obtain Łos theorem in the form:

Lemma 2.7.17. *Let $\varphi(v_1^{i_1}, \ldots, v_r^{i_r})$ be $\Sigma_0^{(n)}$. Then:*

$$\mathbb{D} \models \varphi[\langle f_1, x_1 \rangle, \ldots, \langle f_r, x_r \rangle] \leftrightarrow \langle x_1, \ldots, x_r \rangle \in \pi(e)$$

where $e = \{\langle \vec{z} \rangle | M \models \varphi[f_1(z_1), \ldots, f_r(z_r)]\}$ and $\langle f_l, x_l \rangle \in D_{i_l}$ for $l = 1, \ldots, r$.

Proof: By induction on i we show:

Claim If $i < n$ or $i = n < w$, then the assertion holds for $\Sigma(i)_0$ formulae.

Proof: Let it hold for $j < i$. We proceed by induction on the formula φ.

Case 1 φ is primitive (i.e. φ is $v_i \dot{\in} v_j, v_i \dot{=} v_j, \dot{A} v_i$ or $\dot{B} v_i$ (for $M = \langle J_\alpha^A, B \rangle$). This is immediate by the definition of $\mathbb{D}$.

Case 2 φ is $\Sigma_h^{(j)}$ where $j < i$ and $h = 0$ or 1. If $h = 0$ this is immediate by the induction hypothesis. Let $h = 1$. Then $\varphi = \bigvee u^j \Psi$, where Ψ is $\Sigma_0^{(i)}$. By bound relettering we can assume *w.l.o.g.* that u^i is not in our good sequence $v_1^{i_1}, \dots, v_r^{i_r}$. We prove both directions, starting with $(\rightarrow)$:

Let $\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \dots, \langle f_r, x_r\rangle]$. Then there is $\langle g, y\rangle \in D_j$ such that

$$\mathbb{D} \models \Psi[\langle g, y\rangle, \langle f_1, x_1\rangle, \dots, \langle f_r, x_r\rangle]$$

($u^j, \vec{v}$ being the good sequence for Ψ). Set $e' = \{\langle w, \vec{z}\rangle | M \models \Psi[g(w), \vec{z}(\vec{x})]\}$. Then $\langle y, \vec{x}\rangle \in \pi(e')$ by the induction hypothesis on i. But in M we have:

$$\bigwedge w, \vec{z}(\langle w, \vec{z}\rangle \in e' \rightarrow \langle \vec{z}\rangle \in e).$$

This is a Π_1 statement about e', e. Since $\pi : H \rightarrow_{\Sigma_1} H'$ we can conclude:

$$\bigwedge w, \vec{z}(\langle w, \vec{z}\rangle \in \pi(e') \rightarrow \langle \vec{z} \in \pi(e)).$$

But $\langle y, \vec{x}\rangle \in \pi(e')$ by the induction hypothesis. Hence $\langle \vec{x} \in \pi(e)$. This proves $(\rightarrow)$. We now prove $(\leftarrow)$. Let $\langle \vec{x}\rangle \in \pi(e)$. Let R be the $\Sigma_0^{(j)}$ relation

$$R(w, z_1, \dots, z_r) \leftrightarrow= M \models \varphi[w, z_1, \dots, z_r].$$

Let G be a $\Sigma_0^{(j)}(M)$ map to H^j which uniformizes R. Then G is a spezialization of a function $G'(z_1^{h_1}, \dots, z_r^{h_r})$ such that $h_l \leq j$ for $l \leq j$. Thus G' is a good $\Sigma_0^{(j)}$ function. But

$$f_l(z) = F_l(z, p) \text{ for } z \in \text{dom}(f_l) \text{ for } l = 1, \dots, r$$

where F_l is a good $\Sigma_0^{(k)}$ map to H^{h_l} for $l = 1, \dots, r$ and $j \leq k < i$. (We assume *w.l.o.g.* that the parameter p is the same for all $l = 1, \dots, r_n$.) Define $G''(u^k, w)$ by:

