§2 Preliminaries

§2.1 The strategy

We are given a strongly inaccessible κ and wish to find a generic extension $V[G]$ in which $\kappa = \omega_2$ and $cf(\tau) = \omega$ for every $\tau \in (\omega_1, \kappa)$ which is regular in V. To make things simpler, assume for the moment that GCH holds in V. Our forcing will be an iteration $\langle \mathbb{B}_i \mid i < \kappa \rangle$ s.t. $\bar{\bar{\mathbb{B}}}_i < \kappa$ for $i < \kappa$. We start with $\mathbb{B}_0 = 2$. The next step $\mathbb{B}_1$ should collapse $\tau = \omega_2^V$ to ω_1 while making τ ω-cofinal. But $\mathbb{B}_1$ may not make any regular $\gamma < \kappa$ ω_1-cofinal, since that change would be irreversible. From this we conclude that we need $\bar{\bar{\mathbb{B}}}_1 \geq \omega_{\omega_1}$ and that $\mathbb{B}_1$ must make every regular $\gamma < \omega_{\omega_1}$ ω-cofinal. (Suppose not. Then there is a regular $\gamma < \omega_{\omega_1}$ which does not become ω-cofinal. Let γ be the least such. Then $\gamma = \delta^+$ in V and δ certainly becomes ω-cofinal. By [LF] §4 Lemma 4.1 it follows that γ is collapsed to ω_1. But then γ becomes ω_1-cofinal. Contradiction!)

Happily there is a set of conditions of size ω_{ω_1+1} which collapses ω_{ω_1} to ω_1, adds no reals, makes every regular $\gamma < \omega_{\omega_1}$ ω-cofinal, and leaves ω_{ω_1+1} regular. (Such a forcing is described in the appendix to [LF] §5. We shall, however, employ a variant of that forcing which has better symmetry properties.) In the resulting model ω_{ω_1+1} has become ω_2. We can then repeat the procedure. Proceeding in this way we get complete BA's $\mathbb{B}_0 \subseteq \mathbb{B}_1 \subseteq \cdots \subseteq \mathbb{B}_n \subseteq \cdots$ $(n<\omega)$ s.t. $\mathbb{B}_n$ collapses $\omega_{\omega_1 \cdot n}$ to ω_1. What is, then, the ω-th step? Since we are performing an iteration, $\mathbb{B}_\omega$ must be a completion of $\bigcup_n \mathbb{B}_n$. But, arguing as before, we see that $\mathbb{B}_\omega$ must make every regular $\gamma < \omega_{\omega_1(\omega+1)}$ ω-cofinal and, must, therefore, have cardinality at least $\omega_{\omega_1(\omega+1)}$. Since we are assuming GCH, both the direct limit and the inverse limit of $\langle \mathbb{B}_n \mid n<\omega \rangle$ are too small, so we must find a new limiting procedure. We then continue as before, using the new procedure at limits, until we reach $\omega_{\omega_1 \cdot \omega_1}$. At this point we take the direct limit, which collapses $\omega_{\omega_1 \cdot \omega_1}$ to ω_1 but leaves $\omega_{\omega_1 \cdot \omega_1 + 1}$ regular. Thus $\omega_{\omega_1 \cdot \omega_1 + 1}$ becomes ω_2 in the generic extension by $\mathbb{B}_{\omega_1}$. The next stage $\mathbb{B}_{\omega_1+1}$ must then collapse $\omega_{\omega_1(\omega_1+1)}$ to ω_1. We continue in this fashion, using direct limits at ω_1-cofinal points and the new limiting procedure at other limit points until we reach the first inaccessible limit point θ. $\mathbb{B}_\theta$ is then the direct limit of $\langle \mathbb{B}_i \mid i<\theta \rangle$ and we have

$\theta = \omega_2$ in the $\mathbb{B}_\theta$ - generic extension. If $\theta = \kappa$ we are done. If not, the stage $\mathbb{B}_{\theta+1}$ must make θ ω-cofinal while collapsing $\omega_{\theta+\omega_1}$ to ω_1. We continue in this fashion until we reach κ.

Recall that in §1 we called λ an ω-point of the iteration $\langle \mathbb{B}_i \mid i < \kappa \rangle$ if $\Vdash_{\mathbb{B}_i} cf(\check{\lambda}) = \omega$ for an $i < \lambda$. It is clear from the foregoing that we shall use the new limiting procedure at ω-points while taking direct limits elsewhere. In a variant of this procedure we can selectively make some regular $\gamma < \kappa$ ω_1-cofinal while making the others ω-cofinal. Assuming GCH in V, we can, in fact show that for an arbitrary $A \subset \kappa$ there is a generic extension in which

$\kappa = \omega_2$, no reals are added, and for $\gamma \in (\omega_1, \kappa)$ which is regular in V we have:

$$cf(\gamma) = \begin{cases} \omega_1 & \text{if } \gamma \in A \\ \omega & \text{if } \gamma \notin A \end{cases}.$$

In this case, if e.g. $\omega_2 \in A$, we can simply let $\mathbb{B}_1$ be the collapsing algebra $\mathrm{coll}(\omega_1, \omega_2)$ which collapses ω_2 to ω_1 with countable conditions. Similarly at all successor steps $\mathbb{B}_{\mu+1}$ where $\delta = \omega_2^{V^{\mathbb{B}_\mu}}$ lies in A.

Otherwise the successor stage is, as above, a modification of the forcing described in the appendix to [LF] §5. At an ω-point λ, we must find an appropriate completion $\mathbb{B}_\lambda$ of $\bigcup_{i<\lambda} \mathbb{B}_i$, depending on whether or not $\gamma + V \in A$, where the $\mathbb{B}_i$ $(i < \lambda)$ successive collapse all $\xi < \gamma$ to ω_1.

