

§ 3 The successor step

We are given $\langle \mathbb{B}_i \mid i \leq \mu \rangle$ satisfying (a)-(h) and wish to construct $\mathbb{B}_{\mu+1}$ s.t. (a)-(h) continue to hold.

Let $\delta = \delta_{\mu+1}^+$. Then $\delta = \omega_2^{\mathbb{B}_\mu}$, $2^\delta = \delta$, and $\delta = \beta_\mu^+$ if β_μ exists. Otherwise $\delta = \mu = \delta_\mu^+$ is inaccessible in V . In either case we know that $\mathbb{B}_\mu \subset H_\delta$.

§ 3.1 The first successor case

Suppose that $\delta \in A_0$. Then GCH holds below κ and we wish to make δ ω_1 -cofinal without collapsing $\beta_{\mu+1} = \delta^+$. This we force with $\text{coll}(\omega_1, \omega_2)$ over $V[G]$, where G is \mathbb{B}_μ -generic. In other words we let

$$\mathbb{B}_{\mu+1} \cong \mathbb{B}_\mu * \mathbb{B}^{\dot{\delta}}, \text{ where } \Vdash_{\mathbb{B}_\mu} \mathbb{B}^{\dot{\delta}} = \text{coll}(\omega_1, \omega_2).$$

We verify (a)-(h). (a) is straightforward. (b) holds, since $\Vdash_{\mathbb{B}_\mu} \mathbb{B}^{\dot{\delta}}$ is subcomplete, (c) is immediate, since $\mu+1$ is not a limit point and (a)-(h) hold of $\langle \mathbb{B}_i \mid i \leq \mu \rangle$. Similarly for (d). (e) follows by § 2 Lemma 4.1, (f) follows by § 2 Lemma 4.2. We now sketch the proof of (g). Let \mathbb{B} be \mathbb{B}_h -generic, where $h \leq \mu$,

Set $\tilde{B}_i = B_i / B$ for $h \leq i \leq \mu+1$. (Thus $\langle \tilde{B}_{h+1} \mid 1 \leq i \leq \mu+1-h \rangle$ is the new iteration in $\mathcal{V}[B]$.) We first note that if

B' is B_μ -generic over \mathcal{V} , then

$$B_{\mu+1} / B' \simeq \overset{\circ}{B} B' = BA(\text{coll}(\omega_1, \omega_2))$$

in $\mathcal{V}[B']$. (Recall that $\overset{\circ}{B} \simeq B * \overset{\circ}{B}$ where $\overset{\circ}{B} = BA(\text{coll}(\omega_1, \omega_2))$.)

Now let \tilde{B} be \tilde{B}_μ -generic over $\mathcal{V}[B]$.

$$\text{Then } \tilde{B}_{\mu+1} / \tilde{B} = (B_{\mu+1} / B) / \tilde{B} \simeq B_{\mu+1} / B',$$

where $B' = B * \tilde{B} = \{b \in B_\mu \mid b/B \in \tilde{B}\}$ is

B_μ -generic over \mathcal{V} . Hence there is $\sigma \in \mathcal{V}[B]$ s.t.

$$\Vdash_{\tilde{B}_\mu} \sigma : \tilde{B}_{\mu+1} / \tilde{B} \xrightarrow{\sim} BA(\text{coll}(\omega_1, \omega_2))$$

in $\mathcal{V}[B]$, $\overset{\circ}{B}$ being the canonical generic name). Let

$$\Vdash_{\tilde{B}_\mu} \overset{\circ\circ}{B} = BA(\text{coll}(\omega_1, \omega_2))$$

Define $\sigma : \tilde{B}_{\mu+1} \xrightarrow{\sim} \tilde{B}_\mu * \overset{\circ\circ}{B}$ in $\mathcal{V}[B]$

by $\sigma(a) = \text{that } a' \text{ s.t. } \Vdash_{\tilde{B}_\mu} a' = \sigma(a / \tilde{B})$

Then for $b \in \tilde{B}_\mu$ we have:

$$\sigma(b) = b' \text{ where } \uparrow_{\tilde{B}_\mu} b' = \begin{cases} 1 & \text{if } b' \in B^\circ \\ 0 & \text{if } b' \in B \end{cases}$$

- i.e. $\sigma \uparrow \tilde{B}_\mu$ is the natural injection.

Thus $\tilde{B}_{\mu+1}$ satisfies precisely those conditions which we had placed upon $B_{\mu+1}$ in V . Thus we can carry out all of our proofs in $V[B]$ with

$\langle \tilde{B}_{h+j} \mid j \leq \mu+1-h \rangle$ in place of $\langle B_i \mid i \leq \mu+1 \rangle$.

Finally, we note that, since GCH holds below κ , elementary considerations give us: $B_\mu * B^\circ$ has cardinality $\leq \aleph^+$. Hence we can choose $B_{\mu+1}$

$$\text{st. } B_{\mu+1} \subset H_{\aleph^+}.$$

This completes the first successor case. The second will be much harder, since we shall need $B_{\mu+1} \cong B_\mu * B^\circ$ for a B° which has yet to be defined.

The second successor case

Now suppose $\delta \notin A_0$, where $\delta = \delta_{\mu+1} = \omega_2^{V^{\mathbb{B}}_{\mu+1}}$.

Then δ must acquire cofinality ω at the next stage. But then all regular cardinals in $[\delta, \beta_{\mu+1})$ must become ω -cofinal. Recall that $\beta_{\mu+1}$ is the least β' s.t. either $cf(\beta') = \omega_1$ and $2^{\beta'} = \beta'$, or $\beta' \in A_0$ is regular. In the latter case $\beta' = \beta^+$ is a successor cardinal, $\beta \geq \delta$, and $2^\beta = \beta$, since GCH holds below κ if $A \neq \emptyset$. From now on let β be defined by:

$$\beta = \beta_{\mu+1} \quad \text{if } cf(\beta_{\mu+1}) = \omega_1$$

$$\beta^+ = \beta_{\mu+1} \quad \text{if not,}$$

where β is a cardinal.

Now let \mathbb{B} be \mathbb{B}_μ -generic. We work in $V[\mathbb{B}]$ to define a set of conditions

$\mathbb{P} = \mathbb{P}_{\mathbb{B}}$ which collapses all regular $\kappa \in [\delta, \beta]$ to ω . If $cf(\beta) = \omega_1$, then β^+ becomes ω_2 in $V[\mathbb{B}]^{\mathbb{P}}$. Otherwise β^{++} becomes ω_2 . We then take:

$$\mathbb{B}_{\mu+1} \stackrel{\dot{}}{=} \mathbb{B}_\mu * \mathbb{B} \quad \text{where } \dot{\mathbb{B}} = \text{BA}(\mathbb{P}_{\mathbb{B}}),$$

\mathbb{B} being the canonical generic name.

Let $A \in V$ s.t. $A \in H_\beta$ and $H_\beta^V = L_\beta^A$ ^{*}
 whenever $\gamma \leq \beta$ s.t. $2^\gamma = \gamma$. Set:

Def. $M = L_\beta^{A, B_\mu} = \text{pt} \langle L_\beta[A, B_\mu], A, B_\mu \rangle$.

$N = \langle H_{\beta^+}^V, M, <, \in \rangle$

where $<$ is a well ordering of N .

$Q = H_\beta^V$.

Def Working in $V[B]$ where B is B_μ -
 - generic set:

$M^B = L_\beta^{A, B_\mu, B}$, $N^B = \langle H_{\beta^+}^{V[B]}, M^B, < \rangle$

(Note M^B has the same sets as $H_\beta^{V[B]}$)

$Q^B = Q[B] = \bigcup_{x \in Q} L_x[x, B]$.

(Note $Q^B = H_x^{V[B]}$ since $\beta = \omega_2^{V[B]}$ is

regular in $V[B]$. Hence if B, B' are

B_μ -generic and $V[B] = V[B']$, then

$Q^B = Q^{B'}$.)

Working in $V[B]$ we now define:

Γ^* = the collection of $\langle S, C \rangle$ s.t.

- S is a transitive set
- $S \notin (ZFC^- + \omega_1 \text{ is the largest cardinal})$
- $C < S$ cofinally
- C is countable

(Recall that " $C < S$ cofinally" means $\bigcup C = S$.)

Def For $u = \langle S_u, C_u \rangle, v = \langle S_v, C_v \rangle \in \Gamma_*$ set:

$$\pi: u \triangleleft_* v \iff (\pi: S_u \hookrightarrow S_v \wedge \pi'' C_u = C_v)$$

Def $u \triangleleft_* v \iff \forall \pi \pi: u \triangleleft_* v$

Def For $u = \langle S_u, C_u \rangle \in \Gamma_*$ set $d_u = d_{S_u} = \omega_1^{S_u}$.