$$G''(u, w) \simeq: G'((u)_0^{r-1}, \dots, (u)_{r-1}^{r-1}, w)$$

then G'' is a good $\Sigma_1^{(k)}$ function. Define g by: $\text{dom}(g) = \overset{r}{\underset{i=1}{\times}} \text{dom}(f_i)$ and: $g(\langle \vec{z}\rangle) = G''(\langle \vec{z}\rangle, p)$ for $\langle \vec{z}\rangle \in \text{dom}(g)$. Then $g \in \Gamma^n$ and $g(\langle \vec{z}\rangle) = G(f_1(z_1), \dots, f_r(z_r))$. Hence, letting:

$$e' = \{\langle w, \vec{z}\rangle | M \models \Psi[g(w), \vec{f}(\vec{z})]\},$$

we have:

$$\bigwedge \vec{z}(\langle \vec{z}\rangle \in e \leftrightarrow \langle\langle \vec{z}\rangle, \vec{z}\rangle \in e').$$

This is a Π_1 statement about e, e' in H. Hence in H' we have:

$$\bigwedge \vec{z}(\langle \vec{z}\rangle \in \pi(e) \leftrightarrow \langle\langle \vec{z}\rangle, \vec{z}\rangle \in \pi(e')).$$

But then $\langle\langle\vec{z}\rangle, \vec{z}\rangle \in \pi(e')$. By the induction hypothesis we conclude:

$$\mathbb{D} \models \Psi[\langle g, \langle\vec{z}\rangle\rangle, \langle f_1, x_1\rangle, \ldots, \langle f_r, x_r\rangle].$$

Hence:

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_r, x_r\rangle].$$

QED (Case 2)

Case 3 φ is $\Psi_0 \wedge \Psi_1, \Psi_0 \wedge \Psi_1, \Psi_0 \to \Psi_1, \Psi_0 \leftrightarrow \Psi_1$, or $\neg\Psi$.

This is straightforward and we leave it to the reader.

Case 4 $\varphi = \bigvee u^i \in v_l\chi$ or $\bigwedge u^i \in v_l\chi$, where v_l has type $\geq i$. We display the proof for the case $\varphi = \bigvee u^i \in v_l\chi$. We again assume *w.l.o.g.* that $u' \neq v_j$ for $j = 1, \ldots, r$. Set: $\Psi = (u^i \in v_l \wedge \chi)$. Then φ is equivalent to $\bigvee u^i\Psi$. Using the induction hypothesis for χ we easily get:

$$(*) \quad \begin{array}{ll} \mathbb{D} \models & \Psi[\langle g, y\rangle, \langle f_1, x_i\rangle, \ldots, \langle f_r, x_r\rangle] \\ & \langle y, x_1, \ldots, x_n\rangle \in \pi(e') \end{array}$$

where $e' = \{\langle w, \vec{z}\rangle | M \models \Psi[g(w), \vec{f}(\vec{z})]\}$. Using $(*)$, we consider two subcases:

Case 4.1 $i < n$.

We simply repeat the proof in Case 2, using $(*)$ and with i in place of j.

Case 4.2 $i = n < w$.

(Hence v_l has type n.) For the direction $(\to)$ we can again repeat the proof in Case 2. For the other direction we essentially revert to the proof used initially for Σ_0 liftups.

We know that $e \in H$ and $\langle\vec{x}\rangle \in \pi(e)$, where $e = \{\langle\vec{z}\rangle | M \models \varphi[f_1(z_1), \ldots, f_r(z_r)]\}$. Set:

$$R(w^n, \vec{z}) \leftrightarrow: M \models \Psi[w^n, f_1(z_1), \ldots, f_r(z_r)].$$