In the next section we describe a notion called proudness, which will play a large role in our construction.

§2.2 Proud forcing

In order to simplify our formulations we define:

Def Let $\mathbb{B}$ be a complete Boolean algebra. X is <u>dense in</u> $\mathbb{B}$ iff $X \setminus \{0\}$ is dense in $\mathbb{B} \setminus \{0\}$ — i.e. for every $b \in \mathbb{B} \setminus \{0\}$ there is $a \in X \setminus \{0\}$ s.t. $a \subset b$.

$$d(\mathbb{B}) = \min\{\bar{\bar{X}} \mid X \text{ is dense in } \mathbb{B}\}$$

(Hence $d(\mathbb{B})$ is the smallest size of a set of conditions $\mathbb{P}$ s.t. $\mathbb{B} \simeq BA(\mathbb{P})$.)

Def Let $\mathbb{B}$ be a complete BA and let $G \subset \mathbb{B}$ be $\mathbb{B}$-generic over V. G <u>conforms to</u> σ iff the following hold:

- σ lies in a generic extension of $V[G]$.
- $\sigma : \bar{W} \prec W = H_\theta^V$ for some θ and $\sigma(\bar{\mathbb{B}}) = \mathbb{B}$, where $\bar{W}$ is a countable transitive set in V.
- ~~Let~~ $\bar{G} = \sigma^{-1}{}''G$. Then $\bar{G} \in V$ and $\bar{G}$ is $\bar{\mathbb{B}}$-generic over $\bar{W}$.

<u>Note</u> This definition would become slightly simpler if we took σ as lying in a generic extension which added no reals to V.

We can now define:

Def Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is *proud* iff the following hold:

(a) $\mathbb{B}$ is subcomplete

(b) $d(\mathbb{B}) \leq \omega_2$ in $V[G]$ whenever G is $\mathbb{B}$-generic

(c) For sufficiently large θ:
Let $W = H_\theta$. Let G be $\mathbb{B}$-generic and σ-conforming, where $\sigma : \overline{W} \prec W$, $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$, and $\sigma \in V[G]$. Let $\overline{G}' \in V$ be $\overline{\mathbb{B}}$-generic over $\overline{W}$. Then there is G' s.t. G' is $\mathbb{B}$-generic, $V[G'] = V[G]$, and $\sigma''\overline{G}' \subset G'$.
(Hence G' is σ-conforming.)

Note Call $\mathbb{B}$ *weakly proud* iff (a), (b) hold and for sufficiently large θ there is a parameter $p \in H_\theta$ s.t. (c) holds whenever $p \in \mathrm{rng}(\sigma)$. It is easy to see that weak proudness implies proudness. We shall often tacitly use this fact.

Note If G is $\mathbb{B}$-generic, $\sigma : \overline{W} \prec W = H_\theta$, $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$ and $\overline{G} = \sigma^{-1}{}''G$ is $\overline{\mathbb{B}}$-generic over $\overline{W}$, then σ extends to a unique $\sigma^* : \overline{W}[\overline{G}] \prec W[G]$ s.t. $\sigma \subset \sigma^*$ and $\sigma^*(\overline{G}) = G$.

Def We say that μ verifies the proudness of $\mathbb{B}$ iff $\mathbb{B}$ is proud and some $\theta < \mu$ verifies the subcompleteness of $\mathbb{B}$ and (c) above holds for all cardinals $\theta \geq \mu$.

The condition $\sigma \in V[G]$ is, in fact, superfluous in the definition of proudness, as is shown by:

Lemma 1 Let θ verify the proudness of $\mathbb{B}$. Let G be $\mathbb{B}$-generic and σ-conforming, where $\sigma : \overline{W} \prec W = H_\theta$ and $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$. Let $\overline{G}' \in V$ be $\overline{\mathbb{B}}$-generic over $\overline{W}$. There is a $\mathbb{B}$-generic G' s.t. $V[G'] = V[G]$ and $\sigma''\overline{G}' \subset G'$.

proof.

Let $\sigma(\overline{D}) = D$ where D is dense in $\mathbb{B}$ and has cardinality $\delta = d(\mathbb{B})$ in $V[G]$

(1) $\sigma \restriction \overline{D} \in V[G]$

proof Let $\overline{G} = \sigma^{-1}{}''G$.

σ has a unique extension $\sigma^* : \overline{W}[\overline{G}] \prec W[G]$

s.t. $\sigma^*(\bar{G}) = G$. Let $\sigma^*(\bar{f}) = f$ where $f \in W[G]$ and $f : \omega_1 \xrightarrow{\text{onto}} D$. Then $\sigma \upharpoonright \bar{D} = f \circ \bar{f}^{-1} \in V[G]$. QED (1)

Since $\bar{W}$ is countable it follows from the absoluteness of well foundedness that:

(2) There is $\tilde{\sigma} \in V[G]$ s.t. $\tilde{\sigma} : \bar{W} \prec W$, $\tilde{\sigma}(\bar{\mathbb{B}}, \bar{D}) = \mathbb{B}, D$, and $\tilde{\sigma} \upharpoonright \bar{D} = \sigma \upharpoonright D$.

But then $\tilde{\sigma}``(\bar{D} \cap \bar{G}) = \sigma``(\bar{D} \cap \bar{G}) \subset D \cap G$.

Note that:

$$\bar{G} = \{ b \in \bar{\mathbb{B}} \mid \bigvee d \in \bar{D} \cap \bar{G} \;\; d \subset b \}$$

$$G = \{ b \in \mathbb{B} \mid \bigvee d \in D \cap G \;\; d \subset b \}.$$

Hence $\tilde{\sigma}``\bar{G} \subset G$ – i.e. G conforms to $\tilde{\sigma}$.