The following facts are readily verified and will be stated here without proof.

Fact 1 Let $\langle S, C \rangle \in \Gamma_*$, $d = d_S$. Then

$$S = \{ f(v) \mid f \in C \wedge v < d \}$$

Fact 2 If $\pi: u \triangleleft_* v$, then $d_u \leq d_v$ and

$$\text{rng}(\pi) = \{ f(v) \mid f \in C_v \wedge v < d_u \}$$

Hence:

Fact 3 For any $d \leq d_v$ there is at

most one pair $\langle u, \pi \rangle$ s.t.

$$\pi: u \triangleleft_* v \text{ and } d_u = d.$$

Hence:

Fact 4 Let $u \triangleleft_* v$. There is exactly

one π s.t. $\pi: u \triangleleft_* v$.

Def $\pi u v \stackrel{\text{df}}{=} \text{that } \pi \text{ s.t. } \pi: u \triangleleft_* v$.

Fact 5 $\langle \pi_{uv} \mid u \triangleleft_* v \rangle$ is a continuous commutative system.

Note "continuous" means that if $u_i \triangleleft_* u_j \triangleleft_* v$ for $i \leq j < \lambda$, then the transitive direct limit $\langle u, \langle \pi_{u_i, u} \mid i < \lambda \rangle \rangle$ of $\langle \langle u_i \mid i < \lambda \rangle, \langle \pi_{u_i, u_j} \mid i \leq j < \lambda \rangle \rangle$ exists and

There is $\pi : u \triangleleft_* v$ defined by

$$\pi \pi_{u_i, u} = \pi_{u_i, v} \quad (i < \lambda).$$

Hence:

Fact 6 $\{d_u \mid u \triangleleft_* v \wedge u \neq v\}$ is closed in d_v .

We now define:

Def R is a smooth model iff

• $R = L_{\beta}^{\vec{A}}$ for some $A_1, \dots, A_m \mid \beta$

• R models ZFC - or Zermelo set theory

• $L_{\gamma}^{\vec{A}} = H_{\gamma}^R$ whenever $2^{\beta} = \gamma$ in R .

Def Γ = the set of $\langle R, C \rangle$ s.t.,

• R is a smooth model

• $\langle Q, C \rangle \in \Gamma_*$ where $Q = H_{\omega_2}^R$.

We also write $Q_R = Q_{\langle R, C \rangle} = H_{\omega_2}^R$.

Def Let $u = \langle R_u, C_u \rangle, v = \langle R_v, C_v \rangle \in \Gamma$.

$\pi: u \triangleleft v$ iff

- $\pi: R_u \prec R_v$
- $\pi \upharpoonright Q_u: \langle Q_u, C_u \rangle \triangleleft_* \langle Q_v, C_v \rangle$
- There is R_{uv} s.t. $\langle R_{uv}, \pi \rangle$ is the liftup of $\langle R_u, \pi \upharpoonright Q_u \rangle$

(Hence the map π is wholly determined by $\pi \upharpoonright Q_u$.)

Def $u \triangleleft v$ iff $\forall \pi \pi: u \triangleleft v$.

It follows easily that:

Fact 7 Let $u \triangleleft v$. There is exactly one π s.t. $\pi: u \triangleleft v$.

Def $\pi_{uv} \approx$ that π s.t. $\pi: u \triangleleft v$

Fact 8 $\langle \pi_{uv} \mid u \triangleleft v \rangle$ is a continuous commutative system.

However, the analogue of Fact 3 does not hold for Γ , since $\{u \mid u \triangleleft v\}$ need not be linearly ordered by \triangleleft .

None the less we do have:

Fact 9 Let $u, w \triangleleft v$, $\text{rng}(u, v) \subset \text{rng}(w, v)$.
Then $u \triangleleft w$ and $\pi_{wv} \pi_{uw} = \pi_{uv}$.

The following fact will often be used tacitly:

Fact 10 Let $\pi: \langle R, c \rangle \triangleleft \langle R', c' \rangle$. Let $\gamma \in R$ s.t. $2^\gamma = \gamma$ in R and $\gamma \geq \omega_2^R$.
Let $\pi(\gamma) = \gamma'$, $R = L_{\beta}^{\vec{A}}$, $R' = L_{\beta'}^{\vec{A}'}$.
Set: $\bar{R} = L_{\gamma}^{\vec{A}}$, $\bar{R}' = L_{\gamma'}^{\vec{A}'}$, $\bar{\pi} = \pi \upharpoonright \bar{R}$.

Then $\bar{\pi}: \langle \bar{R}, c \rangle \triangleleft \langle \bar{R}', c' \rangle$.

proof.

Clearly $\bar{\pi}: \bar{R} \triangleleft \bar{R}'$ and \bar{R} models ZFC or Zermelo. Moreover $\bar{\pi} \upharpoonright c = c'$ and

$\Phi_{\bar{R}} = \Phi_R$, $\Phi_{\bar{R}'} = \Phi_{R'}$. Hence

$\bar{\pi} \upharpoonright \Phi_{\bar{R}}: \langle \Phi_{\bar{R}}, c \rangle \triangleleft \langle \Phi_{\bar{R}'}, c' \rangle$.

Claim Let $\bar{\pi}: \bar{R} \rightarrow \bar{R}'$ cofinally. Then $\langle \bar{R}, \bar{\pi} \rangle$ is the liftup of $\langle \bar{R}, \bar{\pi} \upharpoonright \Phi_{\bar{R}} \rangle$.

prf.

We must show that $\bar{\pi}: \bar{R} \rightarrow \bar{R}'$ is $\omega_2^{\bar{R}}$ -cofinal. Let $x \in \bar{R}'$. Then $x \in \pi(w)$ where $w \in \bar{R}$, $\bar{w} < \omega_2$ in \bar{R} . Let

$x \in L_{\pi(w)}^{\vec{A}'}$, where $v \in \bar{R}$. Set

$z = w \cap L_v^{\vec{A}}$. Then $z \in \bar{R}$, $\bar{z} < \omega_2$ in \bar{R}

and $x \in \bar{\pi}(z) = \pi(w) \cap L_{\pi(w)}^{\vec{A}'}$.

QED (Fact 10)

We return now to Q^B, M^B, N^B as defined above. We shall use an infinitary language \mathcal{L}_B on N^B to define an \mathcal{L} -forcing $\mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B}$ in $V[B]$. \mathbb{P}_B is intended to add a \dot{c} s.t. $\langle Q^B, \dot{c} \rangle \in \Gamma_*$ without adding new reals. However, \dot{c} should make not only \aleph ω -cofinal, but every regular $\tau \leq \beta$.

$\mathcal{L} = \mathcal{L}_B$ is the infinitary language on N^B with:

Predicate \in ; Constants \underline{x} ($x \in N^B$), \dot{c}

Axioms : ZFC⁻, $\wedge \underline{x} (\underline{x} \in \underline{x} \leftrightarrow \bigvee_{z \in \underline{x}} \underline{x} = \underline{z})$

for $x \in N^B$, $H_{\omega_1} = \underline{H}_{\omega_1}$, $\dot{c} < \underline{M}^B$, and

(*) $\wedge x \in \underline{M}^B \forall u \in \underline{H}_{\omega_1} \forall \pi (\pi : u \triangleleft \langle \underline{M}^B, \dot{c} \rangle \wedge x \in \text{rng}(\pi) \wedge \Psi(\pi))$ (*)

(This says, in particular, that every $x \in \underline{M}^B$ can be found in the liftup

of a countable \bar{M} by a $\pi' : \bar{M} \rightarrow \underline{M}^B$,

$\pi' : \langle Q_{\bar{M}}, \dot{c} \rangle \triangleleft_* \langle \underline{M}^B, \dot{c} \rangle$.)

* where $\Psi(\pi) = \begin{cases} \text{rng}(\pi) = \underline{M}^B & \text{if } \beta \text{ is regular} \\ \pi = \pi & \text{if not} \end{cases}$

Lemma 1 \mathcal{L} is consistent,
proof.

Let $\sigma: \bar{N} \prec N^B$ where \bar{N} is countable and transitive. Set $\bar{Q} = H_{\omega_2}^{\bar{N}}$. Let $\sigma: \bar{Q} \prec \tilde{Q}$ cofinally, let $\langle \tilde{N}, \tilde{\sigma} \rangle$ be the liftup of $\langle \bar{N}, \sigma \upharpoonright \bar{Q} \rangle$. Let

$k: \tilde{N} \prec N^B$ s.t. $k \tilde{\sigma} = \sigma$ and $k \upharpoonright \tilde{Q} = \text{id}$,

let $\tilde{\mathcal{L}}$ be defined on \tilde{N} like \mathcal{L} on N ,

It suffices to show:

Claim $\tilde{\mathcal{L}}$ is consistent,

since this is a $\Pi_1(\tilde{N})$ statement.