Then R is $\underline{\Sigma}_0^{(n)}$ by Corollary 2.7.16. Moreover $\bigvee w^n R(w^n, \vec{z}) \leftrightarrow \langle\vec{z}\rangle \in e$. Clearly $f_l \in H_M^n$ since $f_l \in \Gamma_n^n$. Let $s \in H_M^n$ be a well odering of $\bigcup \text{rng}(f_l)$. Clearly:

$$\begin{array}{ll} R(w^n, \vec{z}) & \to w^n \in f_l(z_l) \\ & \to w^n \in \bigcup \text{rng}(f_l). \end{array}$$

We define a function g with domain e by:

$$g(\langle\vec{z}\rangle) = \text{the } s\text{–least } w \text{ such that } R(w, \vec{z}).$$

Since R is $\underline{\Sigma}_0^{(n)}$, it follows easily that $g \in H_{\rho^n}^M$. Hence $g \in \Gamma_n^n$. But then

$\bigwedge \vec{z}(\langle\vec{z}\rangle \in e \leftrightarrow \langle\langle\vec{z}\rangle, \vec{z}\rangle \in e')$, where e' is defined as above, using this g.

Hence in H' we have:

$$\bigwedge \vec{z}(\langle\vec{z}\rangle \in \pi(e) \leftrightarrow \langle\langle\vec{z}\rangle, \vec{z}\rangle \in \pi(e')).$$

Since $\langle\vec{x}\rangle \in \pi(e)$ we conclude that $\langle\langle\vec{x}\rangle, \vec{x}\rangle \in \pi(e')$. Hence:

$$\mathbb{D} \models \Psi[\langle g, \langle\vec{x}\rangle\rangle, \langle f_1, x_1\rangle, \ldots, \langle f_r, x_r\rangle].$$

Hence:

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_r, x_r\rangle].$$

QED (Lemma 2.7.17)

Exactly as before we get:

Lemma 2.7.18. *If $\tilde{\in}$ is ill founded, then the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$ does not exist.*

We leave it to the reader and prove the converse:

Lemma 2.7.19. *If $\tilde{\in}$ is well founded, then the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$ exists.*

Proof: We shall again use the term model $\mathbb{D}$ to define an explicit $\Sigma_0^{(n)}$ liftup. We again define:

Definition 2.7.10. $x^* = \pi^*(x) =: \langle \mathrm{const}_x, 0\rangle$, where $\mathrm{const}_x =: \{\langle x, 0\rangle\} =$ the constant function x defined on $\{0\}$.

Using Łoz theorem Lemma 2.7.17 we get:

(1) $\pi^* : M \to_{\Sigma_0^{(n)}} \mathbb{D}$
(where the variables v^i range over J_i on the $\mathbb{D}$ side).

The proof is exactly like the corresponding proof for Σ_0–liftups ((1) in Lemma 2.7.5). In particular we have: $\pi^* : M \to_{\Sigma_0} \mathbb{D}$. Repeating the proof of (2) in Lemma 2.7.5 we get:

(2) $\mathbb{D} \models$ Extensionality.
Hence $\cong$ is again a congruenzrelation and we can factor $\mathbb{D}$, getting:

$$\hat{\mathbb{D}} = (\mathbb{D} \backslash \cong) = \langle \hat{D}, \hat{\in}, \hat{A}, \hat{B} \rangle$$

where

$$\hat{D} =: \{\hat{s} | s \in D\},\ \hat{s} =: \{t | t \cong s\} \text{ for } s \in D$$
$$\hat{s} \hat{\in} \hat{t} \leftrightarrow: s \tilde{\in} t$$
$$\hat{A}\hat{s} \leftrightarrow: \tilde{A}s,\ \hat{B}\hat{s} \leftrightarrow: \tilde{B}s$$

Then $\hat{\mathbb{D}}$ is a well founded identity model satisfying extensionality. By Mostowski's isomorphism theorem there is an isomorphism k of $\hat{\mathbb{D}}$ onto M', where $M' = \langle |M'|, \in, A', B' \rangle$ is transitive. Set:

$$[s] =: k(\hat{s}) \text{ for } s \in D$$
$$\pi'(x) =: [x^*] \text{ for } x \in M$$
$$H_i =: \{\hat{s} | s \in D_i\} (i < n \text{ or } i = n < w).$$

We shall *initially* interpret the variables v^i on the M' side as ranging over H_i. We call this the *pseudo interpretation*. Later we shall show that it (almost) coincides with the intended interpretation. By (1) we have

(3) $\pi' : M \rightarrow_{\Sigma_0^{(n)}} M'$ in the pseudo interpretation. (Hence $\pi' : M \rightarrow_{\Sigma_0^{(n)}} M'$.)