By proudness we conclude that there is G' with: G' is $\mathbb{B}$ – generic, $V[G'] = V[G]$ and $\tilde{\sigma}``\bar{G}' \subset G'$. But $\sigma``(\bar{D} \cap \bar{G}') = \tilde{\sigma}``(\bar{D} \cap \bar{G}') \subset D \cap G'$. Hence $\sigma``\bar{G}' \subset G$ as before. QED (Lemma 1)

For $\mathbb{A} \subseteq \mathbb{B}$ we now define the notion "$\mathbb{B}$ is proud over $\mathbb{A}$". This will be of particular importance in our later applications.

Def Let $\mathbb{A} \subseteq \mathbb{B}$ where $\mathbb{A}, \mathbb{B}$ are complete BA's.
$\mathbb{B}$ is <u>proud over</u> $\mathbb{A}$ iff
(a) $\mathbb{A}$ is proud
(b) $\mathbb{B}$ is subcomplete
(c) $\overline{\overline{d(\mathbb{B})}} \leq \omega_1$ in $V[G]$ if G is $\mathbb{B}$-generic
(d) For sufficiently large θ:
Let G be $\mathbb{B}$-generic and σ-conforming where $\sigma \in V[G]$, $\sigma : \overline{W} \prec W = H_\theta$, and $\sigma(\overline{\mathbb{A}}, \overline{\mathbb{B}}) = \mathbb{A}, \mathbb{B}$. (Hence $G_0 = G \cap \mathbb{A}$ is $\mathbb{A}$-generic and σ-conforming.) Let $\overline{G}_0'$ be $\overline{\mathbb{A}}$-generic over $\overline{W}$, where $\overline{G}_0' \in V$. Let G_0' be $\mathbb{A}$-generic s.t. $V[G_0'] = V[G_0]$ and $\sigma''\overline{G}_0' \subset G_0'$. Let $\overline{G}' \supset \overline{G}_0'$, $\overline{G}' \in V$ be $\overline{\mathbb{B}}$-generic over $\overline{W}$. Then there is $G' \supset G_0'$ s.t. G' is $\mathbb{B}$-generic, $V[G'] = V[G]$, and $\sigma''\overline{G}' \subset G'$.

<u>Note</u> The existence of a G_0' as posited in (d) follows from the proudness of $\mathbb{A}$.

<u>Note</u> Let $\mathbb{B} \neq 2$ be a complete BA. Then $\mathbb{B}$ is proud iff $\mathbb{B}$ is proud over 2.

<u>Note</u> If we define "$\mathbb{B}$ is weakly proud over $\mathbb{A}$" as before, we again get: weak proudness implies proudness. This will often be used tacitly.

Def We say that μ verifies the proudness of $\mathbb{B}$ over $\mathbb{A}$ iff $\mathbb{B}$ is proud over $\mathbb{A}$, some μ verifies the proudness of $\mathbb{A}$, some $\theta < \mu$ verifies the subcompleteness of $\mathbb{B}$, and (c) holds for all $\theta \geq \mu$.

The condition $\sigma \in V[G]$ is again superfluous in the definition of "$\mathbb{B}$ is proud over $\mathbb{A}$", as shown by:

Lemma 2 Let θ verify the proudness of $\mathbb{B}$ over $\mathbb{A}$. Let G be $\mathbb{B}$-generic and σ-conforming, where $\sigma : \bar{W} \prec W = H_\theta$ and $\sigma(\bar{\mathbb{A}}, \bar{\mathbb{B}}) = \mathbb{A}, \mathbb{B}$. Let $\bar{G}_0' \in V$ be $\bar{\mathbb{A}}$-generic over $\bar{W}$ and let G_0' be $\mathbb{A}$-generic s.t. $V[G_0'] = V[G_0]$ and $\sigma''\bar{G}_0' \subset G_0'$. Let $\bar{G}' \supset \bar{G}_0'$ be $\bar{\mathbb{B}}$-generic over $\bar{W}$ s.t. $\bar{G}' \in V$. There is $G' \supset G_0'$ s.t. G' is $\mathbb{B}$-generic, $V[G'] = V[G]$, and $\sigma''\bar{G}' \subset G'$.

The proof is a straightforward modification of the proof of Lemma 1. We leave it to the reader.

Using Lemma 2 we easily get:

<u>Lemma 3</u> Let $\mathbb{C}$ be proud over $\mathbb{B}$ and $\mathbb{B}$ proud over $\mathbb{A}$. Then $\mathbb{C}$ is proud over $\mathbb{A}$.

The proof is straightforward. We again leave it to the reader.

<u>Note</u> Taking $\mathbb{A} = 2$ we get: Let $\mathbb{C}$ be proud over some $\mathbb{B}$. Then $\mathbb{C}$ is proud.

The notion of proudness can be weakened as follows:

<u>Def</u> Weaken the def. of "proud over" by replacing "$d(\mathbb{B}) \leq \omega_1$" with "$d(\mathbb{B}) \leq \omega_2$" in (c). If $\mathbb{A} \subseteq \mathbb{B}$ and $\mathbb{A}, \mathbb{B}$ satisfy the resulting condition, we say that $\mathbb{B}$ is <u>semiproud</u> over $\mathbb{A}$.

$\mathbb{B}$ is <u>semiproud</u> iff it is semiproud over 2.

We then get, as before:

<u>Lemma 3.1</u> Let $\mathbb{C}$ be semiproud over $\mathbb{B}$ and $\mathbb{B}$ proud over $\mathbb{A}$. Then $\mathbb{C}$ is semiproud over $\mathbb{A}$.