We show that $\langle H_{\omega_2}, \tilde{C} \rangle$ models $\tilde{\mathcal{L}}$, where $\tilde{C} = \text{rng}(\sigma \upharpoonright \bar{Q})$. (Note that $\tilde{N} \in H_{\omega_2}$.) All axioms other than

(*) are trivial. We verify (*). Set:

$D =$ the set of $\alpha < \omega_1$ s.t. there is a $u_\alpha = \langle Q_\alpha, C_\alpha \rangle$ with $\langle Q_\alpha, C_\alpha \rangle \triangleleft_{\tilde{\sigma}} \langle \tilde{Q}, \tilde{C} \rangle$

Then D is club in ω_1 and $\alpha_0 = \omega_1^{\bar{Q}}$ is

minimal in D with $u_{\alpha_0} = \langle \bar{Q}, \bar{Q} \rangle$.

Since $\langle \tilde{N}, \tilde{\sigma} \rangle$ is the liftup of $\langle \bar{N}, \sigma \upharpoonright \bar{Q} \rangle$

and $\sigma \upharpoonright \bar{Q} = \pi_{u_{\alpha_0}, \langle \tilde{Q}, \tilde{C} \rangle}$, we see that

$\langle \bar{N}, \pi_{u_{\alpha_0}, u_\alpha} \rangle$ has a transitive

liftup $\langle N_d, \tilde{\sigma}_{d,d} \rangle$. Moreover, there is a map $\tilde{\sigma}_{d, \omega_1} : N_d \hookrightarrow \tilde{N}$ defined by $\tilde{\sigma}_{d, \omega_1}(\tilde{\sigma}_{d,d}(f)\omega_1) = \tilde{\sigma}(f)\omega_1$, where $\nu < d$ and $f \in \bar{N}$ s.t. $f: \omega_1 \rightarrow \bar{N}$.

Set: $\tilde{\sigma}_{d\beta} = (\tilde{\sigma}_{\beta, \omega_1})^{-1} \tilde{\sigma}_{d, \omega_1}$ for $d \leq \beta$,

$d, \beta \in D \cup \{\omega_1\}$. It is clear by these definitions that:

$\langle N_\beta, \tilde{\sigma}_{d\beta} \rangle =$ the liftup of $\langle N_d, \tilde{\sigma}_{d, \omega_1} \rangle$ for $d \leq \beta, d, \beta \in D \cup \{\omega_1\}$ (with $u_{\omega_1} = \langle \tilde{Q}, \tilde{C} \rangle$).

Now set: $\bar{M} = \sigma^{-1}(M)$, $M_d = \tilde{\sigma}_{d,d}(\bar{M})$ for $d \in D \cup \{\omega_1\}$. Then $M_{\omega_1} = \bar{M} = \tilde{\sigma}(\bar{M})$.

Set $\sigma'_{d\beta} = \tilde{\sigma}_{d\beta} \upharpoonright M_d$ ($d \leq \beta, d, \beta \in D \cup \{\omega_1\}$).

Clearly $\bar{M} = \bigcup_{d \in D} \text{rng}(\sigma'_{d, \omega_1})$, so it suffices to show:

Claim $\sigma'_{d, \omega_1} : \langle M_d, C_d \rangle \triangleleft \langle \bar{M}, \tilde{C} \rangle$.

Clearly $\sigma'_{d, \omega_1} : M_d \hookrightarrow \bar{M}$ and

$\sigma'_{d, \omega_1} \upharpoonright C_d = \tilde{C}$. Now let:

$\sigma'_{d, \omega_1} : M_d \rightarrow \bar{M}_d$ cofinally,

(Then $\tilde{M}_\alpha = \tilde{M}$ if β is regular.)

But $\sigma'_{\alpha, \omega_1}$ is $\omega_2^{M_\alpha}$ -cofinal, since

$\tilde{\sigma}'_{\alpha, \omega_1} : N_\alpha \rightarrow \tilde{N}$ is $\omega_2^{N_\alpha}$ -cofinal. (To

see this let $x \in \tilde{M}_\alpha$. Then $x \in \tilde{\sigma}'_{\alpha, \omega_1}(a)$

where $a \in N_\alpha$, $\bar{a} < \omega_2$ in N_α . Since

$\sigma'_{\alpha, \omega_1} : M_\alpha \rightarrow \tilde{M}_\alpha$ cofinally, there

is $b \in M_\alpha$ s.t. $x \in \sigma'_{\alpha, \omega_1}(b)$. But

then $a \cap b \in M_\alpha$ + $x \in \sigma'_{\alpha, \omega_1}(a \cap b)$,

where $\overline{a \cap b} < \omega_2$ in M_α . QED (Lemma 1)

We shall make heavy use of the following lemma:

Lemma 2 Let \mathcal{M} be a solid model of \mathcal{L} .

Let $\langle A_n \mid n < \omega \rangle \in \mathcal{M}$ s.t. $A_n \in M^B$ for

$n < \omega$. Then there is $u = \langle s, c \rangle \in \mathcal{M} \cap H_{\omega_1}$

and $\pi \in \mathcal{M}$ s.t.

$$\pi : \langle s, c \rangle \triangleleft \langle M^B, \bar{c}^{\mathcal{M}} \rangle$$

$$\pi : \langle s, \bar{A}_n \rangle \triangleleft \langle M^B, A_n \rangle \text{ for } n < \omega$$

where $\bar{A}_n =_{\text{df}} \pi^{-1} \ulcorner A_n \urcorner$.

proof of Lemma 2

Set $M^* = \langle M^B, A_1, A_2, \dots \rangle$, Then $M^* \in \mathcal{O}$.

Working in \mathcal{O} we successively pick

$X_i \prec M^*$, $\pi_i : u_i \triangleleft \langle M^B, \dot{c}^{\mathcal{O}} \rangle$ s.t. $u_i \in H_{\omega_1}$

as follows: Let \prec well order H_{ω_1} .

$X_0 =$ the smallest $X \prec M^*$

Let $\langle x_m^j \mid m < \omega \rangle$ enumerate X_j .

$u_i =$ the \prec -least $u \in H_{\omega_1}$ s.t.

$u \triangleleft \langle M^B, \dot{c}^{\mathcal{O}} \rangle$ and $x_m^j \in \text{rng}(\pi_i)$

for $j, m < i$, where $\pi_i = \pi \upharpoonright u, \langle M^B, \dot{c}^{\mathcal{O}} \rangle$.

$X_{i+1} =$ the smallest $X \prec M^*$ s.t.

$$X_0 \cup \text{rng}(\pi_i) \subset X.$$

Let $X = \bigcup_i X_i$, Then $X = \bigcup_i \text{rng}(\pi_i)$

Let $\pi^* : \bar{M}^* \xrightarrow{\sim} X$ where \bar{M}^* is transitive.

Hence $\bar{M}^* \in H_{\omega_1}$. Let $\bar{M}^* = \langle \bar{M}, \bar{A}_1, \bar{A}_2, \dots \rangle$.

It suffices to show:

Claim $\pi : \langle \bar{M}, \bar{c} \rangle \triangleleft \langle M^B, \dot{c}^{\mathcal{O}} \rangle$,

where $\pi = \pi^* \upharpoonright \bar{M}$, $\bar{c} = \pi^{-1} \upharpoonright \dot{c}^{\mathcal{O}}$.

Proof

Clearly $\dot{c}^{\mathcal{O}} \in \text{rng}(\pi_0) \subset X$. Hence

$\pi \upharpoonright \bar{c} : \langle \bar{c}, \bar{c} \rangle \triangleleft_* \langle \bar{c}^B, \dot{c}^{\mathcal{O}} \rangle$ where $\pi^*(\bar{c}) = \dot{c}^{\mathcal{O}}$.

Now let $\pi : \bar{M} \rightarrow \tilde{M}$ cofinally.

It suffices to show:

Claim $\langle \tilde{M}, \tilde{\pi} \rangle$ is the lift up of $\langle \bar{M}, \bar{\pi} \upharpoonright \bar{Q} \rangle$,

prf.

We must show that $\tilde{\pi} : \tilde{M} \rightarrow \tilde{M}$ is $\omega_2^{\tilde{M}}$ -cofinal. Let $x \in \tilde{M}$. Then

$x \in \pi_i(a_i)$ for an $i < \omega$, where $\bar{a}_i < \omega_1$ in M_{a_i} ,

since $\tilde{M} = \bigcup X$ and $X = \bigcup_i \text{rng}(\pi_i)$.