Lemma 2.7.19 then follows from:

Lemma 2.7.20. *$\langle M', \pi' \rangle$ is the $\Sigma^{(n)}$ liftup of $\langle M, \pi \rangle$.*

For $n = 0$ tis was proven in Lemma 2.7.6, so assume $n > 0$. We again us the abbreviation:

$$[f, x] =: [\langle f, x \rangle] \text{ for } \langle f, x \rangle \in D.$$

Defining $\tilde{H}$ exactly as in the proof of Lemma 2.7.6, we can literally repeat our previous proofs to get:

(4) $\tilde{H}$ is transitive.

(5) $[f, x] = \pi(f)(x)$ if $f \in H$ and $\langle f, x \rangle \in D$. (Hence $\tilde{H} = H'$.)

(6) $\pi' \supset \pi$.

(However (7) in Lemma 2.7.6 will have to be proven later.)

In order to see that $\pi : M \rightarrow_{\Sigma^{(n)}} M'$ in the intended interpretation we must show that $H_i = H^i_M$, for $i < n$ and that $H_n \subset H^n_M$. As a first step we show:

(7) H_i is transitive for $i \leq n$.

Proof: Let $s \in H_i, t \in s$. Let $s = [f, x]$ where $f \in \Gamma_i^n$. We must show that $t = [g, y]$ for $g \in \Gamma_i^n$. Let $t = [g', y]$. Then $\langle y, x\rangle \in \pi(e)$ where

$$e = \{\langle u, v\rangle | g'(u) \in f(v)\}.$$

Set:

$$a =: \{u | g'(u) \in \text{rng}(f)\}, g = g' \wedge a.$$

Claim 1 $g \in \Gamma_i^n$.

Proof: $a \subset \text{dom}(q')$ is $\underset{\sim}{\Sigma}_0^{(n)}$. Hence $a \in H$ and $g \in \Gamma^n$. If $i < n$, then $\text{rng}(g) \subset \text{rng}(f) \subset H_M^i$. Hence $g \in \Gamma_i^n$. Now let $i = n$. Then $\text{rng}(f) \in \Gamma_n^n$ and the relation $z = g(y)$ is $\underset{\sim}{\Sigma}_0^{(n)}$. Hence $g \in H_M^n$.

QED (Claim 1)

Claim 2 $t = [g, y]$

Proof:

$$\bigwedge u, v(\langle u, v\rangle \in e \to \langle u, u\rangle \in e')$$

where $e' = \{\langle u, w\rangle | g(u) = g'(w)\}$. Hence the same Π_1 statement holds of $\pi(e), \pi(e')$ in H'. Hence $\langle y, y\rangle \in \pi(e')$. Hence $[g, y] = [g', y] = t$. QED (7)

We can improve (3) to:

(8) Let $\Psi = \bigvee v_{v_1}^{i_1}, \ldots, v_r^{i_r}\varphi$, where φ is $\Sigma_0^{(n)}$ and $i_l < n$ or $i_l = n < w$ for $l = 1, \ldots, r$. Then π' is "Ψ–elementary" in the sense that:

$$M \models \Psi[\vec{x}] \leftrightarrow M' \models \Psi[\pi'(\vec{x})] \text{ in the pseudo interpretation.}$$

Proof: We first prove ($\to$). Let $M \models \varphi[\vec{z}, \vec{x}]$. Then $M' \models \varphi[\pi'(\vec{z}), \pi'(\vec{x})]$ by (3).