However, Lemmas 1 and 2 do not hold if we replace "proud" by "semiproud", but we do obtain the following adaptation of Lemma 2:

<u>Lemma 2.1</u> Let θ verify the semiproperness of $\mathbb{B}$ over $\mathbb{A}$. Let G be $\mathbb{B}$-generic and σ-conforming, where $\sigma : \overline{W} \prec W = H_\theta$, and $\overline{W}$ is countable and transitive, and $\sigma(\overline{\mathbb{A}}, \overline{\mathbb{B}}) = \mathbb{A}, \mathbb{B}$. Assume that σ lies in a generic extension of $V[G]$ which adds no reals and in which $cf(\omega_2^{V[G]}) > \omega$.
Let $\overline{G}_0'$ be $\overline{\mathbb{A}}$-generic over $\overline{W}$ and G_0' be $\mathbb{A}$-generic s.t. $V[G_0'] = V[G_0]$, where $G_0 = G \cap \mathbb{A}$, and let $\sigma''\overline{G}_0' \subset G_0$. Let $\overline{G}' \supset \overline{G}_0$ be $\overline{\mathbb{B}}$-generic over $\overline{W}$. Then there is $G' \supset G_0'$ s.t. G' is $\mathbb{B}$-generic, $V[G'] = V[G]$ and $\sigma''\overline{G}' \subset G'$.

proof
Let D be dense in $\mathbb{B}$ s.t. $\overline{\overline{D}} \leq \omega_2$ in $V[G]$.
We show: <u>Claim</u> $\sigma \upharpoonright D \in V[G]$, the rest of the proof being as before.
Let σ^* extend σ s.t. $\sigma^* : \overline{W}[\overline{G}] \prec W[G]$ and $\sigma^*(\overline{G}) = G$. Let $\sigma^*(\overline{f}) = f$ where $f : \omega_2 \xrightarrow{\text{onto}} D$ in $V[G]$. Since $\omega_2^{V[G]}$ has cofinality $> \omega$ and $\overline{W}$ is countable, we know that $\sup \sigma''\overline{\delta} < \delta$ where $\overline{\delta} = \omega_2^{\overline{W}[\overline{G}]}$, $\delta = \omega_2^{W[G]}$.
Let $\sigma''\overline{\delta} \subset \gamma < \delta$ with $g : \omega_1 \xrightarrow{\text{onto}} \gamma$ in $V[G]$.
Then there is $\alpha < \omega_1$ s.t. $\sigma''\overline{\delta} \subset g''\alpha$.
Thus $g^{-1} \circ (\sigma \upharpoonright \overline{\delta}) \in H_{\omega_1}$, since no reals were added. But:

$$\sigma \upharpoonright \overline{D} = f \circ (\sigma \upharpoonright \delta) \circ \overline{f}^{-1} = f \circ g \circ (\overline{g}^{-1} \circ \sigma \upharpoonright \overline{\delta}) \circ \overline{f}^{-1}.$$

Hence $\sigma \upharpoonright \overline{D} \in V[G]$.

QED (2.1)

We can also strengthen the notion of proudness as follows:

Def Strengthen the def. of "(semi) proud over" by replacing "$V[G'] = V[G]$" in (d) with:

there is $\pi : \mathbb{B} \overset{\sim}{\leftrightarrow} \mathbb{B}$ s.t. $\pi \in V$ and

$G' = \pi''G$.

If $\mathbb{A} \subseteq \mathbb{B}$ and $\mathbb{A}, \mathbb{B}$ satisfy this strengthened condition, we say that $\mathbb{B}$ is symmetrically (semi) proud over $\mathbb{A}$.

$\mathbb{B}$ is symmetrically (semi) proud iff $\mathbb{B}$ is symmetrically proud over 2.

We again get:

Lemma 3.2 Let $\mathbb{C}$ be symmetrically (semi) proud over $\mathbb{B}$ and $\mathbb{B}$ proud over $\mathbb{A}$. Then $\mathbb{C}$ is symmetrically (semi) proud over $\mathbb{A}$.

The appropriate form of Lemma 2 holds for "symmetrically proud" — i.e. we can omit the condition "$\sigma \in V[G]$" from the definition. The precise formulation is:

<u>Lemma 2.2</u> Let θ verify the symmetrical provedness of $\mathbb{B}$ over $\mathbb{A}$. Let G be $\mathbb{B}$-generic and σ-conforming, where $\sigma : \bar{W} \prec W = H_\theta$ and $\sigma(\bar{\mathbb{A}}, \bar{\mathbb{B}}) = \mathbb{A}, \mathbb{B}$. Let $\bar{G}_0'$ be $\bar{\mathbb{A}}$-generic over $\bar{W}$ and let G_0' $\mathbb{A}$-generic s.t. $V[G_0'] = [G_0]$, where $G_0 = G \cap \mathbb{A}$, and $\sigma''\bar{G}_0' \subset G_0'$. Let $\bar{G}' \supset \bar{G}_0'$ be $\bar{\mathbb{B}}$-generic over $\bar{W}$ s.t. $\bar{G}' \in V$. There is $G' \supset G_0'$ s.t. G' is $\mathbb{B}$-generic, $\sigma''\bar{G}' \subset G'$, and there is $\pi : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{B}$ in V s.t. $\pi''G = G'$.

The proof is again left to the reader.

A further notion which will play a role in our construction is:

<u>Def</u> Let $\mathbb{A} \subseteq \mathbb{B}$. $\mathbb{B}$ is <u>symmetrical over</u> $\mathbb{A}$ iff every $\sigma : \mathbb{A} \overset{\sim}{\longleftrightarrow} \mathbb{A}$ extends to a $\sigma' \supset \sigma$ s.t. $\sigma' : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{B}$.

(Thus every $\mathbb{B}$ is symmetrical over 2.)