Hence $a = \pi_i(a_i) \in X$ and $\bar{a} \leq \omega_1$ in M .

Hence $x \in a = \pi(\bar{a})$ for an $\bar{a} \in \bar{M}$

s.t. $\text{card}(\bar{a}) \leq \omega_1$ in \bar{M} . QED (Lemma 2)

" " " " "

We are now ready to define the set of conditions $\mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B}$.

We first set:

Def $\tilde{\mathbb{P}}$ = the set of $p = \langle p_0, p_1 \rangle$ s.t.

$$p_0 = \langle M_p, C^p \rangle \in \Gamma \cap H_{\omega_1}$$

$p_1 = F^p$ is an at most countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $\bar{a} \in M_p, a \in M^B$.

Def For $p \in \tilde{\mathbb{P}}$ let φ_p be the conjunction of:

- $\bullet p_0 \triangleleft \langle \underline{M}^B, \dot{c} \rangle$ (let $\pi_p = \pi_{p_0} \upharpoonright \langle \underline{M}^B, \dot{c} \rangle$)

- $\bullet \pi_p : \langle \underline{M}_p, \underline{a} \rangle \triangleleft \langle \underline{M}^B, \underline{a} \rangle$ for all $\langle a, \bar{a} \rangle \in F^p$

- $\bullet \pi_p : \underline{M}_p \rightarrow \underline{M}^B$ cofinally if β is regular

Def $\mathcal{L}(p) = \mathcal{L} + \varphi_p$.

Def $R^P = \text{rng}(F^P)$, $D^P = \text{dom}(F^P)$

Def $IP = IP_B = IP_{\mathcal{L}_B} = P_{\mathcal{L}_B} \{P \in \tilde{IP} \mid \mathcal{L}(P) \text{ is consistent}\}$

For $p, q \in IP$ set:

$p \leq q$ iff the following hold:

- $R^q \subset R^p$
- $\mathcal{G}_0 \triangleleft P_0$
- $\pi_{\mathcal{G}_0 P_0} : \langle M_q, \bar{a} \rangle \prec \langle M_p, a' \rangle$ whenever $\langle a, \bar{a} \rangle \in F^q$, $\langle a, a' \rangle \in F^p$.

Lemma 3.1 Let $p, q \in IP$. Then $p \leq q$ iff

- $R^q \subset R^p$
- $\mathcal{L}(p) \vdash (\mathcal{L}(q) \wedge \text{rng}(\pi_q) \subset \text{rng}(\pi_p))$.

prf.

(\rightarrow) Let $p \leq q$. Let \mathcal{M} be a solid model of $\mathcal{L}(p)$. It follows easily that $\text{rng}(\pi_q^{\mathcal{M}}) \subset \text{rng}(\pi_p^{\mathcal{M}})$ and $\mathcal{M} \models \mathcal{L}(q)$.

(\leftarrow) Let \mathcal{M} be a solid model of $\mathcal{L}(p)$. Then $\text{rng}(\pi_q^{\mathcal{M}}) \subset \text{rng}(\pi_p^{\mathcal{M}})$. Hence by

Fact 4, $\mathcal{G}_0 \triangleleft P_0$ and $\pi_{\mathcal{G}_0}^{\mathcal{M}} = \pi_p^{\mathcal{M}} \circ \pi_{\mathcal{G}_0 P_0}^{\mathcal{M}}$.

Since $\mathcal{M} \models \mathcal{L}(q)$, we then have:

$$\pi_{\mathcal{G}_0 P_0}^{\mathcal{M}} = (\pi_p^{\mathcal{M}})^{-1} \pi_q^{\mathcal{M}} : \langle M_q, \bar{a} \rangle \prec \langle M_p, a' \rangle$$

for $\langle a, \bar{a} \rangle \in F^q$, $\langle a, a' \rangle \in F^p$.

QED (3.1)

We set: $\pi_{\mathcal{F}^p} = \pi_{\mathcal{F}_0} \upharpoonright P_0$ if $p \leq \mathcal{F}$.

Exactly as in [LF] §0.1 - §0.3 we prove:

Lemma 3.2 Let $p \in \mathcal{I}$, Then

- $(F^p)^{-1}$ is a function
- $\mathcal{A} \upharpoonright R^p$ is closed under set difference, then $F^p: D^p \leftrightarrow R^p$
- $\pi^p = \upharpoonright_{\mathcal{F}^p} F^p \upharpoonright M_p$ is injective into M^B ,

The following lemma expresses a strong form of "revisability" in the sense of [LF].

Lemma 3.3 Let $p \in \mathcal{I}$, let $c \in M_p$ cofinally. Then $p' \in \mathcal{I}$ where:

$$P_0' = \langle M_p, c \rangle, \quad P_1' = P_1.$$

proof

Let \mathcal{M} be a solid model of $\mathcal{L}(p)$. Form \mathcal{M}' by replacing $\dot{c}^{\mathcal{M}}$ with $c' = \pi_p^{\mathcal{M}} \upharpoonright c$.

Claim $\mathcal{M}' \models \mathcal{L}(p')$

We first show: $\mathcal{M}' \models \mathcal{L}$.

Note that if

$$u = \langle \mathcal{Q}_u, c_u \rangle \triangleleft_* \langle \mathcal{Q}_u^B, \dot{c}^{\mathcal{M}'} \rangle \text{ and } \alpha_u \geq \alpha_p,$$

$$\text{then } \langle \mathcal{Q}_p, c_p \rangle \triangleleft_* u \triangleleft_* \langle \mathcal{Q}_p^B, \dot{c}^{\mathcal{M}'} \rangle$$

$$\text{and } \text{rng}(\pi_u, \langle \mathcal{Q}_u^B, \dot{c}^{\mathcal{M}'} \rangle) \supseteq \text{rng}(\pi_p^{\mathcal{M}'} \upharpoonright \mathcal{Q}_p)$$

(where, of course, $\mathcal{Q}_p = H_{\omega_2}^{M_p}$.)

Set: $u' = \langle Q_u, C_{u'} \rangle$ where $C_{u'} = \pi_{u, \langle Q^B, c^M \rangle}^{-1} u' C'$
 $= \pi_{\langle Q_p, C_p \rangle, u} u' C'$. Then $u' \triangleleft_x \langle Q, c' \rangle$ and
 $\pi_{u', \langle Q^B, c^M \rangle} = \pi_{u, \langle Q^B, c^M \rangle}$, as is easily
 seen. But this means that if

$v = \langle S_v, C_v \rangle \triangleleft \langle M^B, c^M \rangle$ with $d_v \geq d_p$,

then $v' = \langle S_v, C_{v'} \rangle \triangleleft \langle M^B, c' \rangle$

where $C_{v'} = \pi_{\langle Q_p, C_p \rangle, \langle Q_v, C_v \rangle} u' C'$

with $\pi_{v', \langle M^B, c' \rangle} = \pi_{v, \langle M^B, c^M \rangle}$,

since $\pi_{v', \langle M^B, c' \rangle}$ is uniquely de-
 termined by $\pi_v \upharpoonright Q_v$. Thus (*) con-
 tinues to hold in \mathcal{M}' . The other
 axioms are trivial.

Our argument shows, in particular,
 that $\pi_p^M = \pi_{\langle M_p, c \rangle, \langle M^B, c' \rangle}$.

Hence $\pi_{\langle M_p, c \rangle, \langle M^B, c' \rangle} : \langle M_p, \bar{a} \rangle \in \langle M^B, a \rangle$

whenever $\langle a, \bar{a} \rangle \in F^p = F^{p'}$.

Thus $\mathcal{M} \models \mathcal{L}(p')$. QED (3,3)

We now prove the main lemma on extendability of conditions.

Lemma 3.4 $IP \neq \emptyset$. Moreover, if $p, q \in IP$ and $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent, then there is an r s.t. $r \leq p, q$. If $R \subset \mathcal{P}(MB)$ is any countable set we may, in fact, choose r s.t. $R \subset R^r$.

proof

To see $IP \neq \emptyset$, let \mathcal{M} be a solid model of \mathcal{L} . Let $u \triangleleft \langle MB, \dot{c}^{\mathcal{M}} \rangle$, $u \in H_{\omega_1}$. Then $p \in IP$ where $p_0 = u$, $p_1 = \emptyset$.