We now prove ($\leftarrow$). Let:

$$M' \models \varphi[[f_1, z_1], \ldots, [f_r, z_r], \pi'(\vec{x})]$$

where $f_l \in \Gamma_{i_l}^n$ for $l = 1, \ldots, r$. Since $\pi'(x) = [\text{const}_x, 0]$, we then have: $\langle z_1, \ldots, z_r, 0 \ldots 0\rangle \in \pi(e)$, where:

$$e = \{\langle u_1, \ldots, u_r, 0 \ldots 0\rangle : M \models \varphi[\vec{f}(\vec{u}), \vec{x}]\}.$$

Hence $e \neq \emptyset$. Hence

$$\bigvee v_1 \ldots v_r M \models \varphi[\vec{f}(\vec{v}), \vec{x}]$$

where $\text{rng}(f_l) \subset H^{i_l}$ for $l = 1, \ldots, r$. Hence $M \models \Psi[\vec{x}]$. QED (8)

If $i < n$, then every $\Pi_1^{(i)}$ formula is $\Sigma_0^{(n)}$. Hence by (8):

(9) If $i < n$ then

$$\pi' : M \to_{\Sigma_2^{(i)}} M' \text{ in the pseudo interpretation.}$$

We also get:

(10) Let $n < w$. Then:

$$\pi' \upharpoonright H_M^n : H_M^n \to_{\Sigma_0} H_n \text{ cofinally.}$$

Proof: Let $x \in H_n$. We must show that $x \in \pi'(a)$ for an $a \in H_M^n$. Let $x = [f, y]$, where $f \in \Gamma_n^n$. Let $d = \text{dom}(f), a = \text{rng}(f)$. Then $y \in \pi(d)$ and:

$$\bigwedge z \in d \ \langle z, 0\rangle \in e$$

where

$$\begin{aligned} e &= \{\langle u, v\rangle | f(u) \in \text{const}_a(v)\} \\ &= \{\langle u, 0\rangle | f(u) \in a\}. \end{aligned}$$

This is a Σ_0 statement about d, e. Hence the same statement holds of $\pi(d), \pi(e)$ in H_n. Hence $\langle z, 0\rangle \in \pi(e)$. Hence $[f, y] \in \pi'(a)$. QED (10)

(**Note:** (10) and (3) imply that $\pi' : M \to_{\Sigma_1^{(n)}} M'$ is the pseudo interpretation, but this also follows directly from (8).)

Letting $M = \langle J_\alpha^A, B\rangle$ and $M' = \langle |M'|, A', B'\rangle$ we define:

$$M_i = \langle H_M^i, A \cap H_M^i, B \cap H_M^i\rangle, M_i' = \langle H_i, A' \cap H_i, B' \cap H_i\rangle$$

for $i < n$ or $i = n < w$. Then each M_i is acceptable. It follows that:

(11) M_i' is acceptable.

Proof: If $i = n$, then $\pi' \upharpoonright M_n : M_n \to_{\Sigma_0} M_n'$ cofinally by (3) and (10). Hence M_n' is acceptable by §5 Lemma 2.5.5. If $i < n$, then $\pi' \upharpoonright M_i : M_i \to_{\Sigma_2^{(i)}} M_i'$ by (9). Hence M_i' is acceptable since acceptability is a Π_2 condition. QED (11)

We now examine the "correctness" of the pseudo interpretation. As a first step we show:

(12) Let $i + 1 \leq n$. Let $A \subset H_{i+1}$ be $\underline{\Sigma}_1^{(i)}$ in the pseudo interpretation. Then $\langle H_{i+1}, A\rangle$ is amenable.