<u>Def</u> Let $\mathbb{A} = \langle \mathbb{A}_i \mid i < \lambda \rangle$ s.t. $\mathbb{A}_i \subseteq \mathbb{A}_j \subseteq \mathbb{B}$ for $i \leq j < \lambda$. $\mathbb{B}$ is <u>symmetrical over</u> $\mathbb{A}$ iff whenever $\sigma : \bigcup_i \mathbb{A}_i \overset{\sim}{\longleftrightarrow} \bigcup_i \mathbb{A}_i$ s.t. $\sigma \upharpoonright \mathbb{A}_i : \mathbb{A}_i \overset{\sim}{\longleftrightarrow} \mathbb{A}_i$ for sufficiently large i, then σ extends to $\sigma' \supset \sigma$ s.t. $\sigma' : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{B}$.

We now apply these notions to the conditions $\mathbb{Q} = \mathrm{coll}(\omega_1, \omega_2)$ – i.e. the countable conditions for collapsing ω_2 to ω_1. It is a straigtforward exercise to show that $\mathbb{Q}$ – or more precisely the complete Boolean algebra $BA(\mathbb{Q})$ – is proud. We now consider:

$$\mathbb{C} = \mathbb{B} * \mathring{\mathbb{C}} \quad , \text{where } \Vdash_{\mathbb{B}} \mathring{\mathbb{C}} = BA(\mathrm{coll}(\omega_1, \omega_2)).$$

Assume, moreover, that $\Vdash_{\mathbb{B}} \check{\delta} = \omega_2$ for a fixed δ. There is a canonical complete embedding $k : \mathbb{B} \xrightarrow{\sim} \mathbb{C}$ defined by: $k(b) =$ that $d \in V^{\mathbb{B}}$ s.t.

$$\Vdash_{\mathbb{B}} (\check{b} \in \mathring{B} \wedge d = 1) \vee (\check{b} \notin \mathring{B} \wedge d = 0)$$

where $\dot{B}$ is the canonical $\mathbb{B}$-generic name. (Recall that $\mathbb{C}$ is the set of $c \in V^{\mathbb{B}}$ s.t. $\Vdash_{\mathbb{B}} c \in \mathring{\mathbb{C}}$. The partial order $\subset_{\mathbb{C}}$ is defined by:

$$c \subset c' \longleftrightarrow \Vdash_{\mathbb{B}} c \subset c'.)$$

For the sake of uniqueness we take $V^{\mathbb{B}}$ as an identity model – i.e. $(\Vdash s = t) \rightarrow s = t$ for $s, t \in V^{\mathbb{B}}$.)

<u>Lemma 4.1</u> Let $\mathbb{B}, \mathbb{C}$ be as above, where $\mathbb{B}$ is semiproud over $\mathbb{A}$ or $\mathbb{B} = \mathbb{A}$ is proud. Then $\mathbb{C}$ is proud over $k''\mathbb{A}$

(Hence $\tilde{\mathbb{C}}$ is proud over $\mathbb{A}$ whenever $l : \mathbb{C} \xrightarrow{\sim} \tilde{\mathbb{C}}$ s.t. $lk = id$.)

proof.

Let C be $\mathbb{C}$-generic and σ-conforming, where $\sigma \in V[C]$, $\sigma : \overline{W} \prec W = H_\theta$, $\theta > 2^\delta$ is big enough to verify the semiproudness of $\mathbb{B}$ over $\mathbb{A}$, and $\sigma(\overline{\mathbb{A}}, \overline{\mathbb{C}}) = \mathbb{A}, \mathbb{C}$. Let $\overline{A}'$ be $\overline{\mathbb{A}}$-generic over $\overline{W}$ and let A' be $\mathbb{A}$-generic s.t. $V[A'] = V[A]$ (where $A = \{a \in \mathbb{A} \mid k(a) \in C\}$), and $\sigma''\overline{A}' \subset A'$. Let $\overline{C}' \supset \overline{k}''\overline{A}'$ be $\overline{\mathbb{C}}$ generic over $\overline{W}$ (where $\sigma(\overline{k}) = k$).

<u>Claim</u> There is $C' \supset k''A'$ s.t. C' is $\mathbb{C}$-generic, $V[C'] = V[C]$, and $\sigma''\overline{C}' \subset C'$.

proof.

We first note that $\mathbb{B} \in rng(\sigma)$. Let $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$. Note that $\mathbb{B}$ is semiproud over $\mathbb{A}$ as verified by θ. Moreover $\sigma \in V[C] = V[B][G]$, where

$B = k^{-1}{}''C$ is $\mathbb{B}$-generic and G is $\mathbb{Q}^B =_{Df}$ $=_{Df} \mathrm{coll}(\omega_1, \omega_2)^{V[B]}$-generic over $V[B]$. Let $\bar{B}' = \bar{k}^{-1}{}''\bar{C}'$. Since B is σ-conforming and $\mathbb{Q}^B$ adds no new countable subsets of $V[B]$, we conclude by Lemma 2.2 that there is $B' \supset A'$ s.t. B' is $\mathbb{B}$-generic, $V[B'] = V[B]$, and $\sigma''\bar{B}' \subset B'$. Hence we need only show that there is $C' \supset k''B'$ s.t. C' is $\mathbb{C}$-generic, $V[C'] = V[C]$, and $\sigma''\bar{C}' \subset C'$. Let $\sigma^* \supset \sigma$ be the unique extension s.t. $\sigma^* : \bar{W}[\bar{B}'] \prec W[B']$ and $\sigma(\bar{B}') = B'$. Clearly $W[\bar{C}'] = W[\bar{B}'][\bar{G}']$ where $\bar{G}'$ is $\mathbb{Q}^{\bar{B}'}$-generic over $\bar{W}[\bar{B}']$. It suffices, then, to find a G' which is $\mathbb{Q}^{B'}$-generic s.t. $V[B'][G'] = V[B][G]$ and $\sigma^*{}''\bar{G}' \subset G'$. Let $\alpha = \omega_1^{\bar{W}}$ and $\bar{\delta} = \omega_2^{\bar{W}[\bar{B}']} = \sigma^{-1}(\delta)$. Let $\bar{f}' = \bigcup \bar{G}'$. Then $\bar{f}' : \alpha \xrightarrow{\text{onto}} \bar{\delta}$. Let $f' = \sigma \circ \bar{f}'$. Then $f' : \alpha \to \delta = \omega_2^{V[B]}$ and $f' \in V[B][G]$. Hence $f' \in V[B]$, since G adds no countable sets. But then $f' \in \mathbb{Q}^{B'}$. (Note that