Now let $\mathcal{M} \models \mathcal{L}(p) \cup \mathcal{L}(q)$. Set:

$$X = \text{rng}(\pi_p^{\mathcal{M}}) \cup \text{rng}(\pi_q^{\mathcal{M}}) \cup F_p \cup F_q \cup R,$$

Then $X \in \mathcal{M}$ is countable in \mathcal{M} with

$X \subset \mathcal{P}(M)$. By Lemma 2 there is

$\langle \bar{m}, \bar{c} \rangle \triangleleft \langle MB, \dot{c}^{\mathcal{M}} \rangle$ s.t. $\langle \bar{m}, \bar{c} \rangle \in H_{\omega_1}$

and $\pi : \langle \bar{m}, \bar{A} \rangle \triangleleft \langle M, A \rangle$ for all $A \in X$,

where $\pi = \pi_{\langle \bar{m}, \bar{c} \rangle, \langle MB, \dot{c}^{\mathcal{M}} \rangle}$ and $\bar{A} = \pi^{-1} \upharpoonright A$.

Define r by: $r_0 = \langle \bar{m}, \bar{c} \rangle$,

$r_1 =$ the set of $\langle A, \bar{A} \rangle$ s.t. $A \in R^p \cup R^q \cup R$

and $\bar{A} = \pi^{-1} \upharpoonright A$.

Then $\mathcal{M} \models \mathcal{L}(r)$. Hence $r \in IP$. But

$$\pi_p^{\mathcal{M}} \upharpoonright p_0 \triangleleft \langle MB, \dot{c}^{\mathcal{M}} \rangle, \quad \pi_r^{\mathcal{M}} \upharpoonright r_0 \triangleleft \langle MB, \dot{c}^{\mathcal{M}} \rangle$$

$$\text{and } \text{rng}(\pi_p^{\mathcal{M}}) \subset \text{rng}(\pi_r^{\mathcal{M}}),$$

Hence $\pi : p_0 \triangleleft r_0$ where $\pi = \pi_r^{nr} \cdot (\pi_p^{nr})^{-1}$,
 by Fact 9. But then, if $\langle a, \bar{a} \rangle \in F^p$,
 $\langle a, a' \rangle \in F^r$, we have:

$$\pi : \langle M_p, \bar{a} \rangle \prec \langle M_r, a' \rangle.$$

Hence $r \leq p$ with $\pi = \pi_{p,r}$.

Similarly $r \leq q$. QED (3.4)

Cor 3.5 p, q are compatible in \mathbb{P}
 iff $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent.

prf.

(\leftarrow) by Lemma 3.4

(\rightarrow) If $r \leq p, q$, then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$.
 QED (3.5)

Cor 3.6 Let $p \in \mathbb{P}$, $R \subset \mathcal{P}(M^B)$ where R is
 countable. There is $q \leq p$ s.t. $R \subset \mathcal{R}^q$.

Cor 3.7 Let $p \in \mathbb{P}$, $u \subset M^B$, where u is
 countable. There is $q \leq p$ s.t. $u \subset \text{rng}(\pi^q)$

Lemma 3.8 Let $p \in \mathbb{P}$, $u \subset M_p$, u finite.

There is $q \leq p$ s.t. $q_0 = p_0$ and $u \subset \text{dom}(\pi^q)$.

prf. Let \mathcal{M} be a solid model of $\mathcal{L}(p)$.

Set: $q_0 = p_0$, $\mathcal{F}^q = \mathcal{F}^p \cup (\pi_p^{nr})^{-1}(u)$.

QED (3.8)

Using the extension lemmas we get:

Lemma 3.9 Let G be IP-generic. Then

(a) $\langle P_p \mid p \in G \rangle, \langle \pi_p q \mid q \leq p \text{ in } G \rangle$ is a directed system with the limit:

$$\langle M^B, C^G \rangle, \langle \pi_p^G \mid p \in G \rangle$$

(More over $\pi_p^G = \bigcup \{ \pi_q \mid q_0 = p_0 \wedge q \leq p \wedge q \in G \}$.)

(b) $\pi_p^G : P_0 \triangleleft \langle M^B, C^G \rangle$ for $p \in G$

(c) $\pi_p^G : \langle M_p, \bar{a} \rangle \triangleleft \langle M^B, \bar{a} \rangle$ whenever $\langle a, \bar{a} \rangle \in FP$.

The proof is left to the reader.

" " " " " "

Using the "reversibility" lemma 3.3 we get:

Lemma 3.9 IP adds no reals.

prf.

Let $\Vdash \dot{f} : \check{\omega} \rightarrow \mathbb{Z}$. It is enough to show:

Claim The set Δ of p n.t. $\forall f p \Vdash \dot{f} = \check{f}$ is dense in IP.

Let $r \in IP$. We construct $q \leq r$ n.t. $q \in \Delta$. Let:

$$N^* = \langle H_\theta, N_1^B, \dot{f}, IP, r, \dots \rangle \text{ where } \theta > 2^B$$

and \triangleleft well orders N^* .

Let $p \in \mathbb{P}$ conform to N^* (as defined in [LF] § 31). Set:

$$\bar{N}^* = \bar{N}^*(p, N^*) = \langle H', N'^B, \langle, f', P', \pi', \dots \rangle$$

Pick $G' \ni \pi$ which is P' -generic over \bar{N}^* .

Define q by: $q_0 = \langle M_p, c^{G'} \rangle$, $q_1 = P_1$

Then $q \in \mathbb{P}$ by the reversibility lemma.

Let $f = f' \circ G'$. It suffices to show:

Claim $q \Vdash \check{f} = \check{f}$ and q is compatible with π .

We first show: $q \Vdash \check{f} = \check{f}$. Suppose not.

Then there is $q' \leq q$ s.t. $q' \Vdash \check{f}(\check{n}) \neq \check{f}(\check{n})$ for some n . Let \mathcal{M} be a solid model of $\mathcal{L}(q')$.

Let $\pi^* \supseteq \pi \upharpoonright_{\mathcal{M}} \cup F \ni \pi$ s.t.

$\pi^* : \bar{N}^* \leftarrow N^*$. Let $\pi' = G'$ s.t.

$\pi' \Vdash_{P'} \check{f}(\check{n}) = \check{f}(\check{n})$. Set $\pi = \pi^*(\pi')$.

Then $\pi \Vdash_{P'} \check{f}(\check{n}) = \check{f}(\check{n})$. Hence π, q are

incompatible. We obtain a contradiction by proving:

Claim $\mathcal{M} \models \mathcal{L}(q') \cup \mathcal{L}(\pi)$.

Pr.

$\mathcal{M} \models \mathcal{L}(q')$ is trivial. We prove $\mathcal{M} \models \mathcal{L}(\pi)$.

Note that $\pi_0 = \pi'_0$ and $F \ni \pi =$ the set

set of $\langle a, \bar{a} \rangle$ s.t. $a = \pi^*(a')$ and

$\langle a', \bar{a} \rangle \in F \pi'$ for some a' . Clearly

$\pi \upharpoonright_{\mathcal{M}}^{G'} : \pi_0 \triangleleft \langle M'^B, c^{G'} \rangle$, since $\pi' \in G'$.

But $g_0 = \langle M^{B'}, c^{G'} \rangle$ and $\pi_g^{U'} : g_0 \triangleleft \langle M^B, c^{U'} \rangle$.

Set $\pi = \pi_g^{U'} \circ \pi_{r'}^{G'} = \pi^* \circ \pi_{r'}^{G'}$. Then $\pi \in U'$

and $\pi : r_0 \triangleleft \langle M^B, c^{U'} \rangle$. It remains only to show:

Claim $\pi : \langle M_{r_1}, a \rangle \triangleleft \langle M^B, a \rangle$ for $\langle a, \bar{a} \rangle \in F_1^r$,

since then $U' \models \mathcal{L}(r_1)$ with $\pi = \pi_{r_1}^{U'}$.

Let $a = \pi^*(a')$. Then $\langle a', \bar{a} \rangle \in F_1^{r'}$. Then

$\pi_{r'}^{G'} : \langle M_{r_1}, \bar{a} \rangle \triangleleft \langle M_{g_1}, a' \rangle$ and

$\pi_g^{U'} : \langle M_{g_1}, a' \rangle \triangleleft \langle M^B, a \rangle$ since $\pi^*(\langle M_{g_1}, a' \rangle) = \langle M^B, a \rangle$

This proves: $g \Vdash f^{\check{}} = \check{f}$. But the last part of the proof shows that for every $r' \in G'$, $r = \pi^*(r')$ is compatible with g' for any $g' \leq g$. i.e. $U' \models \mathcal{L}(g')$

But $r' \in G'$ and $r = \pi^*(r')$ since $\pi^* : N^* \triangleleft N^*$. Hence r is compatible with g . QED (3.9)

An immediate corollary is:

Cor 3.10 Let $\theta \geq 2^B$ be regular. If $G \ni P$ is \mathbb{P} -generic and $c = c^G$, then $\langle H_\theta^{V[G][c]}, c \rangle$ models $\mathcal{L}(P)$.

