

§4 The limit case

§4.1 The 1st limit case

Suppose that λ is an ω -point of the iteration, $\delta = \delta_\lambda$, and we wish to give δ^+ the cofinality ω . Let β be the supremum of the cardinals $< \beta_\lambda$. Then either $\beta = \beta_\lambda$, $2^\beta = \beta$, and $cf(\beta) = \omega_1$, or else we are doing the second construction (hence $2^\beta = \beta$ and $2^\beta = \beta^+$ by GCH) and $\beta_\lambda = \beta^+ \in A$, where $\beta_\lambda < \delta^{(\omega_1)}$. We must find a completion \mathbb{B}_λ of $\bigcup_{i < \lambda} \mathbb{B}_i$ which gives every regular $\tau \leq \beta$ the cofinality ω . If $cf(\beta) = \omega_1$,

we want β^+ to remain a cardinal in $\mathcal{V}^{\mathbb{B}_\lambda}$. Otherwise we want β^+ to have cofinality ω_1 in $\mathcal{V}^{\mathbb{B}_\lambda}$. We know, of course, that $2^\delta = \delta^+$, that every $\gamma < \delta$ is collapsed to ω_1 by some \mathbb{B}_i ($i < \lambda$) and that some \mathbb{B}_i ($i < \lambda$) gives δ cofinality ω .

Set $M = L_\beta^A$, where $L_\gamma^A[A] = H_\gamma$ whenever $\gamma \leq \beta$ and $2^\gamma = \gamma$. Set $Q = \langle L_\beta^A, \langle \mathbb{B}_i \mid i < \lambda \rangle \rangle$,

$N = \langle H_{\beta^+}, Q, M, <, in \rangle$ where $<$ well orders H_{β^+} .

Amimitating the definitions in the second successor case, we set:

Def $\Gamma_* =$ the set of $\langle Q, B, C \rangle$ s.t.

- $Q = \langle L_{\delta}^A, B \rangle$ models Zermelo
- $B = \langle B_i \mid i < \delta_0 \rangle$ where $\delta_0 \leq \delta$ is a limit ordinal; $B_i \in Q$ is a complete BA in Q , B_j is pred over B_i in Q whenever $i < j < \delta_0$ and $i, j \notin A$ in Q ; $\sup_{i < \delta_0} d_i = \delta$ where $d_i = d(B_i) \geq \omega_1$ for $0 < i < \delta_0$ and $B_0 = 2$.
- $B \subset \bigcup_i B_i$ s.t. $B_i = B \cap B_i$ is B_i -generic over Q for $i < \delta_0$
- $C \subset \delta_0$, $\sup C = \delta_0$, $\text{otp}(C) = \omega$, (Set $c_n =$ the n -th element of C).

Def For $u = \langle Q_u, B^u, C^u \rangle$, $v = \langle Q_v, B^v, C^v \rangle$ s.t. $\pi : u \triangleleft_* v \iff$
 $(\pi : Q_u \triangleleft Q_v \wedge \pi'' B^u \subset B^v \wedge \pi'' C^u = C^v)$

Def $u \triangleleft_* v \iff \forall \pi \pi : u \triangleleft_* v$

Fact 1 Let $v = \langle Q_v, B^v, C^v \rangle \in \Gamma_*$, $\alpha \leq d = \text{ht } \omega_1^{Q_v}$. There is at most one pair $\langle u, \pi \rangle$ s.t. $\pi : u \triangleleft_* v$ and $d = d_u$.

proof of Fact 1.

Set $Q^B = Q[B] = \bigcup_{i < \delta_0^Q} Q[B_i]$ for

$\langle Q, B, C \rangle \in \Gamma_*$. Let $\pi : u \triangleleft_* v$ s.t.

$d = d_u$. Then, since $\pi'' B_i^u \subset B_{\pi(i)}^v$ for

$i < \delta_0^u$, π extends to a unique

$\pi_i : Q_u[B_i^u] \rightarrow Q_v[B_{\pi(i)}^v]$ s.t.

$\pi_i(B_i^u) = B_{\pi(i)}^v$. Set $\pi^* = \bigcup_i \pi_i$.

Then $\pi^* : Q^u[B^u] \rightarrow Q^v[B^v]$

cofinally. Set $f_i^u =$ the $Q^u[B^u]$ -least

f s.t. f maps $d = \omega_1^{Q^u}$ onto $L_{d_{c_i}}^{A^u}$. Let

f_i^v be defined similarly for $i < \delta_0^v$.

Then $\pi^*(f_i^u) = f_i^v$ and $\pi^*(f_i^u(\xi)) =$

$\pi(f_i^u(\xi)) = f_i^v(\xi)$ for $i < \omega, \xi <$

Hence $\text{rng}(\pi) = \{ f_i^v(\xi) \mid i < \omega, \xi < \alpha \}$.

QED (Fact 1)

Hence:

Fact 2 Let $u, v \in \Gamma_*$. There is at most

one π s.t. $\pi : u \triangleleft_* v$.

Def Let $u \triangleleft_* v$. $\pi_{u,v} = \pi$ that π s.t.

$\pi : u \triangleleft_* v$.

We easily get:

Fact 3 Let $u \triangleleft_* v \triangleleft_* w$. Then $u \triangleleft_* w$ and

$$\pi_{uw} = \pi_{vw} \cdot \pi_{uv}.$$

(i.e. $\langle \Pi_*, \Pi_* \rangle$ is a commutative system where $\Pi_* = \langle \pi_{u,v} \mid u \triangleleft_* v \rangle$.)

Fact 4 $\langle \Pi_*, \Pi_* \rangle$ is continuous - i.e. if $I \subset \Pi_*$, $R \subset I^2$ s.t. R is directed (i.e. $\forall u, v \in I \exists w \in I \ u, v \triangleleft_* w$) and $u \triangleleft_* v \triangleleft_* w_0$ whenever $u R v$, then there is a unique w_1 s.t. $u \triangleleft_* w_1 \triangleleft_* w_0$ for $u \in I$ and $\langle w_1, \langle \pi_{u,w_1} \mid u \in I \rangle \rangle$ is the direct limit of $\langle I, \langle \pi_{uv} \mid u R v \rangle \rangle$.

(Note It follows that $\pi_{uw_0} = \pi_{w_1 w_0} \pi_{uw_1}$ for $u \in I$.)

Hence:

Fact 5 $\{d_u \mid u \triangleleft_* w\}$ is closed in d_w^{+1}

Def M is a smooth model iff

- $M = L_{\beta}^A$ models ZFC⁻ or Zermelo
- $2^{\omega} = \omega_1$ in M (in particular, ω_1^M exists)

Def Q is a smooth segment of M iff

$Q = \langle L_{\gamma}^A, R_1, \dots, R_n \rangle$ where

- $M = L_{\beta}^A$ is a smooth model
- $\gamma > \omega$ is a limit cardinal in M and $L_{\gamma}[A] = H_{\gamma}^M$
- $R_1, \dots, R_n \in Q$

Def $\Gamma =$ the set of $\langle Q, M, B \rangle$ s.t.

- Q is a smooth segment of M
- $\langle Q, B, C \rangle \in \Gamma_x$ for a C s.t.

$C \in M[B_i]$ for some $i < \gamma_0^Q$.

(Note Each B_i is B_i -generic over M ,

where $Q = \langle L_{\gamma}^A, B \rangle$.)

Def Let $u, v \in \Gamma$, $u = \langle Q_u, M_u, B^u \rangle$, $v = \langle Q_v, M_v, B^v \rangle$.

$\pi : u \triangleleft v$ iff

- $\pi \upharpoonright Q_u : \langle Q_u, C, B^u \rangle \triangleleft_{*} \langle Q_v, \pi''C, B^v \rangle$

for a $C \in M_u[B_i^u]$, $i < \omega$.

- There is M_{uv} s.t. $\langle M_{uv}, \pi \rangle =$ the liftup of $\langle M_u, \pi \upharpoonright Q_u \rangle$.