$\mathbb{Q}^{B'} = \mathrm{coll}(\omega_1\omega_2)^{V[B']} = \mathbb{Q}^B$, since $V[B'] = V[B]$.) Let $F = \bigcup G$ and set $F' = F \upharpoonright (\omega_1 \setminus \alpha) \cup f'$. Set $G' = \{p \in \mathbb{Q}^B \mid p \subset F'\}$. Then G' is easily seen to be $\mathbb{Q}^{B'}$-generic over $V[B']$. But if $p \in \bar{G}'$, then $\sigma(p) \subset f' \subset F'$; hence $\sigma(p) \in G'$. Thus $\sigma''\bar{G}' \subset G$. It is obvious that $V[G'] = V[F'] = V[F] = V[G]$.

QED (Lemma 4.1)

It is fairly easy to see:

<u>Lemma 4.2</u> $\mathbb{B} * \dot{\mathbb{C}}$ is symmetrical over $k''\mathbb{B}$ where $\Vdash_{\mathbb{B}} \dot{\mathbb{C}} = BA(\mathrm{coll}(\omega_1, \omega_2))$.

proof.
Let $\mu : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{B}$. We must find a $\tilde{\mu} : \mathbb{C} \overset{\sim}{\longleftrightarrow} \mathbb{C}$ s.t. $\tilde{\mu} k = k\mu$, where $\mathbb{C} = \mathbb{B} * \dot{\mathbb{C}}$ and $k : \mathbb{B} \to \mathbb{C}$ is the natural injection. μ induces a $\hat{\mu} : V^{\mathbb{B}} \longleftrightarrow V^{\mathbb{B}}$ s.t.

$$\mu([\![\varphi(\vec{t})]\!]) = [\![\varphi(\hat{\mu}(\vec{t}))]\!]$$

for $t_1, \ldots, t_n \in V^{\mathbb{B}}$. (We again take $V^{\mathbb{B}}$ as an identity model.)

Set $\tilde{\mu} = \hat{\mu} \upharpoonright \mathbb{C}$. (Recall $\mathbb{C} = \{c \in V^{\mathbb{B}} \mid \Vdash c \in \mathring{\mathbb{C}}\}$)
Then $\tilde{\mu} : \mathbb{C} \overset{\sim}{\longleftrightarrow} \mathbb{C}$. But $k(b)$ is defined
by $\Vdash_{\mathbb{B}} (\check{b} \in \mathring{B} \wedge k(b) = 1) \vee (\check{b} \notin \mathring{B} \wedge k(b) = 0)$,
where $\mathring{B}$ is the canonical $\mathbb{B}$-generic name.
Note that $[\![\check{b} \in \hat{\mu}(\mathring{B})]\!] = \mu [\![\check{b} \in \mathring{B}]\!] =$
$= \mu(b) = [\![\check{\mu(b)} \in \mathring{B}]\!]$. Hence:
$\Vdash_{\mathbb{B}} (\check{\mu(b)} \in \mathring{B} \wedge \hat{\mu}(k(b)) = 1) \vee (\check{\mu(b)} \notin \mathring{B} \wedge \hat{\mu}(k(b)) = 0)$.
Hence $\hat{\mu}(k(b)) = k(\mu(b))$. QED (4.2)

We can improve Lemma 4.1 to:

<u>Lemma 4.3</u> Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be as in 4.1 where either $\mathbb{B}$ is symmetrically semiproper over $\mathbb{A}$ or $\mathbb{B} = \mathbb{A}$ is symmetrically proper. Then $\mathbb{C}$ is symmetrically proper over $k''\mathbb{A}$.

proof.
Let $\bar{B}', B', \bar{C}', C', \bar{B}, B, \bar{C}, C$ be as in the proof of Lemma 4.1. We show that
There is $\pi \in V$ s.t. $\pi : \mathbb{C} \overset{\sim}{\longleftrightarrow} \mathbb{C}$ and $\pi''C = C'$.
We are given a $\mu : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{B}$ s.t.
$\mu''B = B'$. Let $\mathbb{Q} = \text{coll}(\omega_1, \omega_2)^{V[B]}$
(Recall that $V[B] = V[B']$.) Then
There are G, G' which are $\mathbb{Q}$-generic s.t., letting $\tilde{\mathbb{C}} = BA(\mathbb{Q})$, we have:
$\tilde{C} = \{c \in \tilde{\mathbb{C}} \mid c \cap G \neq \emptyset\}$, $\tilde{C}' = \{c \in \tilde{\mathbb{C}} \mid c \cap G' \neq \emptyset\}$
are both $\tilde{\mathbb{C}}$-generic, and:

$C = B * \tilde{C} = \{c \in \mathbb{C} \mid c^B \in \tilde{C}\}$,

$C' = B' * \tilde{C}' = \{c \in \mathbb{C} \mid c^{B'} \in \tilde{C}'\}$,

(where $\mathbb{C} = \mathbb{B} * \dot{\mathbb{C}}$ = the set of $c \in V^{\mathbb{B}}$ s.t.

$\Vdash c \in BA(\mathrm{coll}(\omega_1, \omega_2)))$.

We return to the construction in 4.1 to show that there is a $\tilde{\pi} \in V[B]$ with $\tilde{\pi} : \tilde{\mathbb{C}} \overset{\sim}{\longleftrightarrow} \tilde{\mathbb{C}}$ and $\tilde{\pi} `` \tilde{C} = \tilde{C}'$. Recall that, letting $F = \bigcup G$, $F' = \bigcup G'$, we have:

$f = F \upharpoonright \alpha$, $f' = F' \upharpoonright \alpha$, $F = F' \upharpoonright (\omega_1 \setminus \alpha) \cup f$,

$F' = F \upharpoonright (\omega_1 \setminus \alpha) \cup f'$, ~~For~~

Set $\Delta = \{p \in \mathbb{Q} \mid \mathrm{dom}(p) \supset \alpha\}$. For $p \in \Delta$ define

$$\pi^*(p) = \begin{cases} p \upharpoonright (\omega_1 \setminus \alpha) \cup f' & \text{if } p \upharpoonright \alpha = f \\ p \upharpoonright (\omega_1 \setminus \alpha) \cup f & \text{if } p \upharpoonright \alpha = f' \\ p & \text{otherwise.} \end{cases}$$

Then Δ is dense in $\mathbb{Q}$ and $\pi^* : \Delta \longleftrightarrow \Delta$ is an isomorphism w.r.t. $\leq_{\mathbb{Q}}$. Hence π^* induces $\tilde{\pi} : \tilde{\mathbb{C}} \overset{\sim}{\longleftrightarrow} \tilde{\mathbb{C}}$ s.t. $\tilde{\pi}([p]) = [\pi^*(p)]$ for $p \in \Delta$. It is then straightforward to show that $\tilde{\pi}(c) \in C'$ for $c \in C$.

But $\tilde{\pi} = \dot{\pi}^B$ for a $\dot{\pi}$ s.t. $\Vdash_{\mathbb{B}} \dot{\pi} : \dot{\mathbb{C}} \overset{\sim}{\longleftrightarrow} \dot{\mathbb{C}}$.

Set $\hat{\pi}(c)$ = that d s.t. $\Vdash_{\mathbb{B}} d = \dot{\pi}(c)$ for $c \in \mathbb{C}$.

Then $\hat{\pi} : \mathbb{C} \overset{\sim}{\longleftrightarrow} \mathbb{C}$. But there is $\mu : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{B}$ s.t. $\mu'' B = B'$. Since $\mathbb{C}$ is symmetrical over $k''\mathbb{B}$, this gives rise to $\tilde{\mu} : \mathbb{C} \overset{\sim}{\longleftrightarrow} \mathbb{C}$ s.t. $\tilde{\mu} k = k\mu$, where $k : \mathbb{B} \to \mathbb{C}$ is the natural injection. We recall that, by the proof of Lemma 4.2, $\tilde{\mu} = \hat{\mu} \upharpoonright \mathbb{C}$, where $\hat{\mu} : V^{\mathbb{B}} \leftrightarrow V^{\mathbb{B}}$ s.t.

$$\mu([\![\varphi(\vec{t})]\!]) = [\![\varphi(\hat{\mu}(\vec{t})]\!]$$

for $t_1, \ldots, t_n \in V^{\mathbb{B}}$. But then

$$\hat{\mu}(t)^{B'} = t^B \, , \text{ since}$$

$$s^B \in t^B \leftrightarrow [\![s \in t]\!] \in B \leftrightarrow$$

$$\leftrightarrow \mu([\![s \in t]\!]) = [\![\hat{\mu}(t) \in \hat{\mu}(t)]\!] \in B'$$

$$\leftrightarrow \hat{\mu}(s)^{B'} \in \hat{\mu}(t)^{B'} .$$

Hence $\tilde{\mu}(c)^{B'} = c^B$ for $c \in \mathbb{C}$. Set:

$\pi = \tilde{\mu} \circ \hat{\pi}$. Then

$$c \in C \leftrightarrow c^B \in \tilde{C} \leftrightarrow \tilde{\pi}(c^B) = \hat{\pi}(c)^B \in \tilde{C}'$$

$$\leftrightarrow (\tilde{\mu}\hat{\pi}(c))^{B'} \in \tilde{C}' \leftrightarrow \pi(c) \in C' .$$

QED (4.3)

§2.3 More on the iteration

We now describe our intended iteration $\langle \mathbb{B}_i \mid i \le \kappa \rangle$ more precisely. We assume κ to be strongly inaccessible and are given an $A_0 \subset \kappa$ which can be empty. If A_0 is not empty, however, we also assume GCH below κ. For $i \le \kappa$ we set:

$\gamma_i =$ the smallest cardinal $\gamma \ge \omega_1$ not collapsed to ω_1 in previous stages (i.e. for all $h < i$ gilt: $\gamma > \omega_1 \wedge \gamma$ is a cardinal in $V^{\mathbb{B}_h}$)

$\beta_i \simeq$ the largest $\beta \ge \gamma_i$ collapsed to ω_1 at the i-th stage (i.e. $\bar{\bar{\beta}} = \omega_1$ in $V^{\mathbb{B}_i}$)

There will be one case in which β_i is undefined.

Recalling the discussion in §2.1 we can define γ_i, β_i in V as follows:

$$\gamma_0 = \beta_0 = \omega_1$$

$$\gamma_{i+1} = \begin{cases} \beta_i^+ & \text{if } \beta_i \text{ exists} \\ \gamma_i & \text{if not} \end{cases}$$

We then define β_{i+1} by:

β_{i+1} = the least cardinal $\beta \geq \gamma_{i+1}$ such that one of the following holds:

- $2^\beta = \beta$ and $cf(\beta) = \omega_1$
- $\beta \in A_0$ is regular.

(Note In the second case we assume GCH below κ, so $2^\beta = \beta$ in either case. It is clear that β is a successor cardinal in the second case.)

For limit λ we set:

$\gamma_\lambda = \sup_{i<\lambda} \gamma_i$

We call λ an ω-point iff $cf(\lambda) = \omega$ or $\omega_1 < cf(\lambda) < \gamma_\lambda$ and $cf(\lambda) \notin A_0$.

If λ is an ω-point we set:

β_λ = the least cardinal $\beta > \gamma_\lambda$ s.t one of the following holds:

- $2^\beta = \beta$ and $cf(\beta) = \omega_1$
- $\beta \in A_0$ is regular.

If λ is not an ω-point, then $\mathbb{B}_\lambda$ will be the direct limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$.

Call λ an ω_1-point iff $cf(\lambda) = \omega_1$ or $\omega_1 < cf(\lambda) < \delta'_\lambda$ s.t. $cf(\lambda) \in A_0$.
If λ is an ω_1-point, set: $\beta_\lambda = \delta'_\lambda$ (since $\mathbb{B}_\lambda$ is the direct limit; hence δ'_λ is collapsed to ω_1, but δ'^+_λ remains a cardinal in $\mathbb{B}_\lambda$).
If λ is neither an ω-point nor an ω_1-point, then $\lambda = \delta'_\lambda$ is inaccessible and β_λ is undefined (since δ'_λ will become ω_2 in $V^{\mathbb{B}_\lambda}$).*)

<u>Note</u> We can verify inductively that $2^{\beta_i} = \beta_i$ whenever β_i exists, Hence Hence $2^{\delta_\lambda} = \delta_\lambda$ for limit λ. Hence λ is strongly inaccessible if it is neither an ω_1-point nor an ω-point.
<u>Note</u> We will always have $\delta_{i+1} = \omega_2^{V^{\mathbb{B}_i}}$.

<u>Def</u> Ac = the set of ξ which are not inaccessible. Hence we shall have: $i \in Ac \leftrightarrow \beta_i$ exists for $i < \kappa$.

*) In this case $\delta_{\lambda+1} = \delta_\lambda$. If $\delta_\lambda \in A_0$, we also have $\beta_{\lambda+1} = \delta_\lambda$. In all other cases $\delta_i \leq \beta_i < \delta_{i+1}$.

We shall verify by induction on i, that γ_i, β_i have the right meaning - i.e

(a) If β_i exists, then in $V^{\mathbb{B}_i}$ we have:

- $\beta_i^+ = \omega_2$ and $cf(\beta_i) = \omega_1$
- $cf(\tau) = \omega$ whenever $\tau \in [\gamma_i, \beta_i)$ is regular in V

If β_i does not exist, then $\gamma_i = \omega_2$ in $V^{\mathbb{B}_i}$.

We shall also verify inductively that $\langle \mathbb{B}_h \mid h \leq i \rangle$ is nicely subcomplete in the sense of § 1. Furthermore we shall verify that $\langle \mathbb{B}_h \mid h \leq i \rangle$ has certain prondness and symmetry properties. The full list of properties:

(b) $\mathbb{B}_i$ is subcomplete

(c) If λ is a limit ordinal which is not an ω-point (in the above sense), then $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$!

(d) ~~Let~~ $\langle \xi_i \mid i < \omega \rangle$ be monotone and cofinal in λ (hence λ is an ω-point). Let $\langle b_i \mid i < \omega \rangle$ be a "thread" in $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$ (i.e. $b_0 \neq 0$ and $b_i = h_{i,i+1}(b_{i+1})$ for $i < \omega$). Then $\bigcap_i b_i \neq 0$ in $\mathbb{B}_\lambda$.

(e) If $i > j$ and $i, j \in Ac$, then $\mathbb{B}_i$ is symmetrically proud over $\mathbb{B}_j$. Moreover, if $j \in Ac$, $i \notin Ac$, then $\mathbb{B}_i$ is symmetrically semiproud over $\mathbb{B}_j$;

(f) $\mathbb{B}_i$ is symmetrical over $\langle \mathbb{B}_j \mid j < i \rangle$.

(g) (a) – (f) hold of $\langle \mathbb{B}_{h+l}/B \mid l \leq i \rangle$ whenever B is $\mathbb{B}_h$ - generic and $h < i$.

~~In~~ The following chapters we shall construct $\langle \mathbb{B}_h \mid h \leq i \rangle$ by induction on i, verifying at each stage that $\langle \mathbb{B}_h \mid h \leq i \rangle$ satisfies (a) – (g). We shall also ensure that:

(h) $\mathbb{B}_i \subset H_{\beta_i^+}$ if β_i exists, and
$\mathbb{B}_i \subset H_{\gamma_i}$ if not.

<u>Note</u> By (a), (h) it is clear that if a cofinality changes at stage i, then it becomes either ω or ω_1. By (b) no reals are added, so it will retain this value for the rest of the iteration. It is clear that the iteration is ω-definite in the sense of §1. Hence the iteration is nicely subcomplete.

Note As stated before, we have: $2^{\underline{\gamma_i}} = \gamma_i$ and $2^{\underline{\beta_i}} = \beta_i$ if β_i exists. Hence $i = \gamma_i$ is strongly inaccessible if β_i does not exist. Since $\bigcup_{h<i} \mathbb{B}_h$ is dense in $\mathbb{B}_i$ and $\overline{\overline{\mathbb{B}_h}} < i$ for $h < i$, it is easily seen that $\mathbb{B}_i$ satisfies the i-CC. Hence $\mathbb{B}_i = \bigcup_{h<i} \mathbb{B}_h$.

Note $\mathbb{B}_0 = 2$, so (a)–(h) are trivial for $\langle \mathbb{B}_i | i \leq 0 \rangle$. Hence we need only consider the successor case and the limit case in the ensuing chapters.