Proof: Suppose not. Then there is $A' \subset H_{i+1}$ such that A' is $\underline{\Sigma}_1^{(i)}$ in the pseudo interpretation, but $\langle H_i, A'\rangle$ is not amenable. Let:

$$A'(x) \leftrightarrow B'(x, p)$$

where B' is $\Sigma_1^{(i)}$ in the pseudoo interpretation. For $p \in M'$ we set:

$$A'_p =: \{x | B'(x,p)\}.$$

Let B be $\Sigma_1^{(i)}(M)$ by the same definition. For $p \in M$ we set:

$$A_p =: \{x | B(x,p)\}.$$

Case 1 $i+1 < n$.

Then $\bigvee p \bigvee a^{i+1} \bigwedge b^{i+1} b^{i+1} \neq a^{l+1} \cap A'_p$ holds in the pseudo interpretation. This has the form: $\bigvee p \bigvee a^{i+1} \varphi(p, a^{i+1})$ where φ is $\Pi_1^{(i+1)}$, hence $\Sigma_0^{(n)}$ in the pseudo interpretation. By (8) we conclude that $M \models \varphi(p, a^{i+1})$ for some $p, a^{i+1} \in M$. Hence $\langle H_M^{i+1}, A_p \rangle$ is not amenable, where A_p is $\underline{\Sigma}_1^{(i)}(M)$.
Contradiction! QED (Case 1)

Case 2 Case 1 fails.

Then $i+1 = n$. Since π' takes H_M^n cofinally to H_n. There must be $a \in H_M^n$ such that $\pi(a) \cap A' \notin H_n$. From this we derive a contradiction. Let $A' = A'_p$ where $p = [f, z]$. Set: $\tilde{B} = \{\langle z, w \rangle | B(w, f(z))\}$. Then $\tilde{B}$ is $\underline{\Sigma}_1^{(i)}(M)$. Set: $b = (d \times a) \cap \tilde{B}$, where $d = \mathrm{dom}(f)$. Then $b \in H_M^n$. Define $g : d \to H_M^n$ by:

$$g(z) =: A_{f(z)} \cap a = \{x \in a | \langle z, x \rangle \in b\}.$$

Then $g \in H_M^n$, since it is rudimentary in $a, b \in H_M^n$. Let $\varphi(u^n, v^n, w)$ be the $\Sigma_0^{(n)}$ statement expressing

$$u = A_w \cap v^n \text{ in } M.$$

Then setting:

$$e = \{\langle v, 0, w \rangle | M \models \varphi[g(v), a, f(z)]\}$$

we have:

$$\bigwedge v \in d \ \langle v, 0, v \rangle \in e.$$

But then the same holds of $\pi(d), \pi(e)$ in H_n. Hence $\langle z, 0, z \rangle \in \pi(e)$. Hence: $[g, z] = A_{[f,z]} \cap \pi(a) \in H_n$.
Contradiction! QED (12)

On the other hand we have:

(13) Let $i+1 < n$. Let $A \subset H_M^{i+1}$ be $\Sigma_1^{(i)}(M)$ in the parameter p such that $A \notin M$. Let A' be $\Sigma_1^{(i)}(M')$ in $\pi'(p)$ by the same $\Sigma_1^{(i)}(M')$ definition in the pseudo interpretation. Then $A' \cap H_{i+1} \notin M'$.

Proof: Suppose not. Then in M' we have:

$$\bigvee a \bigwedge v^{i+1}(v^{i+1} \in a \leftrightarrow A'(v^{i+1})).$$

This has the form $\bigvee a\varphi(a,\pi(p))$ where φ is $\Pi_1^{(i+1)}$ hence $\Sigma_0^{(n)}$. By (8) it then follows that $\bigvee a\varphi(a,p)$ holds in M. Hence $A \in M$. Contradiction! QED (13)

Recall that for any acceptable $M = \langle J_\alpha^A, B\rangle$ we can define ρ_M^i, H_M^i by:

$$\begin{aligned} \rho^0 = {} & \alpha \\ \rho^{i+1} = {} & \text{the least } \rho \text{ such that there is } A \text{ which is} \\ & \underline{\Sigma}_1^{(i)}(M) \text{ with } A \cap \rho \notin M \\ H^i = {} & J_{\rho_i}[A]. \end{aligned}$$

Hence by (11), (12), (13) we can prove by induction on i that:

(14) Let $i < n$. Then

- (a) $\rho_{M'}^i = \rho_i,\ H_{M'}^i = H_i$
- (b) The pseudo interpretation is correct for formulae φ, all of whose variables are of type $\leq i$.

By (9) we then have:

(15) $\pi' : M \to_{\Sigma_2^{(i)}} M'$ for $i < n$.
This means that if $n = \omega$, then π' is automatically Σ^*–preserving. If $n < \omega$, however, it is not necessarily the case that $H_n = H_M^n$, — i.e. the pseudo interpretation is not always correct. By (12), however we do have:

(16) $\rho_n \leq \rho_M^n$, (hence $H_n \subset H_{M'}^n$).
Using this we shall prove that π' is $\Sigma_0^{(n)}$–preserving. As a preliminary we show:

(17) Let $n < w$. Let φ be a $\Sigma_0^{(n)}$ formula containing only variables of type $i \leq n$. Let $v_1^{i_1}, \ldots, v_r^{i_r}$ be a good sequence for φ. Let $x_1, \ldots, x_r \in M'$ such that $x_l \in H_{i_l}$ for $l = 1, \ldots, r$. Then $M \models \varphi[x_1, \ldots, x_r]$ holds in the correct sense iff it holds in the pseudo interpretation.

Proof: (sketch)
Let C_0 be the set of all such φ with: φ is $\Sigma_1^{(i)}$ for an $i < n$. Let C be the closure of C_0 under sentential operation and bounded quantifications of the form $\bigwedge v^n \in w^n\varphi,\ \bigvee v^n \in w^n\varphi$. The claim holds for $\varphi \in C_0$ by (15). We then show by induction on φ that it holds for $\varphi \in C$. In

passing from φ to $\bigwedge v^n \in w^n\varphi$ we use the fact that w^n is interpreted by an element of H_n. QED (17)

Since $\pi''' H^i_M \subset H_i$ for $i \leq n$, we then conclude:

(18) $\pi' : M \to_{\Sigma_0^{(n)}} M'$.

It now remains only the show:

(19) $[f, x] = \pi'(f)(x)$.

Proof: Let $f(x) = G(x, p)$ for $x \in \mathrm{dom}(f)$, where G is $\Sigma_1^{(j)}$ good for a $j < n$. Let $a = \mathrm{dom}(f)$. Let $\Psi(u, v, w)$ be a good $\Sigma_1^{(j)}$ definition of G. Set:

$$e = \{\langle z, y, w\rangle | M \models \Psi[f(z), \mathrm{id}_a(y), \mathrm{const}_p(w)\}.$$

Then $z \in a \to \langle z, z, 0\rangle \in e$. Hence the same holds of $\pi(a), \pi(e)$. But $x \in \pi(a)$. Hence:

$$M' \models \Psi[[f, x], [\mathrm{id}_a, x], [\mathrm{const}_p, x]],$$

where $[\mathrm{id}_a, x] = x, [\mathrm{const}_p, 0] = \pi'(p)$. Hence:

$$[f, x] = G'(x, \pi'(p)) = \pi'(f)(x),$$

where G' has the same $\Sigma_1^{(j)}$ definition. QED (19)

Lemma 2.7.20 is then immediate from (6), (18) and (19).

QED (Lemma 2.7.19)

As a corollary of the proof we have:

Lemma 2.7.21. *Let $\langle M', \pi'\rangle$ be the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$. Then π' is $\Sigma_2^{(i)}$ preserving for $i < n$.*

Finally, we note that we have:

Lemma 2.7.22. *Let $\pi^* \supset \pi$ such that $\pi^* : M \to_{\Sigma_0^{(n)}} M^*$. Then the $\Sigma_0^{(n)}$ liftup $\langle M', \pi'\rangle$ of $\langle M, \pi\rangle$ exists. Moreover there is a unique map $\sigma : M' \to_{\Sigma_0^{(n)}} M^*$ such that $\sigma \upharpoonright H' = \mathrm{id}$ and $\sigma\pi' = \pi^*$.*

Proof: $\tilde{\in}$ is well founded, since:

$$\langle f, x\rangle \tilde{\in} \langle g, y\rangle \leftrightarrow \pi^*(f)(x) \in \pi^*(g)(y).$$

Thus $\langle M', \pi'\rangle$ exists. But for $\Sigma_0^{(n)}$ formulae $\varphi = \varphi(v_1^{i_1}, \ldots, v_r^{i_r})$ we have:

$$\begin{aligned} M' &\models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)(x_r)] \\ &\leftrightarrow \langle x_1, \ldots, x_n\rangle \in \pi(e) \\ &\leftrightarrow M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_r)(x_r)] \end{aligned}$$

where:

$$e = \{\langle z_1, \ldots, z_r\rangle | M \models \varphi[f_1(z_1), \ldots, f_r(z_r)]\}$$

and $\langle f_l, x_l\rangle \in \Gamma^n_{i_l}$ for $l = 1, \ldots, r$. Hence there is a $\Sigma_0^{(n)}$–preserving embedding $\sigma : M' \to M^*$ defined by:

$$\sigma(\pi'(f)(x)) = \pi^*(f)(x) \text{ for } \langle f, x\rangle \in \Gamma^n.$$

But σ is the unique $\sigma : M' \to_{\Sigma_0^{(n)}} M^*$ such that $\sigma \restriction H' = \text{id}$ and $\sigma\pi' = \pi^*$, since, by the definition of $\pi'(f)$ and $\pi^*(f)$ for $f \in \Gamma^n$ we then have:

$$\sigma(\pi'(f)(x)) = \pi^*(f)(x) \text{ for } x \in \pi(\text{dom}(f)).$$

QED (Lemma 2.7.22)

We can improve the result by making stronger assumptions on the map π, vor instance:

Lemma 2.7.23. *Let $\langle M^*, \pi^*\rangle$ be the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$. Let $\pi^* \restriction \rho_M^{n+1} = \text{id}$ and $\mathbb{P}(\rho_M^{n+1}) \cap M^* \subset M$. Then $\rho^n_{M^*} = \sup \pi^{*''} \rho^n_M$.*

(Hence the pseudo interpretation is correct and π^* is $\Sigma_1^{(n)}$ preserving.)

Proof: Suppose not. Let $\tilde\rho = \sup \pi^{*''} \rho^n_M < \rho^n_{M^*}$. Set:

$$H^n = H^n_M = J^{A_M}_{\rho^n_M};\ \tilde H = J^{A_M}_{\tilde\rho}.$$

Then $\tilde H \in M^*$. Let A be $\Sigma^{(n)}(M)$ in p such that $A \cap \rho_M^{n+1} \notin M$. Let:

$$Ax \leftrightarrow \bigvee y^n B(y^n, x),$$

where B is $\Sigma_0^{(n)}$ in p. Let B^* be $\Sigma_0^{(n)}(M^*)$ in $\pi^*(p)$ by the same definition. Then

$$\pi^* \restriction H^n : \langle H^n, B \cap H^n\rangle \to_{\Sigma_1} \langle \tilde H, B^* \cap \tilde H\rangle.$$

Then $A \cap \rho_M^{n+1} = \tilde A \cap \rho_M^{n+1}$, where:

$$\tilde A = \{x | \bigvee_{y^n} \in \tilde H\ B^*(y, x)\}.$$

But $\tilde A$ is $\Sigma_0^{(n)}(M^*)$ in $\pi^*(p)$ and $\tilde H$. Hence

$$A \cap \rho_M^{n+1} = \tilde A \cap \rho_M^{n+1} \in \mathbb{P}(\rho_M^{n+1}) \cap M^* \subset M.$$

Contradiction! QED (Lemma 2.7.13)