Proof.

The only problematical axiom was $H_{\omega_1} = \underline{H}_{\omega_1}$, which is now seen to hold.

QED (3.10)

Def Let $C \subset M^B$ be countable and cofinal,
 $G^C = \{ p \in \mathbb{P} \text{ s.t. } p_0 \triangleleft \langle M^B, C \rangle \}$
 and, letting $\pi = \pi_{p_0, \langle M^B, C \rangle}$, we have:
 $\pi : \langle M_p, \bar{a} \rangle \triangleleft \langle M^B, a \rangle$ whenever $\langle a, \bar{a} \rangle \in \mathbb{P}^P$
 and $\pi : M_p \rightarrow M^B$ is cofinal if β is regular.

Lemma 3.11 Let G be IP-generic. Then
 $G = G^C$ where $C = C^G$.

proof

$G \subset G^C$ is trivial. We prove (\supset)

Let $p \in G^C$. If $p \notin G$ there is $q \in G$ which
 is incompatible with p . But then

$$\langle H_{\theta}^{\mathbb{P}[G]}, C \rangle \models \mathcal{L}(p) \cup \mathcal{L}(q).$$

for regular $\theta \geq 2^B$.

QED (3.11)

Lemma 3.12 Let G be IP-generic. Then

$$\bar{\beta} \leq \omega_1 \text{ in } V[B][G]$$

proof

For each $\bar{\zeta} < \beta$ there is $\langle \bar{m}, \bar{c}, \bar{\zeta} \rangle \in H_{\omega_1}^{\mathbb{P}}$
 s.t. $\langle \bar{m}, \bar{c} \rangle \triangleleft \langle M^B, C^G \rangle$ and $\pi(\bar{\zeta}) = \zeta$

where $\pi = \pi_{\langle \bar{m}, \bar{c} \rangle, \langle M^B, C^G \rangle}$. This maps a

subset of H_{ω_1} onto β . QED (3.12)

Lemma 3.13 Let G be IP-generic. If $\omega_1 < \tau \leq \beta$ and τ is regular in $V[B]$, then $cf(\tau) = \omega$ in $V[B][G]$

proof.

If $\tau = \beta$, then for any $p \in G$ we have $\sup \pi_p^G \restriction \beta_p = \beta$ where β_p is countable.

Now let $\tau < \beta$. Let $p \in G$ st. $\pi_p^G(\bar{\tau}) = \tau$.

Then each $\xi < \tau$ lies in $\pi_p^G(u)$ for a $u \in M_p$ st. $\bar{u} \leq \omega_1$ in M_p . But the set U of such u is countable. Set

$\mu_u = \sup u \cap \bar{\tau}$ for $u \in U$. Then $\mu_u < \bar{\tau}$ and $\{\pi_p^G(\mu_u) \mid u \in U\}$ is cofinal in τ .

QED (3.13)

Cor 3.14 If $\omega_1 < \delta \leq \beta$ and $cf(\delta) \neq \omega_1$,
then $cf(\delta) = \omega$ in $V[B][G]$.

We now recall [LF] § 4 Lemma 4.1 which says:

Fact 11 Let β be a cardinal in an inner model W st. $2^\beta = \beta$ in W . Let $\delta = 2^\beta$ in W . Assume that in V we have:
 $2^\omega = \omega_1$, $\bar{\beta} = \omega_1$, $cf(\beta) = \omega$. Then $\bar{\delta} \leq \omega_1$ in V .

* We are working over $V[B]$, so statements like $cf(\delta) \neq \omega_1$ are understood to be in the sense of $V[B]$.

Hence:

Cor 3.14.1 If $cf(\beta) \neq \omega_1$, then $\overline{\mu} = \omega_1$
in $V[B][G]$, where $\mu = 2^\beta$.

(μ^+ remains a cardinal, however, since $\overline{\mu} \leq \mu$. Hence if $2^\mu = \mu$, we can conclude $cf(\mu) = \omega_1$, since otherwise μ^+ would be collapsed by Fact 11.

In particular, $\mu = \beta^+ + cf(\mu) = \omega_1$
in $V[G]$ if GCH holds in V .)

The case $cf(\beta) = \omega_1$ is quite different as shown by:

Lemma 3.15 Let $cf(\beta) = \omega_1$ in $V[B]$. Then β^+ remains a cardinal in $V[B][G]$ (Hence $\beta^+ = \omega_2$ in $V[B][G]$)
proof.

We imitate the proof of [LF] §4 Lemma 3.1 to show:

Sublemma 3.15.1 $BA(\mathbb{P})$ has a dense subset of size β .

Prf. Wlog in $V[B]$.

Set $H = H_{\mathbb{P}}(B)^+$. Then $\langle H[G], c^G \rangle$

models \mathcal{L} whenever G is \mathbb{P} -generic (interpreting \underline{x} by \dot{x}). Let $\Vdash_{\mathbb{P}} \dot{c} = c^G$,

where \dot{c} is the canonical generic name. We can give every \mathcal{L} sentence ψ an interpretation $\llbracket \psi \rrbracket \in B = BA(\mathbb{P})$ in $H^{\mathbb{P}}$, interpreting \dot{c} by \dot{c} and \underline{x} by \dot{x} .

We then have:

$$\langle H[G], c^G \rangle \models \psi(\underline{x}_1, \dots, \underline{x}_n) \iff$$

$$\iff \llbracket \psi(\underline{x}_1, \dots, \underline{x}_n) \rrbracket \cap G \neq \emptyset$$

for $\underline{x}_1, \dots, \underline{x}_n \in N$ and G a \mathbb{P} -generic set.

Thus it suffices to prove:

Claim For each $p \in \mathbb{P}$ there is an

\mathcal{L} -statement $\psi \in M^B$ s.t. $\llbracket \psi \rrbracket \neq 0$ and

$\llbracket \psi \rrbracket \subset [p]$, ($[p]$ being the smallest $b \in B$ s.t. $p \in b$).

If G is \mathbb{P} -generic, we have;

$$[p] \cap G \neq \emptyset \iff p \in G \iff \langle H[G], c^G \rangle \models \varphi_p.$$

$$\iff \llbracket \varphi_p \rrbracket \cap G \neq \emptyset. \quad \text{Hence}$$

$[p] = \llbracket \varphi_p \rrbracket$ and it suffices to show:

Claim $\prod_{\mathbb{P}} \Psi \rightarrow \varphi_p$ for a $\psi \in M^B$ s.t. $\llbracket \psi \rrbracket \neq \emptyset$

Set $N^* = \langle H, N^B, < \rangle$ where $<$ well orders H .

We may assume w.l.o.g. that p conforms to N^* , since the set of such p is dense in \mathbb{P} . Let G be \mathbb{P} -generic with $p \in G$. Let $\tilde{\beta} = \sup_{\tilde{p}}^G \beta_p$.

Then $\tilde{\beta} < \beta$ since $\tilde{\beta}$ is ω -cofinal.

Set $\tilde{M} = \bigcup_{\tilde{\beta}} A_i \cup B_i \cup B$. For $a \in \mathbb{R}^P$ set $\tilde{a} = a \upharpoonright \tilde{M}$.

Then $\pi_p^G : \langle \tilde{M}, \tilde{a} \rangle \rightarrow \langle \tilde{M}, \tilde{a} \rangle$ is cofinal and Σ_0 -preserving whenever $\langle a, \tilde{a} \rangle \in F_p$.

But then

$$(1) \tilde{a} = \bigcup_{z \in M_p} \pi_p^G(z \upharpoonright \tilde{a}).$$

Let $\langle a_i \mid i < \omega \rangle$ enumerate \mathbb{R}^P in V .

Then $\langle \tilde{a}_i \mid i < \omega \rangle \in H_{\omega_1}$, where $\langle a_i, \tilde{a}_i \rangle \in F_p$.

Moreover $\langle \tilde{a}_i \mid i < \omega \rangle \in M$, since $\tilde{a}_i \in M$

and $\text{cf}(\beta) > \omega$. Let ψ be the sentence:

There are π, σ s.t. $\sigma: \langle \underline{Q}_p, \underline{c}^p \rangle \triangleleft_* \langle \underline{Q}^B, \underline{c}^B \rangle \wedge$
 $\wedge \langle \tilde{M}, \pi \rangle$ is the liftup of $\langle \underline{M}_p, \sigma \rangle \wedge$
 $\wedge \bigwedge_{i < \omega} \underline{\tilde{a}}_i = \bigcup_{z \in \underline{M}_p} \pi(z \cap \underline{a}_i)$,

Clearly $\psi \in M^B$. Moreover,

(2) $\llbracket \psi \rrbracket \neq 0$, since $\langle H[G], c^G \rangle \models \psi$

(since then ψ holds with $\langle \sigma = \pi_p^G \upharpoonright Q_p, \pi = \pi_p^G \rangle$).