Fact 6 Let $u, v \in \Gamma$. There is at most one π s.t. $\pi: u \triangleleft v$.

proof.

Let $\pi: u \triangleleft v$. We show that π is unique. Let i be least s.t. δ_0^u is ω -cofinal in M_u . Let $C =$ the least $C \in M_u[B_i^u]$ s.t. $C \subset \delta_0^u = \sup C$ and $\text{otp}(C) = \omega$. Let $i' = \pi(i)$.

Then $\pi[B_i^u] \subset B_{i'}^v$, where $\pi(B_i^u) = B_{i'}^v$. Hence

there is $\pi' \supset \pi$ s.t. $\pi': M_u[B_i^u] \triangleleft M_v[B_{i'}^v]$ and $\pi(B_i^u) = B_{i'}^v$. Let $C' = \pi'(C)$. Then

C', i' are defined from $\langle Q_v, M_v, B^v \rangle$ as C, i were defined from $\langle Q_u, M_u, B^u \rangle$. Set:

$\bar{u} = \langle Q_u, B^u, C \rangle$, $\bar{v} = \langle Q_v, B^v, C' \rangle$. Then

\bar{u} is defined from u + \bar{v} is defined from v by the same def. But clearly

\bar{v} by the same def. But clearly

$\pi \upharpoonright Q_u: \bar{u} \triangleleft_* \bar{v}$; hence $\pi \upharpoonright Q_u = \pi \bar{u} \bar{v}$.

Thus $\pi \upharpoonright Q_u$ depends only on the pair u, v .

But then so does π , since \bar{u} depends only on $\pi \upharpoonright Q_u$. QED (Fact 6)

Def $u \triangleleft v \iff \exists \pi \pi: u \triangleleft v$

For $u \triangleleft v$ we set:

$\pi_{uv} =$ that π s.t. $\pi: u \triangleleft v$

We easily get:

Fact 7 $\langle \Gamma, \Pi \rangle$ is a continuous commutative system, where $\Pi = \langle \pi_{uv} \mid u \triangleleft v \rangle$

Hence:

Fact 8 $\{\alpha_u \mid u \triangleleft v\}$ is closed in α_v

Fact 9 Let $u, v \triangleleft w$, $\text{rng}(\pi_{uw}) \subset \text{rng}(\pi_{v,w})$.

Then $u \triangleleft v$ and $\pi_{vw} \pi_{uv} = \pi_{uw}$.

Let Q, M, N be as defined above, let $B = \langle B_i \mid i < \delta_0 \rangle$ where $\delta_0 = \lambda$ be the iteration up to λ . Suppose $B \subset \bigcup_i B_i$ s.t. $B_i = B \cap B_i$ is B_i -generic for $i < \lambda$. Then $\langle Q, M, B \rangle \in \Gamma$.

As a preliminary to defining B_λ we define:

Let \mathcal{L} be the language on N with:

Predicate \in ; Constants \underline{x} ($x \in N$), \mathbb{B}°

Axioms ZFC⁻, $\bigwedge \underline{x} (\underline{x} \in \underline{x} \leftrightarrow \bigvee_{\underline{z} \in \underline{x}} \underline{x} = \underline{z})$,

$H_{\omega_1} = \underline{H}_{\omega_1}$, $\langle \underline{Q}, \underline{M}, \mathbb{B}^\circ \rangle \in \Gamma$ and:

(*) For each $\underline{\beta} < \underline{\beta}$ there are u, π s.t. $u \in H_{\omega_1}$,

$\pi: u \triangleleft \langle \underline{Q}, \underline{M}, \mathbb{B}^\circ \rangle \wedge \underline{\beta} \in \text{rng}(\pi) \wedge \Psi$,

where $\Psi = \begin{cases} \sup \pi'' \beta_u = \underline{\beta} & \text{if } \beta \text{ is regular,} \\ u = u & \text{if not} \end{cases}$

Note $\sup \pi'' \beta = \underline{\beta}$ says the same as $M_u, \langle \underline{Q}, \underline{M}, \mathbb{B}^\circ \rangle = \underline{M}$, recalling

$\langle M_{u \cup v}, \pi \rangle =$ the lift up of $\langle M_u, \pi \upharpoonright Q_u \rangle$
for $u \triangleleft v$.)

If B is \mathbb{B}_i -generic for some $i < \aleph_0$, we can,

in $V[B]$, define: $\mathbb{B}/B = \langle \mathbb{B}_{i+j}/B \mid j < \aleph_0 - i \rangle$

$Q^B = \langle Q[B], B \rangle = \langle L_{\aleph_1}^{A, B}, \mathbb{B}/B \rangle$

$M^B = \langle M[B], B \rangle = L_{\beta}^{A, B}$

$N^B = \langle N[B], B \rangle = \langle H_{\beta^+}^{V[B]}, \epsilon, Q^B, M^B, <, \dots \rangle$

where $<$ well orders N^B .

We can then define a language \mathcal{L}_B over N^B as before in $\mathcal{V}[B]$ with Q^B, M^B in place of Q, M .

Let $B \subset \bigcup_i B_i$ s.t. $B_i = B \cap B_i$ is B_i -generic for $i < \delta_0$, we set:

$B/B_{i_0} = \{b/B_{i_0} \mid b \in B\}$, where $b \rightarrow b/B_{i_0}$ is the canonical projection of $\bigcup_i B_i$ onto $\bigcup_j B_j/B_{i_0}$.

(We shall follow the convention in §3 of saying that $\mathcal{M} = \langle |\mathcal{M}|, \in^{\mathcal{M}}, B^{\mathcal{M}} \rangle$ is a solid model of \mathcal{L} iff \mathcal{M} is solid, $N \subset \text{wfcore}(\mathcal{M})$, and \mathcal{M} becomes a model of \mathcal{L} if we interpret \underline{x} by x for $x \in N$.)
We then get:

Lemma 0

(a) Let $\mathcal{M} = \langle |\mathcal{M}|, \in^{\mathcal{M}}, B \rangle$ is a solid model of \mathcal{L} , then $B \subset \bigcup_i B_i$ and $B_i = B \cap B_i$ is B_i -generic over \mathcal{V} for $i < \delta_0$.

(b) Let $B \subset \bigcup_i B_i$ s.t. $B_i = B \cap B_i$ is B_i -generic for $i < \delta_0$. Let $i_0 < \delta_0$. Then $\mathcal{M} = \langle |\mathcal{M}|, \in^{\mathcal{M}}, B \rangle$ is a solid model of \mathcal{L} iff $\mathcal{M}_{i_0} = \langle |\mathcal{M}|, \in^{\mathcal{M}}, B/B_{i_0} \rangle$ is a solid model of $\mathcal{L}_{B_{i_0}}$.

The proof is straightforward.

Lemma 1 \mathcal{L} is consistent,

proof.

Since δ_0 is an ω -point, there is an $i < \delta_0$ s.t. $\text{lt}_{\mathbb{B}_i} \delta_0$ is ω -cofinal. By Lemma 0 it suffices to show that if B is \mathbb{B}_i -generic, then \mathcal{L}_B is consistent.

(this being a statement in $V[B]$). Hence we may assume w.l.o.g. that δ_0 is ω -cofinal in V . Let $C = \{c_i \mid i < \omega\}$ where $\langle c_i \mid i < \omega \rangle$ is monotone and cofinal in δ_0 . Set:

$\mathbb{B}' =$ the inverse limit of $\langle \mathbb{B}_{c_m} \mid m < \omega \rangle$.

Then \mathbb{B}' is subcomplete by the ω -case of the iteration lemma for subcomplete algebras. Let B be \mathbb{B}' -generic. Set

$\mathbb{B}_\xi = B \cap \mathbb{B}_\xi$ for $\xi < \delta_0$. In $V[B]$ let

$\sigma : \bar{N}[B] \prec N[B]$, $\sigma(\bar{B}) = B$, $\sigma(\bar{C}) = C$;

where \bar{N} is countable + transitive.

Let $\sigma(\bar{Q}) = Q$. Clearly we have:

(1) $\sigma \upharpoonright \bar{Q} : \langle \bar{Q}, \bar{B}, \bar{C} \rangle \triangleleft_* \langle Q, B, C \rangle$

Set: $\langle \tilde{N}, \tilde{\sigma} \rangle =$ the liftup of $\langle \bar{N}, \sigma \upharpoonright \bar{Q} \rangle$

Then $\tilde{\sigma} : \bar{N} \prec \tilde{N}$ cofinally and there

is a unique $\tilde{k} : \tilde{N} \prec N$ s.t. $\tilde{k} \upharpoonright Q = \text{id}$

and $\tilde{k} \circ \tilde{\sigma} = \sigma$. Let $\tilde{\mathcal{L}}$ be defined

over \tilde{N} the way \mathcal{L} was defined over N .
 (with $\tilde{M} = k^{-1}(M)$ taking the place of M).

Then $k : \langle \tilde{N}, \tilde{\mathcal{L}} \rangle \prec \langle N, \mathcal{L} \rangle$ and it suffices
 to prove:

Claim $\tilde{\mathcal{L}}$ is consistent.

We show, in fact, that $\langle H_{\omega_2}^V[B], B \rangle$
 models $\tilde{\mathcal{L}}$. The proof is a virtual
 repetition of the corresponding step
 in the proof of §3 Lemma 1. The
 details are left to the reader.

QED (Lemma 1)

The same proof obviously yields:

Cor 1.1 Let B be \mathbb{B}_i -generic ($i < \aleph_0$).
 Then \mathcal{L}_B is consistent.

Exactly as in §3 we get:

Lemma 2 Let \mathcal{M} be a solid model of \mathcal{L} .

Let $\langle A_n \mid n < \omega \rangle \in \mathcal{M}$ s.t. $A_n \subset M$ for $n < \omega$.

There is $u = \langle Q_u, M_u, B_u \rangle \in \mathcal{T} \cap H_{\omega_1}$ s.t.

- $u \triangleleft \langle Q, M, B \rangle$

- $\pi : \langle M_u, \bar{A}_n \rangle \prec \langle M, A_n \rangle$ for $n < \omega$

where $\pi = \pi_u, \langle Q, M, B \rangle$, $\bar{A}_n = \pi^{-1} \llcorner A_n$.

Cor 2.1 Let B be \mathbb{B}_i -generic ($i < \aleph_0$),
 Lemma 2 holds of \mathcal{L}_B in $V[B]$
 (with M^B in place of M).

We now define the conditions $IP = IP_{\mathcal{L}}$.

Def $\tilde{IP} =$ the set of $\langle p_0, p_1 \rangle$ s.t.

- $p_0 = \langle \underline{Q}_p, \underline{M}_p, B^p \rangle \in \Gamma \wedge H_{\omega_1}$
- $p_1 = F^p$ is a countable set of pairs $\langle a, \bar{a} \rangle$
 s.t. $\bar{a} \subset M_p, a \subset M$.

Def For $p \in \tilde{IP}$ let φ_p be the conjunction
 of • $p_0 \triangleleft \langle \underline{Q}, \underline{M}, B \rangle$

- $\pi_p : \langle \underline{M}_p, \underline{a} \rangle \triangleleft \langle \underline{M}, \underline{a} \rangle$ for $\langle a, \bar{a} \rangle \in F^p$
 where $\pi_p = \pi_{p_0} \upharpoonright \langle \underline{Q}, \underline{M}, B \rangle$.

Def $IP = IP_{\mathcal{L}} = \{ p \in \tilde{IP} \mid \mathcal{L}(p) \text{ is consistent} \}$

where $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. For $p, q \in IP$ set:

$p \leq q$ iff the following hold:

- $R^q \subset R^p$ where $R^p = \text{rng}(F^p), D^p = \text{dom}(F^p)$
- $\pi_q \triangleleft \pi_p$
- $\pi_q \upharpoonright p_0 : \langle \underline{M}_q, \bar{a} \rangle \triangleleft \langle \underline{M}_p, \bar{a}' \rangle$ whenever
 $\langle a, \bar{a} \rangle \in F^q, \langle a, \bar{a}' \rangle \in F^p$.

• $\sup \pi_p \upharpoonright \langle \underline{Q}, \underline{M}, B \rangle = \pi_q$ if B is regular

Exactly as before:

Lemma 3.1 Let $p, q \in IP$. Then $p \leq q$ iff

- $R^q \subset R^p$
- $\mathcal{L}(p) \vdash (\mathcal{L}(q) \wedge \text{rng}(\pi_q) \subset \text{rng}(\pi_p))$

Def $\pi_q^p = \pi_q \circ p$ for $p \leq q$.

Lemma 3.2 Let $p \in IP$. Then

- $(F^p)^{-1}$ is a function
- If R^p is closed under set difference, then $F^p: D^p \leftrightarrow R^p$
- $\pi^p =_{\text{df}} F^p \upharpoonright M_p$ is injective into M .

Lemma 3.3 $IP \neq \emptyset$. Moreover, if $q, p \in IP$ and $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent, there is $r \leq p, q$.
Moreover, for any countable $R \subset \mathcal{P}(M)$ we can choose r s.t. $R \subset R^r$.

Cor 3.4 p, q are compatible in $IP \iff \mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent

Cor 3.5 Let $p \in IP$. Let $R \subset \mathcal{P}(M)$ be countable. There is $q \leq p$ s.t. $R \subset R^q$.

Cor 3.6 Let $p \in IP$. Let $u \subset M$ be countable. There is $q \leq p$ s.t. $u \subset \text{rng}(\pi^q)$.

Lemma 3.7 Let $p \in IP$, $u \in M_p$, u finite.
There is $q \leq p$ s.t. $q_0 = p_0$, $u \in \text{rng}(\pi_q)$.

Lemma 3.8 Let G be IP -generic. Then

(a) $\langle \langle Q_p, M_p, B^p \rangle \mid p \in G \rangle$, $\langle \pi_{pq} \mid q \leq p \text{ in } G \rangle$
is a directed system with limit:

$$\langle Q, M, B^G \rangle, \langle \pi_p^G \mid p \in G \rangle.$$

(Moreover: $\pi_p^G = \bigcup \{ \pi_{pq} \mid q \leq p, |q| = |p|, q \in G \}$)

(b) $p_0 \triangleleft \langle Q, M, B^G \rangle$ with $\pi_p = \pi_{p_0}^G$, $\langle Q, M, B^G \rangle$

(c) $\pi_p^G : \langle M_p, \bar{a} \rangle \triangleleft \langle M, a \rangle$ for $\langle a, \bar{a} \rangle \in FP$.

The proofs are exactly as in §3.

Imitating the proof of §3 Lemma 6.13 (Case 2), we get the reversibility lemma:

Lemma 4.1 Let $p \in IP$, let \bar{B} be s.t.

- $\bar{B} \subset \bigcup_{i < \delta_0^p} B_i^p$ and $\bar{B}_i = \bar{B} \cap B_i^p \in B_i^p$ - generic over Q_p for $i < \delta_0^p$
- $Q_p[\bar{B}] = Q_p[B^p]$

Then $p' \in IP$ where $p'_0 = \langle Q_p, M_p, \bar{B}' \rangle$, $p'_1 = p_1$.

proof. (sketch)

Let $M = \langle |M|, B^{M^*} \rangle$ be a solid model of $\mathcal{L}(p)$,

Set $B = \bigcup_{i < \delta_0^p} \pi_p(B_i)$. Then $Q[B] = Q[B^{M^*}]$

and $B \subset \bigcup_{i < \delta_0} B_i$ s.t. $B_i = B \cap B_i \in B_i$ -

- generic over Q (hence over V) for $i < \delta_0$.

Set $M' = \langle |M|, B \rangle$.

Claim M' models $\mathcal{L}(p')$.

We first show that M' models \mathcal{L} , all axioms are trivial except $(*)$. We verify $(*)$.

Let $i < \delta_0^p$ s.t. $\prod_{B_i}^{M_p} \delta_0^p$ is co-final.

Then, since $\pi_p^{M_p} \upharpoonright_{B_i} \bar{B}_i \subset B_{\pi(i)}$, π_p^M extends

to $\pi^*: M_p^{\bar{B}_i} \prec M_{\pi(i)}^{B_{\pi(i)}}$. Let $\bar{C} \in M_p^{\bar{B}_i}$ s.t.

$\bar{C} \subset \delta_0^p = \pi_p(\bar{C})$, $ot_p(\bar{C}) = \omega$. Let

$\pi^*(\bar{C}) = C$. It follows easily that

$\pi_p \upharpoonright_{Q_p} \langle Q_p, B, \bar{C} \rangle \triangleleft_* \langle Q, B, C \rangle$.

But then $\pi_p : \langle Q_p, M_p, \bar{B} \rangle \triangleleft \langle Q, M, B \rangle$.

Now let $\xi \in B$. By Lemma 2 there is

$\pi_1 : u \triangleleft \langle Q, M, B^{or} \rangle$ s.t. $u \in H_{u_1}$,

$\pi_1 \in \mathcal{M}$ and $\text{rng}(\pi_p) \cup \{\xi\} \subset \text{rng}(\pi_1)$.

Set $\pi_0 = \pi_1^{-1} \pi_p$. It follows that

$$\pi_0 : P_0 \triangleleft u, \pi_1 : u \triangleleft \langle Q, M, B^{or} \rangle$$

$$\text{and } \pi_p = \pi_1 \circ \pi_0.$$

If we set $B' = \bigcup_{i < \delta_0^p} \pi_0(B_i) = \bigcup_{i < \delta_0} \pi_1^{-1}(B_i)$,

we have $\pi_0 : P_0' \triangleleft u' \wedge \pi_1 : u' \triangleleft \langle Q, M, B \rangle$,

where $u' = \langle Q_{u'}, M_{u'}, B' \rangle$. (Recall that

$P_0' = \langle Q_p, M_p, \bar{B} \rangle$. But $\xi \in \text{rng}(\pi_1)$, which

proves (*). Hence $\mathcal{M}' \models \mathcal{L}$. Since

$\pi_p^{or} : P_0' \triangleleft \langle Q, M, B \rangle$ and $P_1' = P_1$, we

see that \mathcal{M} models $\mathcal{L}(P')$.

QED(4.1)

It turns out, however, that the

$Q_p[\bar{B}] = Q_p[BP]$ is too restrictive for our

purposes. We formulate another re-

visability principle which evades

the restriction:

Lemma 4.2 Let θ be big enough to verify the productness of \mathbb{B}_i over \mathbb{B}_i for all $i < j < \delta_0$

s.t. $i, j \in A_c$. Suppose moreover that $\theta > 2^{\mathbb{B}}$. Let $N^* = \langle H_\theta, N, \langle, \mathbb{P}, \mathbb{B}, m \rangle$,

let p conform to N^* . Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, \langle, \bar{\mathbb{P}}, \bar{\mathbb{B}}, m \rangle$.

(Hence $\bar{\mathbb{B}} = \mathbb{B}^p$.) Let $\bar{B}' \subset \bigcup_c \bar{\mathbb{B}}_c$ s.t. $\bar{B}'_i = \bar{B}' \cap \bar{\mathbb{B}}_i$ is $\bar{\mathbb{B}}_i$ -generic over \bar{N} for $i < \delta_0^p$. Then $p' \in \mathbb{P}$ where

$$p'_0 = \langle \mathbb{Q}_p, M_p, \bar{B}' \rangle \quad | \quad -p'_1 = p_1$$

Proof

Let $\mathcal{M} = \langle \mathcal{M}, \dot{\mathbb{B}}^{\mathcal{M}} \rangle$ be a rooted model of $\mathcal{L}(p)$. Set $B = \dot{\mathbb{B}}^{\mathcal{M}}$. Then $\pi = \pi_p^{\mathcal{M}}$ extends

to $\pi^v : \bar{N}^* \prec N^*$ s.t. $\pi \cup F^p \subset \pi^*$. We assume that \mathcal{M} (and hence π^*) lies

in a generic extension of V . Note that B_i is \mathbb{B}_i -generic over \mathcal{Q} , hence

over V for $i < \delta_0$. Let $\langle \bar{\delta}_i \mid i < \omega \rangle$ be a monotone cofinal sequence in $\bar{\delta}_0 = \delta_0^p$ s.t. $\bar{\delta}_i \in A_c$ in \bar{N}^* for $i < \omega$.

Set $\delta_i = \pi(\bar{\delta}_i)$. Then $\langle \delta_i \mid i < \omega \rangle$ is monotone and cofinal in δ_0 and

$\delta_i \notin A_c$ for $i < \omega$.

But then B_{δ_i} is pro-d over B_{δ_h} for $h < i < \omega$, where B_{δ_i} is π^* -conforming.

Then we can successively form

B'_{δ_i} s.t. B'_{δ_i} is B_{δ_i} -generic,

$\pi^* \bar{B}'_{\delta_i} \subset B'_{\delta_i}$, and $V[B'_{\delta_i}] = V[B_{\delta_i}]$.

Set: $B' = \bigcup_i B'_{\delta_i}$. Then $B'_3 = B' \cap B_3$

is B_3 -generic for $3 < \delta_0$. Let

$\bar{C} \in M_p[\bar{B}'_{i_0}]$ s.t. $\bar{C} < \delta_0 = \sup \bar{C}$ and

$\sup \bar{C} = \omega$, π^* extends uniquely

to $\tilde{\pi}: \bar{N}^*[\bar{B}'_{i_0}] \rightarrow N^*[B'_{\pi(i_0)}]$ s.t.

$\tilde{\pi}(\bar{B}'_{i_0}) = B'_{\pi(i_0)}$, since $\pi^* \bar{B}'_{i_0} \subset B'_{\pi(i_0)}$.

Let $\tilde{\pi}(\bar{C}) = C$. Then $\pi^* \bar{C} = C$.

It follows easily that:

$$(1) \pi \upharpoonright \bar{Q}_p: \langle Q_p, \bar{B}', \bar{C} \rangle \triangleleft_* \langle Q, B', C \rangle,$$

Hence:

$$(2) \tilde{\pi}: \langle Q_p, M_p, \bar{B}' \rangle \triangleleft \langle Q, M, B' \rangle,$$

where $P'_0 = \langle Q_p, M_p, \bar{B}' \rangle$.

Now set: $N'_1 = \langle \omega_1, B' \rangle$. It

follows as in Lemma 4.1 that

$$N'_1 \models \mathcal{L}(P'). \quad \text{QED (4.2)}$$

By a virtual repetition of the proof of §3 Lemma 3, using 4.2, we get:

Lemma 4.3 \mathbb{P} admits no reals

But then as before:

Lemma 4.4 Let $\theta \geq 2^B$ be regular, $A \in G \ni P$ is \mathbb{P} -generic and $B = B^G$, then $\langle H_\theta^{V[G]}, B \rangle$ models $\mathcal{L}(P)$.

Def Let $B = \bigcup_{i < \delta_0} B_i$ s.t. $B_i = B \cap B_i$ is B_i -generic for $i < \delta_0$.

$G^B =$ the set of $P \in \mathbb{P}$ s.t.

$P_0 \triangleleft \langle Q, M, B \rangle$ and, letting

$\pi = \pi_{P_0}, \langle Q, M, B \rangle$ we have:

$\pi: \langle M_P, \bar{a} \rangle \prec \langle M, a \rangle$ for all $\langle a, \bar{a} \rangle \in F^P$.

Exactly as in §3 Lemma 3.11 we have:

Lemma 4.5 Let G be \mathbb{P} -generic,

Then $G = G^B$ where $B = B^G$,

Hence as in §3 Lemma 3.12;

Lemma 4.6 Let G be \mathbb{P} -generic,

Then $\bar{\beta} \leq \omega_1$ in $V[G]$

Def We define a homomorphism of $\bigcup_i B_i$ into $BA(\mathbb{P})$ by:

$$k(b) = \llbracket b^v \in B^G \rrbracket_{\mathbb{P}}$$

Lemma 4.7 k is injective

prf.

Suppose not. Then $k(b) = 0$ for a $b \in \bigcup_i B_i$, s.t. $b \neq 0$. But $\mathcal{L} + \underline{b} \in \mathring{B}$ is con-

sistent. (To see this choose B s.t. $b \in B$ in the proof of Lemma 1.) Let \mathcal{M} be a solid model of $\mathcal{L} + \underline{b} \in \mathring{B}$.

Then $b \in B^{\mathcal{M}}$. By Lemma 2 there is $u \in \Gamma \wedge H_{\omega_1}$ s.t. $u \triangleleft \langle \mathcal{Q}, \mathcal{M}, B \rangle$

and $\pi(b^-) = b$ for some b^- where

$\pi = \pi_{u, \langle \mathcal{Q}, \mathcal{M}, B \rangle}$. Define p by:

$p_0 = u$, $p_1 = \{ \langle b, b^- \rangle \}$. Then \mathcal{M} models

$\mathcal{L}(p)$. Hence $p \in \mathbb{P}$. But if $G \ni p$ is \mathbb{P} -generic, then $b \in B^G$, hence

$$0 \neq \llbracket p \rrbracket \subset \llbracket b^v \in B^G \rrbracket = \sigma(b).$$

Contr!

QED (4.7)

We now show that $BA(\mathbb{P})$ is, in fact, generated by $\{k(b) \mid b \in \bigcup_{i < \aleph_0} B_i\}$.

As a preliminary we prove:

Lemma 4.8 Let G be \mathbb{P} -generic, $B = B^G$.

Then $G \in V[B]$.

proof.

(1) Let $u \in H_{\omega_1} \cap \Gamma$, $\pi: u \triangleleft \langle Q, M, B \rangle$.

Then $u \in V[B]$.

proof.

Let $\bar{3}$ be least s.t. $\| \text{cf}(\aleph_0^u) = \omega$. Then $\bar{3} \in \text{rng}(\pi)$ since $\pi: M_u \triangleleft M$, $\pi(B^u) = B$.

Let $\pi(\bar{3}) = \bar{3}$, let $\bar{c} \in M_u$, $\bar{c} \subset \aleph_0^u = \text{sup}(\bar{c})$,

$\text{otp}(\bar{c}) = \omega$. π extends uniquely to

$\pi^*: M[B_{\bar{3}}^u] \triangleleft M[B_{\bar{3}}]$ s.t. $\pi^*(B_{\bar{3}}^u) = B_{\bar{3}}$,

since $\pi^* B_{\bar{3}}^u \subset B_{\bar{3}}$. Let $C = \pi^*(\bar{c})$.

Then $C = \pi^* \bar{c}$ and

$\pi \upharpoonright Q_u: \langle Q_u, B^u, \bar{c} \rangle \triangleleft_* \langle Q, B, C \rangle$,

where $\langle Q, B, C \rangle \in V[B]$. Hence

$\text{rng}(\pi \upharpoonright Q_u) =$ the smallest $X \triangleleft \langle Q, B, C \rangle$

s.t. $\omega_1^{Q_u} \cup C \subset X$.

Hence $\pi \upharpoonright Q_u \in V[B]$. But then

$\pi \in V[B]$, since, letting $\pi: M_n \rightarrow \tilde{M}$
 cofinally we have:

$\langle \tilde{M}, \pi \rangle =$ the liftup of $\langle M_n, \pi \upharpoonright M_n \rangle$.

QED (1)

But then by Lemma 4.5 we have

$$P \in G \leftrightarrow V[B] \models P \in G^B.$$

Hence $G \in V[B]$, QED (4.8)

As a corollary of the proof:

Lemma 4.9 $BA(IP)$ is generated by

$$\{k(b) \mid b \in \bigcup_{i < \aleph_0} B_i\}.$$

proof.

Let $\check{B} \in V^{IP}$ s.t. $\Vdash_{IP} \check{B} = B^{\check{G}}$.

Standard methods show:

$$\Vdash_{IP} [\varphi(\check{x}_1, \dots, \check{x}_n, \check{B}) \in IB^*],$$

for all $x_1, \dots, x_n \in V$ + all $Z \in$
 $\text{-Formulae } \varphi$, where IB^* is the
 complete subalgebra of $BA(IP)$
 generated by $\{k(b) \mid b \in \bigcup_{i < \aleph_0} B_i\}$.

By the proof of Lemma 4.8,
 however:

$\models (\check{p} \in \check{G} \leftrightarrow (\check{p} \in G^{\check{B}})_{V[\check{B}]})$; hence

$[p] = \llbracket \check{p} \in \check{G} \rrbracket = \llbracket (\check{p} \in G^{\check{B}})_{V[\check{B}]} \rrbracket \in B^*$, where

$BA(\mathbb{P})$ is generated by $\{[p] \mid p \in \mathbb{P}\}$.

QED (4.9)

We now estimate the size of $BA(\mathbb{P})$:

Lemma 4.10 $\overline{BA(\mathbb{P})} \leq \beta_\lambda^+$.

proof.

If $\beta \neq \beta_\lambda$, then $\beta_\lambda = \beta^+ = 2^\beta$, since GCH

then holds below κ . But $\overline{\mathbb{P}} \leq 2^\beta$ and

hence the result follows. Now let

$\beta = \beta_\lambda$. Then $cf(\beta) = \omega_1$. It suffices

to show:

Sublemma 4.10.1 $BA(\mathbb{P})$ has a dense

subset of size β .

proof (sketch)

The proof is virtually identical to that of § 3 Sublemma 3.15.1. Set $H = H_{(2^\beta)^+}$. Then

$\langle H[G], B \rangle$ models \mathcal{L} whenever G is \mathbb{P} -

generic and $B = B^G$. We can give every

\mathcal{L} -sentence ψ an interpretation $\llbracket \psi \rrbracket \in$

$BA(\mathbb{P})$ in $H^{\mathbb{P}}$, interpreting \check{B} by \check{B}

where $\check{B} \in H^{\mathbb{P}}$, $\models_{\mathbb{P}}^H \check{B} = B^G$, G being the canonical generic name, $\underline{\kappa}$ is

(interpreted by \bar{x}). It then suffices to show:

Claim For each $p \in IP$ there is an L -statement $\psi \in M$ s.t. $\llbracket \psi \rrbracket \neq 0$ and $\llbracket \psi \rrbracket \subset [p]$ in $BA(IP)$.

The proof is an almost literal repetition of the corresponding step in the proof of §3 Sublemma 3.15.1. We leave the details to the reader. QED (4.10)

Lemma 4.11 Let G be IP-generic. Then $cf(\bar{\alpha}) = \omega$ in $V[G]$ whenever $\bar{\alpha} \in [\delta_\lambda^1, \beta]$ is regular in V .

proof.

Let $\bar{\alpha} = \pi_p^G(\bar{\alpha})$. Then $\sup \pi_p^G \bar{\alpha} = \bar{\alpha}$, since $\pi_p^G: M_p \rightarrow \tilde{M}$ is δ_p -cofinal, where $\pi_p^G(\delta_p) = \delta^* = \delta_\lambda^1$. QED (4.11)

Lemma 4.12 Let G be IP-generic. Then

(a) $\bar{\beta}_\lambda = \omega_1 = cf(\beta_\lambda)$ in $V[G]$

(b) $\beta_\lambda^+ = \omega_2$

proof.

$\bar{\beta} \leq \omega_1$ since $\beta \subset \bigcup_{p \in G} \text{rng}(\pi_p^G)$,

where $\pi_p^G = \pi_{p_0}^G \langle Q, M, B^G \rangle$ depends only on $p_0 \in H_{\omega_1}$.