We show:

(3) $\langle H[G], c^G \rangle \models \psi \rightarrow \varphi_p$

whenever G is IP-generic.

Let $\langle H[G], c^G \rangle \models \psi$. Let $\sigma: \langle \underline{Q}_p, \underline{c}^p \rangle \triangleleft_* \langle \underline{Q}, \underline{c}^G \rangle$

and $\langle \tilde{M}, \pi \rangle =$ the liftup of $\langle \underline{M}_p, \sigma \rangle$.

It remains only to show:

$\pi: \langle \underline{M}_p, \underline{a} \rangle \triangleleft \langle M^B, a \rangle$ whenever

$\langle a, \underline{a} \rangle \in F^p$, since then we have

$\pi: \langle \underline{M}_p, \underline{c}^p \rangle \triangleleft \langle M^B, \underline{c}^G \rangle$ and

hence: $p \in G^{c^G} = G$ with $\pi = \pi_p^G$.

Let $b = \{ \vec{z} \in M \mid \langle M, a \rangle \models \chi(\vec{z}) \}$. Then $b \in \mathbb{R}^p$

by the N^* -conformity of p . Let

$\langle b, \bar{b} \rangle \in F^p$. Then by N^* -conformity:

$$\bar{b} = \{ \vec{z} \in M_p \mid \langle \underline{M}_p, \underline{a} \rangle \models \chi(\vec{z}) \}.$$

Hence: $\langle \underline{M}_p, \underline{a} \rangle \models \chi(\vec{z}) \leftrightarrow \vec{z} \in \bar{b} \leftrightarrow$

$\leftrightarrow \pi(\vec{z}) \in \bar{b}^\sim = \bar{b} \cap \tilde{M} \leftrightarrow \langle M, a \rangle \models \chi(\pi(\vec{z}))$,

since $\bar{b}^\sim = \bigcup_{u \in M_p} \pi(u \cap \bar{b})$. QED (3.15)

Note $cf(\beta) = \omega_1$ is the only case to consider if $A_0 = 0$.

We also note that we could have defined \mathcal{L} (and hence $IP = IP_{\mathcal{L}}$) somewhat differently: Let \mathcal{L}' be like \mathcal{L} except that in (*) we omit: $\text{Urng}(\pi) = M$ if β is regular, and instead add the axiom:

(*) If β is regular, then whenever $u \in H_{\omega_1}$ and $\pi: u \triangleleft \langle \underline{m}, \underline{c} \rangle$, we have:
 $\text{sup } \text{On} \cap \text{rng}(\pi) < \beta$.

It turns out that \mathcal{L}' is also consistent. If $IP' = IP_{\mathcal{L}'}$ and β is regular, we can modify the proof of Lemma 3.1. to get: $IB' = BA(IP')$ contains a dense subset of size β . Hence $\beta^+ = \omega_2$ and $cf(\beta) = \omega_1$ in $V[G']$, where G' is IP' -generic. We omit the proof, since this is not relevant to the present paper.

We are now ready to prove that IP is subcomplete. Since we are working in $V[B]$ we shall again write V for $V[B]$ and - for the sake of simplicity - we also write Q, M, N for Q^B, M^B, N^B .

Lemma 4 \mathbb{P} is subcomplete.

prf: (We work in $\mathcal{V}[\mathcal{B}]$)

Let $W = L_{\tau}^{A'}$ where $2^{\beta} < \theta < \tau$, τ is regular,

and $H_{\theta} \subset W$. Let $\sigma: \bar{W} \prec W$ s.t. \bar{W} is countable and full with:

$$\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{M}, \bar{\alpha}, \bar{\lambda}_i) = \theta, \mathbb{P}, M, \alpha, \lambda_i \quad (i=1, \dots, n)$$

where $\mathbb{P} \in H_{\lambda_i}$ (hence $N \in H_{\lambda_i}$), $\lambda_i < \theta$, and λ_i

is regular for $i=1, \dots, n$. Let \bar{G} be $\bar{\mathbb{P}}$ -generic over \bar{W} .

Claim There is $g \in \mathbb{P}$ s.t. whenever $G \ni g$ is \mathbb{P} -generic, then there is $\sigma_0 \in \mathcal{V}[\mathcal{B}]$ with:

(a) $\sigma_0: \bar{W} \prec W$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{P}}, \bar{M}, \bar{\alpha}, \bar{\lambda}_i) = \theta, \mathbb{P}, M, \alpha, \lambda_i \quad (i=1, \dots, n)$

(c) $\sup \sigma_0 \text{''} \bar{\lambda}_i = \sup \sigma \text{''} \bar{\lambda}_i \quad (i=0, \dots, n)$,

where $\bar{\lambda}_0 = 0 \cap \bar{W}$

(d) $\sigma_0 \text{''} \bar{G} \subset G$.

We first show by standard methods:

Sublemma 4.1 Let σ be least s.t. $L_{\sigma}(W)$ is admissible. The following language \mathcal{L}^* on $L_{\sigma}(W)$ is consistent:

Predicate \in , Constants \underline{x} ($x \in L_{\sigma}(W)$), $\underline{\sigma}$

Axioms: ZFC⁻, $\bigwedge \sigma (\sigma \in \underline{x} \leftrightarrow \bigvee_{z \in \underline{x}} \sigma = \underline{z})$ for $x \in L_{\sigma}(W)$,

$\underline{\sigma}: \bar{W} \prec \underline{W}$, $\underline{\sigma}(\bar{\theta}, \bar{\mathbb{P}}, \bar{M}, \bar{\alpha}, \bar{\lambda}_i) = \underline{\theta}, \underline{\mathbb{P}}, \underline{M}, \underline{\alpha}, \underline{\lambda}_i \quad (i=1, \dots, n)$,

$\sup \underline{\sigma} \text{''} \bar{\lambda}_i = \sup \sigma \text{''} \bar{\lambda}_i \quad (i=0, \dots, n)$, and:

$\underline{\sigma} \upharpoonright \bar{Q}: \bar{Q} \prec \underline{Q}$ cofinally (where $\sigma(\bar{Q}) = \underline{Q}$),

$\langle \underline{N}, \underline{\sigma} \upharpoonright \underline{N} \rangle$ is the liftup of $\langle \bar{N}, \underline{\sigma} \upharpoonright \bar{Q} \rangle$

Note \mathcal{L}^* does not posit that $H_{w_1} = H_{w_2}$,

pr. f. (sketch) of 4.1

Let \mathcal{L}_0 be like \mathcal{L}^* except that the axiom

$$\sup \sigma \text{ " } \bar{\lambda}_i = \underline{\sup \sigma \text{ " } \bar{\lambda}_i} \text{ (} i=0, \dots, n \text{)}$$

is replaced by:

$$\sup \sigma \text{ " } \bar{\lambda}_i = \lambda_i \text{ (} i=0, \dots, n \text{) (where } \lambda_0 =_{\text{def}} \tau \text{)}$$

Let $\sigma \upharpoonright \bar{Q} : \bar{Q} < \tilde{Q}$ cofinally ($\bar{Q} = H_{w_2}^{\bar{w}}$) and

let $\tilde{\sigma} : \bar{w} < w$ be the liftup of \bar{w} by $\sigma \upharpoonright \bar{Q}$.

Let $k : \tilde{w} < w$ s.t. $k \upharpoonright \tilde{Q} = \text{id}$ and $k \tilde{\sigma} = \sigma$,

let $\tilde{\mathcal{L}}_0$ be defined on $L_{\tilde{\sigma}}(\tilde{w})$ like \mathcal{L}_0 on

$L_{\sigma}(w)$ in the obvious sense, where $\tilde{\sigma}$ is

least s.t. $L_{\tilde{\sigma}}(\tilde{w})$ is admissible. (More

precisely, \mathcal{L}_0 is defined in the parameter

w and parameters $Q, M, N, \theta, \varepsilon, \lambda_i, \dots \in w$,

$\tilde{\mathcal{L}}_0$ has the same definition over $L_{\tilde{\sigma}}(\tilde{w})$

in the parameter \tilde{w} and the parameters

$k^{-1}(Q), k^{-1}(M), \dots, k^{-1}(\varepsilon), k^{-1}(\lambda_i)$ ($i=1, \dots, n$).