If $\beta = \beta_\lambda$, then $\text{cf}(\beta) = \omega_1$ in V , hence in $V[G]$. Moreover, β^+ remains a cardinal in $V[G]$, since $\text{BA}(\mathbb{P})$ has a dense subset of size β .

Now let $\beta_\lambda = \beta^+$. Then $\text{cf}(\beta) = \omega$ in $V[G]$, since either β is regular or $\text{cf}(\beta) = \omega$ in V .

We also know $\beta^+ = 2^\beta$, since GCH holds below κ . By [LF] §4 Lemma 4.1 (Fact 11 of §3 in this paper) we conclude that

$\overline{\beta_\lambda} = \omega_1$ in $V[G]$. But β_λ^+ remains a cardinal in $V[G]$ since $\overline{\mathbb{P}} \leq \beta^+ = \beta_\lambda$.

Hence $\text{cf}(\beta_\lambda) = \omega_1$ in $V[G]$, since otherwise another application of §3 Fact 11

would give: $\overline{\beta_\lambda^+} \leq \omega_1$ in $V[G]$,

QED (Lemma 4.12)

We choose σ, IB_λ (recall $\lambda = \delta_0$) s.t.

$$\begin{array}{ccc} \sigma : BA(\mathbb{P}) & \xrightarrow{\sim} & IB_\lambda \\ k \uparrow & \nearrow & \\ \bigcup_i IB_i & & \end{array}$$

and $IB_\lambda \subset H_{\beta+}$. IB_λ is the completion of $\bigcup_i IB_i$ which we sought. However, we must still verify many of its properties.

Now let $i_0 < \delta_0$ + let B be IB_{i_0} - generic.

$$\text{Set: } IB^+ = IB \langle IB_\lambda \rangle = \langle IB_i \mid i \leq \lambda \rangle,$$

We shall have to prove that the properties we have shown to hold of IB^+ also hold of $IB^+/B = \langle IB_i/B \mid i \leq \delta_0 - i_0 \rangle$

in $V[B]$. We know by the induction hypothesis that the salient properties of IB hold of $IB/B = \langle IB_i/B \mid i < \delta_0 - i_0 \rangle$ in

$\mathcal{V}[B]$. But in $\mathcal{V}[B]$ we can use the language \mathcal{L}_B over N^B to construct conditions IP_B exactly as IP was constructed from \mathcal{L} . As we have established properties of B, IP , we can repeat our proofs to verify the same properties of $B/B, IP_B$ in $\mathcal{V}[B]$. Now let

$k_B : \bigcup_{h < \delta_0 - i_0} B_{i+h}/B \rightarrow BA(IP_B)$ be defined in $\mathcal{V}[B]$ as k was defined in \mathcal{V} . We then define $\sigma^B, B_{\lambda-i_0}^B$ in $\mathcal{V}[B]$ like σ, B_λ in \mathcal{V} . We have:

$$\begin{array}{ccc}
 BA(IP_B) & \xleftrightarrow[\sigma^B]{\sim} & B_{\lambda-i_0}^B \\
 \uparrow k_B & \nearrow & \\
 \bigcup_i B_i/B & &
 \end{array}$$

It suffices to prove:

Lemma 5.1 There is $\mu : BA(IP)/B \xleftrightarrow{\sim} B_{\lambda-i_0}^B$

st. $\mu(b/B) = b/B$ for $b \in \bigcup_i B_i$.

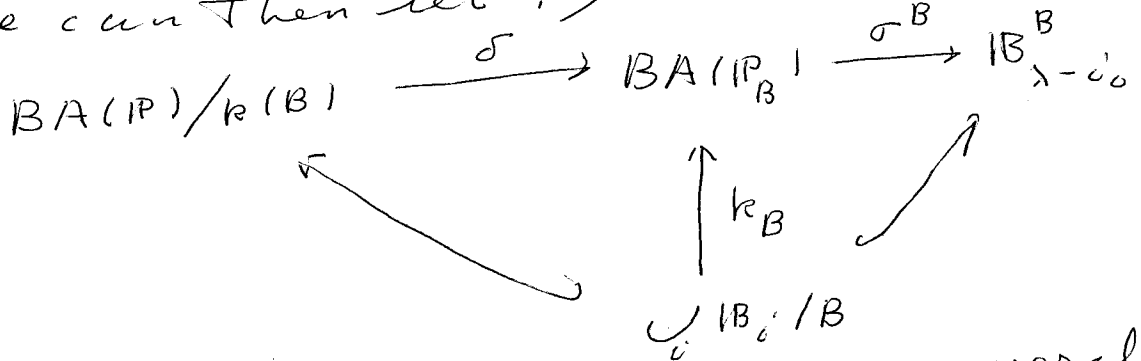
proof

This is equivalent to:

Claim There is $\delta: BA(\mathbb{P})/k(B) \xrightarrow{\sim} BA(\mathbb{P}_B)$
 s.t. $\delta(k(b)/k(B)) = k_B(b/B)$ for $b \in \cup_i B_{i_0}$,

where $k(B) =_{\text{df}} k^{\text{cl}} B$.

We can then set: $\mu = \sigma^B \circ \delta$.



The proof stretches over several lemmas and closely follows that of §3 Cor 6.9.1. \mathbb{P}_B plays the same role as the \mathbb{P}_B of that proof. The role of \mathbb{P}' is played by:

$$\mathbb{P}' =_{\text{df}} \{ p \in \mathbb{P} \mid i_0 \in \text{rang}(\pi^p) \}$$

Note that \mathbb{P}' is dense in \mathbb{P} .

As before, we assign to each $p \in \mathbb{P}'$ a potential element p^* of \mathbb{P}_B :

$$p^* = \langle \mathbb{Q}_p^B, M_p^B, B^p/B_{i_p}^p \rangle, \quad p_1^* = p_1,$$

where $\pi^p(i_p) = i_0$ and

$$B^p/B = \{ b/B \mid b \in B^p \}$$
 whenever B is

$B_{i_0}^p$ - generic over M_p .

Lemma 5.1.1 Let B be B_{i_0} -generic. Let $p, q \in \mathbb{P}'$, $[p]/k(B), [q]/k(B)$ are compatible in $\mathbb{B}A(\mathbb{P})/k(B)$ iff p^*, q^* are compatible in \mathbb{P}_B .

Proof.

(\Leftarrow) Suppose not. Then $[p] \wedge [q]/k(B) = 0$. Hence $[p] \wedge [q] \wedge k(b) = 0$ for a $b \in B$. Let $\mathcal{M}^* = \langle \text{val}^*, B^* \rangle$ be a solid model of $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$. Then $\mathcal{M} = \langle \text{val}^*, B' \rangle$ a solid model of $\mathcal{L}(p) \cup \mathcal{L}(q)$ where $B' = B^* \cdot B^* = \{b \in \bigcup_i B_i \mid b/B \in B'\}$, by Lemma 0. Hence there is $\mathcal{M} \leq p, q$ s.t. $\mathcal{M} \models \mathcal{L}'(\mathcal{M})$ and $b \in \text{rng}(\pi^{\mathcal{M}})$, let $\pi^{\mathcal{M}}(\bar{b}) = b$. Then $\bar{b} \in B^{\mathcal{M}}$. Hence $\mathcal{M} \models \bar{b} \in B^{\mathcal{M}}$. Hence $[\mathcal{M}] \subset \llbracket \bar{b} \in B^{\mathcal{M}} \rrbracket = k(b)$, where $[\mathcal{M}] \wedge k(b) = 0$. Contr!

(\Leftarrow) Suppose not. Then $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$ is inconsistent. Hence there is $b \in B$ s.t. (1) $b \Vdash_{B_{i_0}} (\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*))$ is inconsistent.

Let \tilde{G} be $\mathbb{B}A(\mathbb{P})/k(B)$ -generic s.t. $[p]/k(B), [q]/k(B) \in \tilde{G}$. Set $G = \{p \in \mathbb{P} \mid [p]/k(B) \in \tilde{G}\}$

Then G is \mathbb{P} -generic with $p, q \in G$ and $B = B_{i_0}^G$. Let $\mathcal{M} \in G$ s.t. $\mathcal{M} \leq p, q$ and $b \in \text{rng}(\pi^{\mathcal{M}})$. Let $\pi^{\mathcal{M}}(\bar{b}) = b$

Then $\bar{b} \in B^x$. Let $\mathcal{M}' = \langle \mathcal{M}', B' \rangle$ be a solid model of $\mathcal{L}(\mathcal{M})$. Then $\mathcal{M} = \langle \mathcal{M}', B'/B \rangle$ is a solid model of $\mathcal{L}_B(p^*) \cup \mathcal{L}_B(q^*)$, where $b \in B'$ and B' is B_{i_0} -generic, contradicting (1). Contr! QED (5.1.1)

As a corollary:

Cor 5.1.2 $[p] / k(B) \neq 0 \iff p^* \in IP_B$

for $p \in IP'$ and B_{i_0} -generic B .

Lemma 5.1.3 $\{[p^*] \mid [p] / k(B) \neq 0\}$

is dense in $BA(IP_B)$.

Proof. (We follow the proof of §3 Lemma 6.4)

By the same argument as before, using Lemma 0,

$\{p^* \mid [p] / k(B) \neq 0\}$ is the same as the set \hat{IP} of $p \in IP_B$ s.t. $i_0 \in \text{rng}(\pi p)$ and

$f p \in V$. We show:

Claim Let $q \in IP_B$. There is $p \in \hat{IP}$ s.t.

$[p] \subset [q]$.

Let $A = \langle a_i \mid i < \omega \rangle \in V[B]$ enumerate P^q as before. Let $D \subset \beta$ s.t. A is

$\langle M^B, D \rangle$ -definable, where $D \in M[B]$.

Let $D = \hat{D}^B$ and set, as before,

$E = \{ \langle \nu, b \rangle \mid b \in B_{i_0} \wedge \nu < \beta \wedge b \Vdash \check{\nu} \in \hat{D} \}$

Then A is $\langle M^B, E \rangle$ -definable as before,

for $\forall [B]$ define:

$$N^* = \langle H_\theta, N^B, M^B, Q^B, B, E, A, \dots \rangle, \text{ where } \theta > (2^B)^+$$

is a cardinal and $<$ well orders N^* . Let

$p \leq q$ s.t. p conforms to N^* . Again set:

$$\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, \bar{M}, \bar{Q}, \bar{B}, \bar{E}, \bar{A}, \dots \rangle,$$

Then $\bar{M} = M_p, \bar{Q} = Q_p, \bar{B} = B^p$ and \bar{A} is $\langle \bar{M}, \bar{E} \rangle$ -definable by the same definition.

Form p', p'' by:

$$p'_0 = p_0, p'_1 = \{ \langle a, \bar{a} \rangle \in Fp \mid a \in R^{\bar{a}} \}$$

$$p''_0 = p_0, p''_1 = \{ \langle E, \bar{E} \rangle \} \text{ where } \langle E, \bar{E} \rangle \in Fp,$$

Then $p' \leq q$ in \mathbb{P}_B and $p'' \in \hat{\mathbb{P}}_B$. We show:

Claim $[p''] \subset [p']$ in $BA(\mathbb{P}_B)$.

pf. Let $G \ni p''$ be \mathbb{P}_B -generic. It suffices to show:

Claim $p' \in G$.

Since $p'_0 = p''_0$ we have:

$$\bar{\pi} : p'_0 \triangleleft \langle Q^B, M^B, B^G \rangle \text{ where } \bar{\pi} = \pi_{p''}^G,$$

We need only show:

$$\bar{\pi} : \langle M_{p'}, \bar{a} \rangle \triangleleft \langle M^B, a \rangle \text{ for } \langle a, \bar{a} \rangle \in Fp',$$

as before, $a = A(i)$ is $\langle M^B, E \rangle$ -definable and

$\bar{a} = \bar{A}(i)$ is $\langle M_{p'}, \bar{E} \rangle$ -definable by the same definition, where $\bar{\pi} : \langle M_{p'}, \bar{E} \rangle \triangleleft \langle M^B, E \rangle$.

QED (5.1.3)

Lemma 5.14 $[P]/k(B) \subset [q]/k(B)$ in $BA(P)/B$

iff $[P^*] \subset [q^*]$ in $BA(P_B)$

for $P, q \in IP'$

Proof.

$$S_2 + A = \{ [P]/k(B) \mid P \in IP' \} \setminus \{0\}$$

$$A' = \{ [P^*] \mid P \in A \} = \{ [P^*] \mid P \in IP' \} \setminus \{0\}$$

Since A is dense in $BA(P)$ we have

$$a \subset b \iff \bigwedge c \in A (c \cap b = 0 \rightarrow c \cap a = 0)$$

for $a, b \in BA(P)/k(B)$. In particular, this holds for $a, b \in \{ [P]/k(B) \mid P \in IP' \}$.

Similarly:

$$a \subset b \iff \bigwedge c \in A' (c \cap a = 0 \rightarrow c \cap b = 0)$$

for $a, b \in BA(P_A)$ (in particular for $a, b \in \{ [P^*] \mid P \in IP' \}$).

But:

$$[P]/k(B) \cap [q]/k(B) = 0 \iff$$

$$\iff [P^*] \cap [q^*] = 0$$

for $P, q \in IP'$ by Lemma 5.1.1.

The conclusion is immediate.

Q.E.D. (5.14)

Cor 5.15 There is a unique

$$\delta: BA(P)/k(B) \xrightarrow{\sim} BA(P_B) \text{ s.t.}$$

$$\delta([P]/k(B)) = [P^*] \text{ for } P \in IP'$$

We complete the proof of 5.1 by showing:

Lemma 5.1.6 $\delta(k(b)/k(B)) = k_B(b/B)$

for $b \in \cup_i B_i$.

proof.

It suffices to show that if F is a $BA(\mathbb{P})/k(B)$ -generic filter and $F^* = \delta''F$, then

$$k(b)/k(B) \in F \iff k_B(b/B) \in F^*$$

Set: $G = \{p \in \mathbb{P} \mid [p]/k(B) \in F\}$. Then

G is \mathbb{P} -generic. Hence there is

$p \in \mathbb{P}' \cap G$ s.t. $b \in \text{rng}(\pi^p)$. Let

$$\pi^p(\bar{b}) = b. \text{ Then } \bar{b} \in B^p \iff b \in B^G \iff$$

$$\iff k(b)/k(B) = \llbracket \bar{b} \in B^G \rrbracket / k(B) \in F, \text{ since}$$

$$F = \{a/k(B) \mid a \in BA(\mathbb{P}) \wedge a \upharpoonright G \neq 0\}.$$

$$\text{But } \bar{b} \in B^p \iff \bar{b}/B_{i_p}^p \in B^p/B_{i_p}^p = B^{p*}.$$

Set $G^* = \{p \in \mathbb{P}_B \mid [p] \in F^*\}$. Then

G^* is \mathbb{P}_B -generic. But $\pi_{p^*}^{G^*}(B_{i_p}^p) = B$,

since $\pi_{p^*}^{G^*}: M_p^{B_{i_p}^p} \prec M_B$. Hence

$$\pi_{p^*}^{G^*}(\bar{b}/B_{i_p}^p) = b/B. \text{ And}$$

$k(b)/k(B) \in F$, then $\bar{b} \in B^p$ and

$$\bar{b}/B_{i_p}^p \in B^{p*}. \text{ Hence } b/B = \pi_{p^*}^{G^*}(\bar{b}/B_{i_p}^p) \in$$

$$\in B^{G^*}. \text{ Hence } k_B(b/B) =$$

$$= \llbracket \bar{b}/B \in B^{G^*} \rrbracket \in F^*.$$

Similarly, if $k(b)/k(B) \notin F$, then

$$k_B(b/B) \notin F^*. \text{ QED (Lemma 5.1)}$$

Lemma 6 B_λ is subcomplete.

Proof.

There is $i_0 < \lambda$ s.t. $\text{cf}(\lambda^{i_0}) = \omega$. Since

$$B_\lambda \cong B_{i_0} * \overset{\circ}{B} \quad \text{where} \quad \text{cf} \overset{\circ}{B} = \overset{\vee}{B}_\lambda / B$$

($\overset{\circ}{B}$ being the canonical generic name), and B_{i_0} is subcomplete, it suffices to

show that $B_\lambda \setminus B$ is subcomplete in $V[B]$ whenever B is B_{i_0} -generic.

But $B_\lambda \setminus B \cong BA(\mathbb{P}_B)$, so it suffices to show that \mathbb{P}_B is subcomplete

in $V[B]$. But then it suffices to prove the subcompleteness of \mathbb{P} .

under the assumption $i_0 = 0$, since the same proof would apply to

\mathbb{P}_B in $V[B]$.

Let $W = L_{\bar{\tau}}^A$ where $2^B < \theta < \tau$, τ is regular, $H_\theta \subset W$, and W verifies the proneness of

$B_{\bar{\tau}}$ over $B_{\bar{\tau}}$ for $\bar{\tau} < \bar{\tau} < \delta_0$. Let $\sigma: \bar{W} \rightarrow W$ s.t. \bar{W} is countable and full. Let

$\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{Q}, \bar{M}, \bar{N}, \bar{\tau}, \bar{\lambda}_i) = (\theta, \mathbb{P}, Q, M, N, \tau, \lambda_i)$

($i = 1, m, n$) where $\mathbb{P} \in H_\lambda$ (hence $N \in H_\lambda$)

and $\lambda_i < \theta$ for $i = 1, m, n$. Let \bar{G} be

$\bar{\mathbb{P}}$ -generic over \bar{W} .

Claim There is $g \in \mathbb{P}$ s.t. whenever $G \ni g$ is \mathbb{P} -generic, there is $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \bar{W} \prec W$

(b) $\sigma_0(\bar{\theta}, \bar{P}, \bar{Q}, \bar{M}, \bar{N}, \bar{\alpha}, \bar{\lambda}_i) = \theta, P, Q, M, N, \alpha, \lambda_i$
 ($i=1, \dots, m$)

(c) $\sup \sigma_0 \upharpoonright \bar{\lambda}_i = \sup \sigma \upharpoonright \bar{\lambda}_i$ ($i=0, \dots, m$)

where $\bar{\lambda}_0 = 0_{m+1} \bar{W}$

(d) $\sigma_0 \upharpoonright \bar{G} \in G$,

Let $\langle \bar{c}_i \mid i < \omega \rangle \in \bar{M}$ be ^{monotone and} cofinal in $\bar{\delta}_0 = \sigma^{-1}(\delta_0)$.
 Set $c_i = \sigma(\bar{c}_i)$. Then $\langle c_i \mid i < \omega \rangle$ is monotone and cofinal in δ_0 . Let $\sigma(\bar{B}) = B$, where $B = \langle B_i \mid i < \delta_0 \rangle$. Then $\sigma(\bar{B}_{c_n}) = B_{c_n}$ for $n < \omega$.
 Let $\tilde{B} = \bigcup_{i < n} B_i$ be the inverse limit of $\langle B_{c_n} \mid n < \omega \rangle$. Modifying slightly the proof of the ω -case of the iteration theorem for subcompleteness, we get:

Fact Let $\bar{B} = \bigcup_n \bar{B}_{c_n}$ s.t. $\bar{B}_n = \bar{B} \cap \bar{B}_{c_n}$ is \bar{B}_{c_n} -generic over \bar{W} for $n < \omega$. There is a $b \in \tilde{B} \setminus \{0\}$ s.t. whenever $\tilde{B} \ni b$ is \tilde{B} -generic, then there is $\tilde{\sigma} \in V[\tilde{B}]$ s.t.

(a) $\tilde{\sigma} : \bar{W} \prec W$

(b) $\tilde{\sigma}(\bar{\theta}, \bar{m}, \bar{\lambda}_i) = \theta, m, \lambda_i$ ($i=1, \dots, m$)

(c) $\sup \tilde{\sigma} \upharpoonright \bar{\lambda}_i = \sup \sigma \upharpoonright \bar{\lambda}_i$ ($i=0, \dots, m$)

(d) $\tilde{\sigma} \upharpoonright \bar{B}_n \in \tilde{B}_n = \tilde{B} \cap B_{c_n}$ for $n < \omega$.

Making use of this fact we get:

Sublemma 6.1 Let \mathcal{L} be least int. $L_{\mathcal{L}}(W)$ is admissible. The following language \mathcal{L}^* on $L_{\mathcal{L}}(W)$ is consistent:

Predicate : \in , Constants \underline{x} ($x \in L_{\mathcal{L}}(W)$), $\bar{\sigma}$, \bar{B}

Axioms : ZFC⁻, $\bigwedge \sigma (v \in \underline{x} \leftrightarrow \bigvee_{z \in \underline{x}} v = \underline{z})$,

$\bar{\sigma} : \bar{W} \prec W$, $\bar{\sigma} \upharpoonright \bar{M} : \langle \bar{Q}, \bar{M}, \bar{B} \rangle \triangleleft \langle Q, M, B \rangle$,

$\bar{\sigma}(\bar{\theta}_i, \bar{p}_i, \bar{q}_i, \bar{m}_i, \bar{n}_i, \bar{z}_i, \bar{\lambda}_i) = \theta_i, p_i, q_i, m_i, z_i, \lambda_i$ ($i=1, \dots, m$)

$\sup \bar{\sigma} \text{ " } \bar{\lambda}_i = \sup \sigma \text{ " } \lambda_i$ ($i=0, \dots, m$)

[Note These axioms do not say that $\omega_1 = \bar{\omega}_1$.]

proof (sketch) We follow the proof of §3 Lemma 4.1.

We first let \mathcal{L}_0 be like \mathcal{L}^* except that the axiom $\sup \bar{\sigma} \text{ " } \bar{\lambda}_i = \sup \sigma \text{ " } \lambda_i$ is replaced by: $\sup \bar{\sigma} \text{ " } \bar{\lambda}_i = \lambda_i$ ($i=0, \dots, m$), where $\lambda_0 = \bar{\omega}_1$.

Let $\tilde{\sigma}, \tilde{B}$ be as in the above Fact. Let $\tilde{H} = H_{\tilde{\sigma}, \tilde{B}}^{\bar{W}}$ where $\sigma(\bar{\delta}) = \delta$. Let $\tilde{\sigma} \upharpoonright \tilde{M} \prec H'$ cofinally, and let $\langle W', \sigma' \rangle$ be the lift up of $\langle \bar{W}, \tilde{\sigma} \upharpoonright \tilde{H} \rangle$. Note that

$\tilde{\sigma} \upharpoonright \bar{Q} : \langle \bar{Q}, \bar{B}, \bar{C} \rangle \triangleleft_* \langle Q, B, C \rangle$,

where $\bar{C} = \{\bar{c}_i \mid i < \omega\}$, $C = \{c_i \mid i < \omega\}$,

Hence $\sigma' \upharpoonright \bar{M} : \langle \bar{Q}, \bar{M}, \bar{B} \rangle \triangleleft \langle Q, M', B \rangle$,

where $\sigma'(M') = M'$. There is a canonical

$k : W' \prec W$ s.t. $k\sigma' = \sigma$ and $k \upharpoonright H' = \text{id}$.

Let \mathcal{L}'_0 be defined on $L_{\mathcal{L}'}(W')$ the way

\mathcal{L}_0 was defined on $L_{\mathcal{L}}(W)$, where