Then $\langle H_{w_2}, \tilde{\sigma} \rangle$ models $\tilde{\mathcal{L}}_0$. Assume

w.l.o.g. $\lambda_0 > \dots > \lambda_n$ and let

$$\sigma \upharpoonright H_{\bar{\lambda}_m}^{\bar{w}} : H_{\bar{\lambda}_m}^{\bar{w}} < H' \text{ cofinally.}$$

(Here $H_{\bar{\lambda}_m}^{\bar{w}} = \bar{w}$ if $m=0$.) Let $\sigma' : \bar{w} < w'$

be the liftup of \bar{w} by $\sigma \upharpoonright H_{\bar{\lambda}_m}^{\bar{w}}$. Let

$k' : w' < w$ s.t. $k' \upharpoonright H' = \text{id}$ and $k' \sigma' = \sigma$.

There is then $k : \tilde{w} < w'$ s.t.

$\tilde{k} \uparrow \tilde{Q} = \text{id}$ and $\tilde{k} \tilde{\sigma} = \sigma'$. We then have $k' \tilde{k} = k$. Let δ' be least s.t. $L_{\delta'}(W')$ is admissible and let \tilde{L}'_0 be defined over $L_{\delta'}(W')$ as L_0 was defined over $L_{\delta}(W)$ (in the obvious sense). The statement that \tilde{L}'_0 is consistent in $\text{TT}_1(L_{\delta'}(\tilde{W}))$ in the parameter \tilde{W} and parameters $\vec{p} \in \tilde{W}$. The statement that \tilde{L}'_0 is consistent in $\text{TT}_1(L_{\delta'}(W'))$ in W' and $k'(\vec{p})$. Hence \tilde{L}'_0 is consistent. Note that $k'(N) = N$. Let \mathcal{M} be a solid model of \tilde{L}'_0 which lies in some generic extension $V[G]$ of V . Let $\mu > \varepsilon$ be regular in $V[G]$. Then $\langle H_{\mu}^{V[G]}, k' \circ \sigma' \rangle$ models \tilde{L}'_0 , where $\sigma' = \sigma \upharpoonright \mathcal{M}$. QED (4.1)

Now let $N^* = \langle H_{\delta}, W, N, \sigma, \lambda_1, \dots, \lambda_m, \iota, IP, \dots \rangle$ where $\delta > \sigma_{\text{on } W}$. Let p conform to N^* . Set: $\bar{N}^* = \bar{N}^*(p, N^*) = \langle H', W', N', \sigma', \lambda'_1, \dots, \lambda'_m, \iota', IP', \dots \rangle$. Let \bar{L}^* be defined in \bar{N}^* like L^* in N^* . Let $\mathcal{M} \in H_{W'}$ be a solid model of \bar{L}^* . Set: $\sigma^* = \sigma \upharpoonright \mathcal{M}$. Set $\bar{Q} = c \bar{G}$, $c' = \sigma^* \circ c$. Since $\sigma^* \uparrow \bar{Q} : \bar{Q} \rightarrow Q'$ cofinally (where $Q' = Q_p$ is defined in \bar{N}^* like Q in N^*), we have: $q \in IP$ where q is defined by:

$$q_0 = \langle M_p, c' \rangle, \quad q_1 = P_1.$$

We show that this q satisfies the claim.

Let $G \ni q$ be IP-generic. Note that, since

$\langle N', \sigma^* \upharpoonright \bar{N} \rangle$ is the liftup of $\langle \bar{N}, \sigma^* \upharpoonright \bar{Q} \rangle$

and $\sigma^* \upharpoonright \bar{Q} : \bar{Q} \triangleleft Q'$ with $\sigma^* \bar{c} = c' = c_q$, we

have: $\langle \bar{m}, \bar{c} \rangle \triangleleft q_0 = \langle M_q, c' \rangle$ with

$\pi_{\langle \bar{m}, \bar{c} \rangle, q_0} = \sigma^* \upharpoonright \bar{m}$. But $q_0 \triangleleft \langle M, c \rangle$ with

$\pi_{q_0, \langle M, c \rangle} = \pi_q^G$. (Here $c = c^G$.) Then

$\langle \bar{m}, \bar{c} \rangle \triangleleft \langle M, c \rangle$ with $\pi_{\langle \bar{m}, \bar{c} \rangle, \langle M, c \rangle} = \pi_q^G \circ \sigma^* \upharpoonright \bar{m}$.

Now let $\pi^* \supset \pi_q^G \cup F q_{1,t}$, $\pi^* : N^* \triangleleft N^*$.

Set $\sigma_0 = \pi^* \sigma^*$. Then (a)-(c) are readily established. We show:

(d) $\sigma_0 \bar{G} \subset G$.

Let $\bar{r} \in \bar{G}$, $r = \sigma_0(\bar{r})$. Then $r_0 = \bar{r}_0$. But then

$r_0 \triangleleft \langle \bar{m}, \bar{c} \rangle$ and $\pi_{r_0, \langle \bar{m}, \bar{c} \rangle} = \pi_{\bar{r}_0}^{\bar{G}}$, since $\bar{r} \in \bar{G}$.

Hence $r_0 \triangleleft \langle M, c \rangle$ and $\pi_{r_0, \langle M, c \rangle} = \sigma_0 \circ \pi_{\bar{r}}^{\bar{G}}$.

Since $\pi_{\langle \bar{m}, \bar{c} \rangle, \langle M, c \rangle} = \sigma_0 \upharpoonright \bar{m}$ by the above.

It remains only to show:

Claim $\sigma_0 \pi_{\bar{r}_0}^{\bar{G}} : \langle M_r, \bar{a} \rangle \triangleleft \langle M, a \rangle$

whenever $\langle a, \bar{a} \rangle \in F^{\bar{r}}$.

Prf

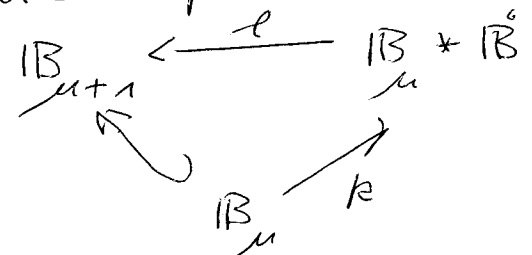
Let $\langle a, \bar{a} \rangle = \sigma_0(\langle a', \bar{a}' \rangle)$ where $\langle a', \bar{a}' \rangle \in F^{\bar{r}}$.

Then $\pi_{\bar{r}}^{\bar{G}} : \langle M_r, \bar{a} \rangle \triangleleft \langle \bar{m}, a' \rangle$. But

$\sigma_0 \upharpoonright \bar{m} : \langle \bar{m}, a' \rangle \triangleleft \langle M, a \rangle$ since $\sigma_0(\langle \bar{m}, a' \rangle) = \langle M, a \rangle$.

QED (Lemma 4)

Note that $\overline{BA(\mathbb{P}_B)} \leq 2^{\beta_{\mu+1}}$, since either $\beta = \beta_{\mu+1}$, cf $|\beta| = \omega_1$, and $BA(\mathbb{P}_B)$ has a dense subset of size β , by Sublemma 3.15.1, or else $\beta_{\mu+1} = \beta^+$, $\overline{\mathbb{P}_B} = 2^\beta = \beta^+$ (since then GCH holds below κ). Now let $\dot{\mathbb{I}}B = BA(\mathbb{P}_B)$, $\mathbb{I}B_\mu$ being the canonical generic name. We then form $\mathbb{I}B_\mu * \dot{\mathbb{I}}B$, which also has cardinality $\leq 2^{\beta_{\mu+1}}$, since $\mathbb{I}B_\mu$ has cardinality $\leq 2^{\beta_\mu} \leq \beta_{\mu+1}$, since $2^{\beta_\mu} = \beta_{\mu+1}$. Let $k: \mathbb{I}B_\mu \rightarrow \mathbb{I}B_\mu * \dot{\mathbb{I}}B$ be the natural injection. Choose $\mathbb{I}B_{\mu+1} \supseteq \mathbb{I}B_\mu$ s.t. there is an isomorphism l with:



We ensure that $\mathbb{I}B_{\mu+1} \subset H_{\beta_{\mu+1}}^+$. By Lemma 4 we know that $\mathbb{I}B_{\mu+1}$ is subcomplete. However, we have found it necessary to devise another representation of $\mathbb{I}B_{\mu+1}$ in order to elicit its deeper properties. Working in V define as before:

$$Q = H_\beta, M = L_\beta^A(\mathbb{I}B_\mu), N = \langle H_{\beta^+}, M, \langle \cdot, \cdot \rangle \rangle.$$

We then define a class Π' of triples as follows: