§1 An iteration theorem

We adopt the notation and terminology of [SPSC]§1. Just as there we understand an <u>iteration</u> to be a sequence $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ of complete Boolean algebras s.t.

- $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$; $\mathbb{B}_0 = \{0, 1\}$
- $\mathbb{B}_\lambda$ is generated by $\bigcup_{i<\lambda} \mathbb{B}_\lambda$ for limit $\lambda < \alpha$

It is customary to take the limit algebra $\mathbb{B}_\lambda$ as either the direct limit or inverse limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$. For our applications, however, we must consider other completions - indeed $\mathbb{B}_\lambda$ will sometimes have a larger cardinality than even the inverse limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$.

Def An iteration $\mathbb{B}$ is ω-<u>definite</u> iff the question, whether a given ordinal λ acquires cofinality ω at stage i is decided in V - i.e.

$$V b \; b \Vdash_i \mathrm{cf}(\check{\lambda}) = \omega \quad \rightarrow \quad \Vdash_i \mathrm{cf}(\check{\lambda}) = \omega .$$

By an <u>ω-point</u> of $\mathbb{B}$ we mean a λ s.t. $V i < \lambda \; \Vdash_i \mathrm{cf}(\check{\lambda}) = \omega$.

<u>Def</u> We call an iteration $\langle \mathbb{B}_i \mid i < \alpha \rangle$ <u>nicely subcomplete</u> iff it is ω-definite and:

(a) Each $\mathbb{B}_i$ is subcomplete

(b) Let $\lambda < \alpha$ and let $\langle \xi_i \mid i < \omega \rangle$ be monotone and cofinal in λ. Let $\langle b_i \mid i < \omega \rangle$ be a "thread" in $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$ – i.e. $b_0 \neq 0$ and $h(b_{ij}) = b_i$ for $i \leq j < \omega$. Then $\bigcap_i b_i \neq 0$ in $\mathbb{B}_\lambda$.

(c) If $\lambda < \alpha$ is a limit ordinal which is not an ω-point, then $\mathbb{B}_\lambda$ is the minimal completion of $\bigcup_{i<\lambda} \mathbb{B}_i$.

(d) If G is $\mathbb{B}_{\xi+1}$ - generic, then $\bar{\bar{\xi}}, \bar{\bar{\mathbb{B}}}_\xi \leq \omega_1$ in $V[G]$.

(e) If G_i is $\mathbb{B}_i$ - generic, then (a)–(d) hold of $\langle \mathbb{B}_{i+j}/G_i \mid j < \alpha - i \rangle$

<u>Note</u> (b) does <u>not</u> require that the set of such $b = \bigcap_i b_i$ be dense in $\mathbb{B}_\lambda \setminus \{0\}$. If we required that, then $\mathbb{B}_\lambda$ would simply be the inverse limit of $\langle \mathbb{B}_i \mid i < \lambda \rangle$.

__Theorem__ Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be nicely subcomplete. Let α be a limit ordinal which is not an ω-point. Then the direct limit $\mathbb{B}_\alpha$ of $\mathbb{B}$ is subcomplete.

__Note__ This theorem holds with "subproper" (and presumably even "semi subproper") in place of "subcomplete".

The proof is essentially a modification of the proof of the main iteration Theorem (Theorem 5) in [SPSC] §2. However, the argument in Case 3.2 of that proof did not adapt well to this setting, so we devised an alternative argument which, we believe, could have been used to good effect in the original proof.

Let Θ be big enough to verify the subcompleteness of $\mathbb{B}_i$ for $i < \alpha$. Let $N = L_\tau^A$ where $\tau > \Theta$ is regular and $H_\Theta \subset N$. Let $\sigma : \bar{N} \prec N$, where $\bar{N}$ is countable and full. Suppose that $\sigma(\bar{\mathbb{B}}) = \mathbb{B}$ where $\bar{\mathbb{B}} = \langle \bar{\mathbb{B}}_i \mid i < \bar{\alpha} \rangle$ and $\sigma(\bar{\mathbb{B}}_{\bar{\alpha}}) = \mathbb{B}_\alpha$, where $\bar{\mathbb{B}}_{\bar{\alpha}}$ is the direct limit of $\bar{\mathbb{B}}$. Let $\tilde{\alpha} = \sup \sigma''\bar{\alpha}$. Then $\tilde{\alpha} < \alpha$, since $\tilde{\alpha}$ is an ω-point and α is not.

Let $\bar{r} \in \bar{N}$, $\bar{\lambda}_i \in \bar{N}$ $(i = 1, \dots, n, n \geq 0)$ s.t. $\sigma(\bar{r}, \bar{\theta}, \bar{\lambda}_i) = r, \theta, \lambda_i$ where $\lambda_i < \theta$ is regular and $\mathbb{B}_\alpha \in H_{\lambda_i}$. Let $\bar{G}$ be $\bar{\mathbb{B}}_{\bar{\alpha}}$ - generic over $\bar{N}$. We must show that there is $b \in \mathbb{B}_\alpha \setminus \{0\}$ with the property that if $G \ni b$ is $\mathbb{B}_\alpha$ - generic, then there is $\sigma' \in V[G]$ with the following properties:

(a) $\sigma' : \bar{N} \prec N$

(b) $\sigma'(\bar{r}, \bar{\theta}, \bar{\mathbb{B}}, \bar{\mathbb{B}}_\alpha, \bar{\lambda}_i) = \bar{r}, \theta, \mathbb{B}, \mathbb{B}_\alpha, \lambda_i$ $(i = 1, \dots, n)$

(c) $\sup \sigma' `` \bar{\lambda}_i = \tilde{\lambda}_i =_{df} \sup \sigma `` \bar{\lambda}_i$

$(i = 0, \dots, n$, where $\bar{\lambda}_0 = On \cap \bar{N}$)

(d) $\sigma' `` \bar{G} \subset G$.

Just as in the earlier proof in [SPSC] we consider two cases:

<u>Case 1</u> $cf(\alpha) < \alpha$ or $\alpha \leq \bar{\bar{\mathbb{B}}}_i$ for some $i < \alpha$.

In either case there will be $\eta < \alpha$ s.t. $\Vdash_\eta cf(\check{\alpha}) \leq \omega_1$. But α is not an ω-point, so $\Vdash_\eta cf(\check{\alpha}) = \omega_1$. It suffices to show that $\mathbb{B}_\alpha / G_\eta$ (which is the direct limit of $\langle \mathbb{B}_{\eta+i} / G_\eta \mid i < \alpha - \eta \rangle$) is subcomplete

in $V[G_\eta]$, since then $\mathbb{B}_\alpha \simeq \mathbb{B}_\eta * \dot{\mathbb{B}}$ is subcomplete, where $\Vdash_\eta \dot{\mathbb{B}} \simeq \check{\mathbb{B}}_\alpha / \dot{G}_\eta$.
Thus we may assume w.l.o.g. that $\eta = 0$.

Let $f : \omega_1 \to \alpha$ be monotone and cofinal s.t. $\sigma(\bar{f}) = f$. Let $\langle \nu_i \mid i < \omega \rangle$ be monotone and cofinal in $\omega_1^{\bar{N}}$. Set:

$$\bar{\xi}_i = \bar{f}(\nu_i) \quad , \quad \xi_i = f(\nu_i) \quad (i < \omega).$$

Then $\sigma(\bar{\xi}_i) = \xi_i$ and, in fact, $\sigma'(\bar{\xi}_i) = \xi_i$ for any $\sigma' : \bar{N} \prec N$ s.t. $\sigma'(\bar{f}) = f$. We shall assume w.l.o.g. that $\bar{\xi}_0 = \xi_0 = 0$.

For each $j = 0, \ldots, n$ choose a sequence $\langle \bar{\xi}_i^j \mid i < \omega \rangle$ which is monotone and cofinal in $\bar{\lambda}_j$. Set: $\xi_i^j = \sigma(\bar{\xi}_i^j)$.
Our strategy will be to construct by induction on $k < \omega$ a $c_k \in \mathbb{B}_k \setminus \{0\}$ and a $\dot{\sigma}_k \in V^{\mathbb{B}_k}$ s.t. $\langle c_k \mid k < \omega \rangle$ is a thread through $\langle \mathbb{B}_{\xi_k} \mid k < \omega \rangle$ (i.e. $h_{\xi_i \xi_k}(c_k) = c_i$ for $i \le k < \omega$) and s.t. the following holds:

(*) Let $G_k \ni c_k$ be $\mathbb{B}_{\xi_k}$ - generic. Set: *)
$\sigma_j = \sigma_j^{G_k} = \sigma_j^{G_i}$ for $j \leq k$, where
$G_j = G_k \cap \mathbb{B}_{\xi_j}$. Then:

(a) $\sigma_0 = \sigma$

(b) $\sigma_k : \bar{N} \prec N$

(c) $\sigma_k(\bar{\tau}, \bar{\theta}, \bar{\mathbb{B}}, \bar{\mathbb{B}}_{\bar{\alpha}}, \bar{\lambda}_i) = \tau, \theta, \mathbb{B}, \mathbb{B}_\alpha, \lambda_i$ $(i = 1, \ldots, n)$

(d) $\sup \sigma_k{}'' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n)$

(e) $\sigma_k(x_j) = \sigma_j(x_j)$ for $j \leq k$, where $\langle x_j \mid j < \omega \rangle$ is a fixed enumeration of $\bar{N}$.

(f) Let $i = 0, \ldots, n$ and let $k = j + 1$ s.t. $\sigma_j(\bar{\xi}^i_m) \leq \xi^i_k < \sigma_j(\bar{\xi}^i_{m+1})$. Then
$\sigma_k(\bar{\xi}^i_l) = \sigma_j(\bar{\xi}^i_l)$ for $l \leq m+1$.

(g) $\sigma_k(\bar{f}) = f$

(h) $\sigma_k{}'' \bar{G}_k \subset G_k$, where $\bar{G}_k = \bar{G} \cap \bar{\mathbb{B}}_{\bar{\xi}_k}$.

*) <u>Note</u> As in [SPSC] we assume that the "natural injection" of $V^{\mathbb{B}_i}$ into $V^{\mathbb{B}_j}$ $(i \leq j)$ is the identity. Hence, $t^{G_j} = t^{G_i}$ if $t \in V^{\mathbb{B}_i}$, G_j is $\mathbb{B}_j$ - generic, and $G_i = G_j \cap \mathbb{B}_i$. This also implies

that $[\![\varphi(t_1,\dots,t_n)]\!]_{\mathbb{B}_i} = [\![\varphi(t_1,\dots,t_n)]\!]_{\mathbb{B}}$, if $t_1,\dots,t_n \in V^{\mathbb{B}_i}$ and φ is a Σ_0 formula.

<u>Note</u> By (g) we have $\sigma_k(\bar{\xi}_i) = \xi_i$ for $i<\omega$. However, although we know that σ_k takes $\bar{\lambda}_\ell$ cofinally to $\tilde{\lambda}_\ell$ for $\ell = 0,\dots,n$, we do <u>not</u> necessarily have $\sigma_k(\bar{\xi}_i^\ell) = \xi_i^\ell$.

Before constructing c_k, σ_k, we show that (*) will prove the claim. Let $c = \bigwedge_k c_k$. Then $c \in \mathbb{B}_{\tilde{\alpha}} \setminus \{0\} \subset \mathbb{B}_\alpha \setminus \{0\}$. Let $G \ni c$ be $\mathbb{B}_\alpha$-generic. Set: $\sigma_k = \sigma_k^G$ for $k<\omega$. By (e) we can define a new map $\sigma' : \bar{N} \prec N$ by:

$$\sigma'(x) = \sigma_i(x) \text{ for sufficiently large } i.$$

Hence:

(1) $\sigma'(\bar{\tau}, \bar{\theta}, \bar{\mathbb{B}}, \bar{\mathbb{B}}_{\bar{\alpha}}, \bar{\lambda}_i) = \tau, \theta, \mathbb{B}, \mathbb{B}_\alpha, \lambda_i$ $(i = 1,\dots,n)$

Using (d), (f) we show:

(2) $\sup \sigma'{}''\bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n)$

prf.

$(\leq)$ is immediate by (d). But if $k = j' + 1$ and m is as in (f), then $\sigma'(\bar{\xi}^i_{m+1}) > \xi^i_k$.

Hence $(\geq)$ holds. QED(2)

(3) $\sigma'{}''\bar{G}_k \subset G_k$ for $k < \omega$.

proof.

Let $\bar{b} \in G_k$, $b = \sigma'(\bar{b})$. Then $b = \sigma_i(\bar{b})$ where i can be chosen $\geq k$. Hence $b \in G_i \cap \mathbb{B}_{\xi_k} = G_k$. QED(3)

Hence:

(4) $\sigma : \bar{G} \subset G$.

proof.

$\bigcup_i \bar{\mathbb{B}}_{\bar{\xi}_i} \setminus \{0\}$ is dense in $\bar{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$. By genericity, therefore, $\bigcup_i \bar{G}_i$ is dense in $\bar{G}$. Let $\bar{b} \in \bar{G}$, $\bar{a} \subset \bar{b}$, where $\bar{a} \in \bigcup_i \bar{G}_i$. Then $a = \sigma(\bar{a}) \subset b = \sigma(\bar{b})$ where $a \in \bigcup_i G_i \subset G$,

QED(4).

This proves the claim. It remains only to construct $c_k, \dot{\sigma}_k$ and verify $(*)$.

We set: $c_0 = 1$, $\dot{\sigma}_0 = \check{\sigma}$.

Now let $c_k, \dot\sigma_k$ be given. Let $G_k \ni c_k$ be $\mathbb{B}_{\xi_k}$ - generic. Set $\sigma_j = \dot\sigma_j^{G_j}$, $G_j = G_k \cap \mathbb{B}_{\xi_j}$ for $j \leq k$. Then σ_k has a unique extension $\tilde\sigma_k \supset \sigma_k$ s.t.

$$\tilde\sigma_k : \bar N[\bar G_k] \prec N[G_k] \text{ and } \tilde\sigma_k(\bar G_k) = G_k.$$

Set $\mathbb{B}^* = \mathbb{B}_{\xi_{k+1}} / G_k$, $\bar{\mathbb{B}}^* = \bar{\mathbb{B}}_{\bar\xi_{k+1}} / \bar G_k$.

Then $\mathbb{B}^*$ is subcomplete in $V[G_k]$ and $\tilde\sigma_k(\bar{\mathbb{B}}^*) = \mathbb{B}^*$. It follows easily that there is $c^* \in \mathbb{B}^* \setminus \{0\}$ s.t. whenever $G^* \ni c^*$ is $\mathbb{B}^*$ - generic over $V[G_k]$, then there is $\sigma^* \in V[G_k][G^*]$ s.t.

(b′) $\sigma^* : \bar N[\bar G_k] \prec N[G_k]$, $\sigma(\bar G_k) = G_k$

(c′) $\sigma^*(\bar s, \bar\theta, \bar{\mathbb{B}}, \bar{\mathbb{B}}_{\bar\alpha}, \bar{\mathbb{B}}^*, \bar\lambda_i) = s, \theta, \mathbb{B}, \mathbb{B}_\alpha, \mathbb{B}^*, \lambda_i$ $(i = 1, m, n)$

(d′) $\sup \sigma^{*} {}'' \bar\lambda_i = \tilde\lambda_i = \sup \tilde\sigma_k {}'' \bar\lambda_i$ $(i = 0, m, n)$

(e′) $\sigma^*(x_j) = \tilde\sigma_k(x_j)$ for $j \leq k+1$

(f′) Let $\tilde\sigma_k(\bar\xi_m^i) \leq \xi_{k+1}^i < \xi_{m+1}^i$.
Then $\sigma^*(\bar\xi_l^i) = \tilde\sigma_k(\bar\xi_l^i)$ for $l \leq m+1$

(g′) $\sigma^*(\bar f) = f = \tilde\sigma_k(f)$

(h′) $\sigma^{*} {}'' \bar G^* \subset G^*$ where $\bar G^* = \bar G_{k+1} / \bar G_k =$
$=_{df} \{b / \bar G_k \mid b \in \bar G_{k+1}\}$.

But $c^* = \dot{c}^{G_k}$ where – w.l.o.g. – it is forced that the above holds of $c^* = \dot{c}^{G_k}$ whenever $c_k \in G_k$. Thus

$$c_k \Vdash_{\xi_k} \dot{c} \in (\check{\mathbb{B}}_{\xi_{k+1}} / \dot{G}_k) \setminus \{0\}$$

W.l.o.g. we may also assume:

$$\Vdash_{\xi_k} (\check{c}_k \notin \dot{G}_k \to \dot{c} = 0 \text{ in } \check{\mathbb{B}}_{\xi_{k+1}} / \dot{G}_k).$$

Hence $\Vdash_{\xi_k} \dot{c} \in \check{\mathbb{B}}_{\xi_{k+1}} / \dot{G}_k$. Hence there is a unique $c \in \mathbb{B}_{\xi_{k+1}}$ s.t.

$$\Vdash_{\xi_k} \check{c}/\dot{G}_k = \dot{c}.$$ We set: $c_{k+1} = c$.

Then $h_{\xi_k \xi_{k+1}}(c_{k+1}) = [\![\check{c}/\dot{G}_k \neq 0]\!]_{\xi_k} = c_k$.

Now let $G_{k+1} \ni c_{k+1}$ be $\mathbb{B}_{\xi_{k+1}}$-generic.

Set $\mathbb{B}^* = \mathbb{B}_{\xi_{k+1}} / G_k$, $G^* = G_{k+1} / G_k$.

Then $c^* = \dot{c}^{G_k} = c_{k+1}/G_k \in G^*$ and hence there is $\sigma^* \in V[G_k][G^*] = V[G_{k+1}]$ satisfying (b′)–(h′).

Hence $\sigma' = \sigma^* \upharpoonright \bar{N}$ satisfies (b)–(h) (with $k+1, \sigma'$ in place of k, σ_k). We may assume $\sigma' = \dot{\sigma}^{G_{k+1}}$, where (a)–(h) hold of σ' whenever $c_{k+1} \in G_{k+1}$, Set: $\dot{\sigma}_{k+1} = \dot{\sigma}$. This completes the construction. QED (Case 1)

<u>Case 2</u> Case 1 fails.

Note that α is regular and $\mathbb{B}_i \in H_\alpha$ for $i < \alpha$. We extend the sequence $\lambda_1, \ldots, \lambda_n$ by setting: $\bar{\lambda}_{n+1} = \bar{\alpha}$ (hence $\lambda_{n+1} = \alpha$, $\tilde{\lambda}_{n+1} = \tilde{\alpha}$). For $j = 0, \ldots, n+1$ let $\langle \bar{\xi}_i^j \mid i < \omega \rangle$ be monotone and cofinal in $\bar{\lambda}_j$ with $\bar{\xi}_0^j = 0$. Set: $\xi_i^j = \sigma(\bar{\xi}_i^j)$. We also write: $\bar{\xi}_i = \bar{\xi}_i^{n+1}$, $\xi_i = \xi_i^{n+1}$. Our strategy is essentially the same as before: We construct $c_k, \dot{\sigma}_k$ s.t. $\langle c_k \mid k < \omega \rangle$ is a thread in $\langle \mathbb{B}_{\xi_k} \mid k < \omega \rangle$ and c_k forces that $\dot{\sigma}_k : \bar{N} \prec N$. The intention is, again, that if $c = \bigcap_k c_k \in G$ and G is $\mathbb{B}_\alpha$-generic, then we will be able to define the desired embedding σ' from $\langle \sigma_k \mid k < \omega \rangle$, where $\sigma_k = \dot{\sigma}_k^G$. However, we now have nothing corresponding to the function $\bar{f}$, which was used previously to define $\langle \bar{\xi}_i \mid i<\omega \rangle$. Although we will be able to enforce $\sup \sigma_k{}''\bar{\alpha} = \tilde{\alpha}$, we will no longer be able to enforce $\sigma_k(\bar{\xi}_i) = \xi_i$ for $i<\omega$. This will necessitate a number of changes in the proof.

We shall inductively construct $c_k \in \mathbb{B}_{\xi_k}$ and $\dot{\sigma}_k \in V^{\mathbb{B}_{\xi_k}}$ s.t.

(I) $h_{\xi_i \xi_k}(c_k) \supseteq c_i$ for $k = i+1$

(II) Let $G \ni c_k$ be $\mathbb{B}_{\xi_k}$-generic. Set $G_\nu = G \cap \mathbb{B}_\nu$ for $\nu \leq \xi_k$. For $j \leq k$ set $\sigma_j = \dot{\sigma}_j^G = \dot{\sigma}_j^{G_{\xi_j}}$. Then:

(a) $\sigma_0 = \sigma$

(b) $\sigma_k : \bar{N} \prec N$

(c) $\sigma_k(\bar{s}, \bar{\theta}, \bar{\mathbb{B}}, \bar{\mathbb{B}}_{\bar{\alpha}}, \bar{\lambda}_i) = s, \theta, \mathbb{B}, \mathbb{B}_\alpha, \lambda_i$ $(i = 1, \ldots, n+1)$

(d) $\sup \sigma_k{}''\bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n+1)$

(e) $\sigma_k(x_j) = \sigma_j(x_j)$ $(j \leq k)$, where $\langle x_j \mid i<\omega \rangle$ is a fixed enumeration of $\bar{N}$

(f) Let $i = 0, \dots, n+1$ and $k = j+1$ s.t. $\sigma_j(\bar{\xi}^i_m) \le \xi^i_k < \sigma_j(\bar{\xi}^i_{m+1})$. Then $\sigma_k(\bar{\xi}^i_\ell) = \sigma_j(\bar{\xi}^i_\ell)$ for $\ell \le m+1$.

(g) Let $k = j+1$. Let m_k = that m s.t. $\sigma_j(\bar{\xi}_m) \le \xi_k < \sigma_j(\bar{\xi}_{m+1})$. (Hence m is maximal s.t. $\sigma_k(\bar{\xi}_m) \le \xi_k$ by (f).) Set $\bar{\mu}_k = \bar{\xi}_m$, $\mu_k = \sigma_k(\bar{\mu}_k) = \sigma_j(\bar{\mu}_k)$. Then $\sigma_k{}'' \bar{G}_{\bar{\mu}_k} \subset G_{\mu_k}$ where $\bar{G}_\nu =_{Df} \bar{G} \cap \mathbb{B}_\nu$. Moreover $\sigma_k = \sigma_j$ if $\mu_k < \xi_j$.

(a) – (f) are unchanged – except that (c), (d), (f) now hold for $i = n+1$ as well. Our old condition (g) (which said that $\sigma_k(\bar{f}) = f$) is missing. The new condition (g) takes the place of the old condition (h). Before constructing e_k, σ_k, we show that (I) (II) implies the theorem. Let $c = \bigwedge_k c_k$ and let $G \ni c$ be $\mathbb{B}_\alpha$ – generic. We again define $\sigma' : \bar{N} \prec N$ by:

$\sigma'(x) = \sigma_i(x)$ for sufficiently large i, As before:

(1) $\sigma'(\bar{s}, \bar{\theta}, \bar{\mathbb{B}}, \mathbb{B}_{\bar{\alpha}}, \bar{\lambda}_i) = s, \theta, \mathbb{B}, \mathbb{B}_\alpha, \lambda_i$
$(i = 1, \ldots, n+1)$

(2) $\sup \sigma'{}''\bar{\lambda}_i = \tilde{\lambda}_i \quad (i = 0, \ldots, n+1)$

Clearly $\sigma_k{}''\bar{G}_{\bar{\mu}_k} \subset G_{\mu_k}$ for $k < \omega$, so we get:

(3) $\sigma'{}''\bar{G}_{\bar{\mu}_k} \subset G_{\mu_k}$

as before. By (f) it is easily seen that $\sup\limits_{i<\omega} \mu_i = \bar{\alpha}$. Hence as before:

(4) $\sigma'{}''\bar{G} \subset G$,

It remains only to construct $c_k, \dot{\sigma}_k$.

In addition to $c_k, \dot{\sigma}_k$ we shall define $b_k \in \mathbb{B}_{\xi_k}$ s.t.

(III) $c_k \subset b_k$

(IV) (a)–(g) hold whenever $G \ni b_k$ is $\mathbb{B}_{\xi_k}$-generic

(V) If $k = j+1$, then $h_{\xi_j \xi_k}(b_k) \supset c_j$

Def Let $k = j+1$. For $\nu \le \xi_k$, $i < \omega$ set:

$$a^{i,\nu} = a_k^{i\nu} =_{Df} [\![\dot{\sigma}_j(\check{\xi}_i) = \check{\nu} \wedge \check{\xi}_k < \dot{\sigma}_j(\check{\bar{\xi}}_{i+1})]\!]_{\mathbb{B}_{\xi_j}}$$

Now let $b_k, \dot{\sigma}_k$ be given, where $k = j+1$.

We also assume as part of the ind. hypothesis:

(VI). If $\nu \leq \xi_j$, then

$$b_k \cap a_k^{i\nu} = b_j \cap a_j^{i\nu} \cap [\![\dot{\sigma}_j(\check{\bar{\xi}}_{i+1}) > \check{\xi}_k]\!]_{\xi_j}$$

(VII) $\quad b_k \cap a^{i\nu} \cap [\![\dot{\sigma}_k(\check{x}) = \check{y}]\!]_{\xi_j} \in \mathbb{B}_\nu$

for all $x \in \bar{N}$, $y \in N$, $\nu \leq \xi_k$

Note that (VI) is vacuously true for $k=1$ and (VII) holds for $k=1$ since $a_1^{i\nu} = 1$ if $i = 0, \nu = \xi_1$; $a_1^{i\nu} = 0$ otherwise.

To facilitate the construction we also define

Def Let $k = j+1$, $\ell > k$.

$$a_{k\ell}^{i\nu} = a_k^{i\nu} \cap [\![\check{\xi}_{\ell-1} < \dot{\sigma}_j(\check{\bar{\xi}}_{i+1}) \leq \check{\xi}_\ell]\!]_{\mathbb{B}_{\xi_j}}$$

Thus $a_k^{i\nu} = \bigcup_{\ell > k} a_{k,\ell}^{i\nu}$ and

$$a_{k\ell}^{i\nu} \cap a_{k,\ell'}^{i'\nu'} = 0 \text{ if } \langle i, \nu, \ell\rangle \neq \langle i', \nu', \ell'\rangle.$$

For $a = a_{k\ell}^{i\nu} \neq 0$ we shall define a $d = d_{k\ell}^{i\nu} \in \mathbb{B}_{\xi_\ell}$ and $a\,\dot{\sigma} = \dot{\sigma}_{k\ell}^{i\nu} \in V^{\mathbb{B}_{\xi_\ell}}$ s.t.

(VIII) $\quad h_{\nu, \xi_\ell}(d) = b_k \cap a_{k\ell}^{i\nu}$

($\boxed{IX}$) Let $G \ni d$ be $\mathbb{B}_{\xi_\ell}$ - generic set;

$\sigma_h = \dot\sigma_h^G$ for $h \le k$ and $\sigma' = \dot\sigma^G$ where $\dot\sigma = \dot\sigma_{k\ell}^{iv}$.

Then:

(b) $\sigma' : \bar N \prec N$

(c) $\sigma'(\bar\kappa, \bar\theta, \bar{\mathbb{B}}, \bar{\mathbb{B}}_{\bar\alpha}, \bar\lambda_i) = \kappa, \theta, \mathbb{B}, \mathbb{B}_\alpha, \lambda_i$ $(i = 1, \dots, n+1)$

(d) $\sup \sigma'{}'' \bar\lambda_i = \tilde\lambda_i = \sup \sigma_k{}'' \bar\lambda_i$ $(i = 0, \dots, n+1)$

(e) $\sigma'(x_i) = \sigma_k(x_i)$ for $i \le \ell$

(f) Let $\sigma_k(\bar\xi^i_m) \le \xi_\ell < \sigma_k(\bar\xi^i_{m+1})$ $(i = 0, \dots, n+1)$

Then $\sigma'(\bar\xi^i_h) = \sigma_k(\bar\xi^i_h)$ for $h \le \ell$.

(g) Let $\sigma_k(\bar\xi_m) = \mu \le \xi_\ell < \sigma_k(\bar\xi_{m+1})$; $\bar\mu = \bar\xi_m$.

Then $\sigma'{}'' \bar G_{\bar\mu} \subset G_\mu$ where $\bar G_{\bar\mu} = \bar G \cap \bar{\mathbb{B}}_{\bar\mu}$, $G_\mu = G \cap \mathbb{B}_\mu$.

(h) Let μ be as in (g). Let $G_\nu = G \cap \mathbb{B}_\nu$.

Then d/G_ν, $[\![\dot\sigma(\check x) = \check y]\!]_{\xi_\ell}/G_\nu \in \mathbb{B}_\mu/G_\nu$

for $x \in \bar N$, $y \in N$.

We define $\dot d = \dot d^{iv}_{k\ell}$, $\dot\sigma = \dot\sigma^{iv}_{k\ell}$ as follows:

<u>Case A</u> $\nu < \xi_j$ (where $k = j+1$).

Recall that $a^{iv}_k \cap b_k = a^{iv}_j \cap b_j \cap [\![\dot\sigma_j(\bar\xi_{i+1}) > \check\xi_k]\!]_{\xi_j} \supset a^{iv}_{j\ell}$.

We set: $d = d^{iv}_{j\ell}$, $\dot\sigma = \dot\sigma^{iv}_{j\ell}$.

Then $d \in \mathbb{B}_{\xi_\ell}$. $\underline{\text{VIII}}$ follows, since

$h_{\nu\xi_\ell}(d) = b_j \cap a^{iv}_{j\ell} = b_k \cap a^{iv}_{k\ell}$

by ($\underline{\text{VI}}$) and the induction hypothesis.

(Note that $j > 0$, since $a_1^{i\nu} = 0$ for $\nu < \xi_1$)
But since $\nu < \xi_j$, we know that if $G \ni a_k^{i\nu} \cap b_k$ is $\mathbb{B}_{\xi_k}$-generic, then $\sigma_j = \sigma_k$, where $\sigma_h = \dot{\sigma}_h^G$ for $h \leq k$. But $\sigma' = \dot{\sigma}^G = (\dot{\sigma}_{j'\ell}^{i\nu})^G$, so (IX) follows by the ind. hypothesis.

<u>Case B</u> $\xi_j \leq \nu$.
We first note that by (VII) there is a $\dot{\sigma}^{i\nu} = \dot{\sigma}_k^{i\nu} \in V^{\mathbb{B}_\nu}$ s.t.
$[\![\dot{\sigma}^{i\nu}(\check{x}) = \check{y}]\!]_\nu = b_k \cap a_k^{i\nu} \cap [\![\dot{\sigma}_k(\check{x}) = \check{y}]\!]_{\xi_k}$
for $x \in \bar{N}$, $y \in N$. Hence
$b_k \cap a_k^{i\nu} \Vdash_\nu \dot{\sigma} : \check{\bar{N}} \prec \check{N}$ and $(\dot{\sigma}^{i\nu})^{G_\nu} = \dot{\sigma}_k^G$
whenever $G \ni b_k \cap a_k^{i\nu}$ is $\mathbb{B}_{\xi_k}$-generic and $G_\nu = G \cap \mathbb{B}_\nu$. Thus $\dot{\sigma}_k^G$ depends only upon G_ν.
Now let $G_\nu \ni b_k \cap a_{k\ell}^{i\nu}$ be $\mathbb{B}_\nu$-generic. Set:
Set $G_\eta = G \cap \mathbb{B}_\eta$ for $\eta \leq \nu$. Set:
$\sigma_k = (\dot{\sigma}^{i\nu})^{G_\nu}$, $\sigma_h = \dot{\sigma}_h^{G_\nu}$ for $h < k$.
Let $\sigma_k(\bar{\xi}_m) \leq \xi_\ell < \sigma_k(\bar{\xi}_{m+1})$. (Hence $m \geq i+1$, since $\sigma_k(\bar{\xi}_{i+1}) \leq \xi_\ell$..)

Set: $\bar{\mu} = \bar{\xi}_m$, $\mu = \sigma_k(\bar{\mu})$. Let $\bar{\nu} = \bar{\xi}_i$ (hence $\nu = \sigma_k(\bar{\nu})$). Then $\sigma_k(\bar{\mathbb{B}}_{\bar{\nu}}) = \mathbb{B}_\nu$, $\sigma_k{}''\bar{G}_{\bar{\nu}} \subset G_\nu$, where $\bar{G}_{\bar{\nu}} = \bar{G} \cap \bar{\mathbb{B}}_{\bar{\nu}}$. Hence σ_k has a unique extension $\tilde{\sigma}_k \supset \sigma_k$ s.t.

$$\tilde{\sigma}_k : \bar{N}[\bar{G}_{\bar{\nu}}] \prec N[G_\nu],\ \tilde{\sigma}(\bar{G}_\nu) = G_\nu.$$

Set: $\mathbb{B}^* = \mathbb{B}_\mu / G_\nu$, $\bar{\mathbb{B}}^* = \mathbb{B}_{\bar{\mu}} / \bar{G}_{\bar{\nu}}$.

Then $\tilde{\sigma}_k(\bar{\mathbb{B}}^*) = \mathbb{B}^*$. But $\mathbb{B}^*$ is sub-complete in $V[G_\nu]$. Hence there is $d^* \in \mathbb{B}^* \setminus \{0\}$ and a $\tilde{\sigma} \in V[G_\nu]^{\mathbb{B}^*}$ s.t.

(**) If $G^* \ni d^*$ is $\mathbb{B}^*$-generic over $V[G_\nu]$, and $\sigma^* = \tilde{\sigma}^{G^*}$, then:

(b) $\sigma^* : \bar{N}[\bar{G}_{\bar{\nu}}] \prec N[G_\nu]$, $\sigma^*(\bar{G}_{\bar{\nu}}) = G_\nu$

(c) $\sigma^*(\bar{s}, \bar{\theta}, \bar{\mathbb{B}}, \bar{\mathbb{B}}_\alpha, \bar{\mathbb{B}}^*, \bar{\lambda}_i) = s, \theta, \mathbb{B}, \mathbb{B}_\alpha, \mathbb{B}^*, \lambda_i$ $(i = 1, \ldots, n+1)$

(d) $\sup \sigma^*{}''\bar{\lambda}_i = \tilde{\lambda}_i = \sup \sigma_k{}''\bar{\lambda}_i$ $(i = 0, \ldots, n+1)$

(e) $\sigma^*(x_i) = \sigma_k(x_i)$ for $i \leq \ell$

(f) Let $\sigma_k(\bar{\xi}^i_m) \leq \xi^i_\ell < \sigma_k(\bar{\xi}^i_{m+1})$. Then $\sigma^*(\bar{\xi}^i_h) = \sigma_k(\bar{\xi}^i_h)$ for $h \leq \ell$ $(i = 0, \ldots, n+1)$

(g) $\sigma^*{}''\bar{G}^* \subset G^*$, where $\bar{G}^* = \bar{G}_{\bar{\mu}} / \bar{G}_{\bar{\nu}} =$
$=_{\text{df}} \{ b/\bar{G}_{\bar{\nu}} \mid b \in \bar{G}_{\bar{\mu}} \}$

We may suppose w.l.o.g. that $\dot{d}^* = \dot{d}^{G_\nu}$ and $\tilde{\sigma} = \ddot{\sigma}^{G_\nu}$, where $(**)$ holds of $d^* = \dot{d}^{G_\nu}$, $\tilde{\sigma} = \ddot{\sigma}^{G_\nu}$ whenever $G_\nu \ni b_k \cap a^{i\nu}_{k\ell}$ is $\mathbb{B}_\nu$-generic.
We may also suppose w.l.o.g. that $\neg(b_k \cap a^{i\nu}_{k\ell}) \Vdash_\nu \dot{d} = 0$. Then $\Vdash_\nu \dot{d} \in \check{\mathbb{B}}_{\xi_\ell} / \dot{G}_\nu$.
Hence there is $d \in \mathbb{B}_{\xi_\ell}$ s.t.
$\Vdash_\nu \check{d}/\dot{G}_\nu = \dot{d}$. Set: $d^{i\nu}_{k\ell} = d$.
Now let $G \ni b_k \cap a^{i\nu}_{k\ell} \cap d$ be $\mathbb{B}_{\xi_\ell}$-generic.
Set $G_\eta = G \cap \mathbb{B}_\eta$ for $\eta \leq \xi_\ell$. Then
$G_\nu \ni b_k \cap a^{i\nu}_{k\ell}$ is $\mathbb{B}_\nu$-generic. Set
$\mathbb{B}^* = \mathbb{B}_\mu / G_\nu$, $G^* = G_\mu / G_\nu$, where μ is defined as above from $\dot{\sigma}_k^G = (\dot{\sigma}_k^{i\nu})^{G_\nu}$.
Let $\sigma' = ((\ddot{\sigma})^{G_\nu})^{G^*} \restriction \bar{N}$. Clearly $\sigma' = \dot{\sigma}^G$ for
$$\dot{\sigma}^G = ((\ddot{\sigma})^{G_\nu})^{G^*} \restriction \bar{N}' \text{ \underline{whenever}}$$
$G \ni b_k \cap a^{i\nu}_k \cap d$ is $\mathbb{B}_{\xi_\ell}$-generic and G_ν, G^* are defined as above.
Set: $\dot{\sigma}^{i\nu}_{k\ell} = \dot{\sigma}$.

We verify $(\underline{\mathrm{VIII}})$, $(\underline{\mathrm{IX}})$

To see ($\boxed{\text{VIII}}$) note that:

$$h_{\nu \xi_\ell}(d) = [\![\check{d}/\dot{G}_\nu \neq 0]\!]_\nu = [\![\dot{d} \neq 0]\!]_\nu = (b_k \cap a^{iv}_{k\ell})$$

($\boxed{\text{IX}}$) (b)–(f) are then immediate.

To see (g) note that for $\sigma^* = \tilde{\sigma}^{G^*}$, where $G \ni d$ in $\mathbb{B}_{\xi_\ell}$ – generic and $G^* = G_\mu / G_\nu$ where μ is defined as above, then if $\sigma^*(\bar{b}) = b$, $\bar{b} \in \bar{G}_{\bar{\mu}}$, then

$\bar{b}/\bar{G}_\nu \in \bar{G}^* = \bar{G}_{\bar{\mu}} / \bar{G}_\nu$ and $\sigma^*(\bar{b}/\bar{G}_\nu) =$

$= b/G_\nu \in G_\mu / G_\nu$. (Hence $b \in G_\mu$, where $b \in \mathbb{B}_\mu$)

But $\sigma' = \sigma^* \upharpoonright \bar{N}$, QED (g).

We now prove (h).

$d/G_\nu = \dot{d}^{G_\nu} \in \mathbb{B}_\mu / G_\nu$ is immediate.

Let $u = [\![\dot{\sigma}(\check{x}) = \check{y}]\!]_{\xi_\ell}$. Then

$$u \in G \longleftrightarrow u/G_\nu \in G/G_\nu$$

$$\longleftrightarrow \tilde{\sigma}^{G^*}(x) = y$$

$$\longleftrightarrow [\![\tilde{\sigma}(\check{x}) = \check{y}]\!]_{\mathbb{B}^*} \in G_\mu / G_\nu \subset G/G_\nu$$

But this holds for every $G \ni G_\nu$.

Hence $u/G_\nu = [\![\tilde{\sigma}(\check{x}) = \check{y}]\!]_{\mathbb{B}^*} \in \mathbb{B}^* = \mathbb{B}_\mu / G_\nu$.

QED ($\boxed{\text{IX}}$).

This completes the construction of $d = d^{i\nu}_{kl}$, $\dot\sigma = \dot\sigma^{i\nu}_{kl}$ for $i < \omega$, $\nu \le \xi_k$, $k < l$.

* * * * *

We define c_k, b_{k+1}, $\dot\sigma_{k+1}$ as follows:

Def $c_k =_{Df} b_k \cap \bigcup_{a^{i\nu}_{kl} \neq 0} (a^{i\nu}_{kl} \cap h_{\xi_k \xi_l}(d^{i\nu}_{kl}))$

Set: $\tilde a^{i\nu}_k = a^{i\nu}_{k,k+1}$, $e^{i\nu}_k = \bigcup_{l > k+1} a^{i\nu}_{k,l}$.

(Hence $\tilde a^{i\nu} \cup e^{i\nu} = a^{i\nu}$, $\tilde a^{i\nu} \cap e^{i\nu} = 0$)

Set: $\tilde d^{i\nu} = \tilde d_k = d_{k,k+1}$.

We define:

Def $b_{k+1} = \bigcup_{\tilde a^{i\nu} \neq 0} (b_k \cap \tilde a^{i\nu}_k \cap \tilde d^{i\nu}_k) \cup \bigcup_{e^{i\nu}_k \neq 0} (b_k \cap e^{i\nu}_k)$

Def $\dot\sigma_{k+1} = \dot\sigma$ where $\dot\sigma \in V^{\mathbb{B}_{\xi_{k+1}}}$ s.t.

$(b_k \cap \tilde a^{i\nu}_k \cap \tilde d^{i\nu}_k) \Vdash_{\xi_{k+1}} \dot\sigma = \dot\sigma^{i\nu}_{k,k+1}$

$b_k \cap e^{i\nu}_k \Vdash_{\xi_{k+1}} \dot\sigma = \dot\sigma_k$, Clearly $c_k \in \mathbb{B}_{\xi_k}$ and $c_k \subset b_k$. We verify (I):

(1) $h_{\xi_j \xi_k}(c_k) \supset c_j$

proof.

$h_{\xi_j \xi_k}(c_k) = \bigcup_{a_{kl}^{i\nu} \neq 0} h_{\xi_j \xi_k}(b_k \cap a_{kl}^{i\nu} \cap h_{\xi_k \xi_l}(d_{kl}^{i\nu})),$

so it suffices to show:

<u>Claim</u> $h_{\xi_j \xi_k}(b_k \cap a_{kl}^{i\nu} \cap h_{\xi_k \xi_l}(d_{kl}^{i\nu})) \supset c_j \cap a_{kl}^{i\nu}$,

since then $h_{\xi_j \xi_k}(c_k) \supset c_j \cap \bigcup_{k,l}^{i\nu} a_{kl}^{i\nu} = c_j$,

<u>Case 1</u> $\nu \geq \xi_j$

$h_{\xi_j \xi_k}(b_k \cap a_{kl}^{i\nu} \cap h_{\xi_k \xi_l}(d_{kl}^{i\nu})) =$

$= h_{\xi_j \nu}(b_k \cap a_{kl}^{i\nu} \cap h_{\nu \xi_l}(d_{kl}^{i\nu}))$ since $b_k \cap a_{kl}^{i\nu} \in \mathbb{B}_\nu$

$= h_{\xi_j \nu}(b_k \cap a_{kl}^{i\nu})$ since $h_{\nu \xi_l}(d_{kl}^{i\nu}) = b_k \cap a_{kl}^{i\nu}$

$= h_{\xi_j \xi_k}(b_k \cap a_{kl}^{i\nu}) = h_{\xi_j \xi_k}(b_k) \cap a_{kl}^{i\nu} \supset c_j \cap a_{kl}^{i\nu}.$

<u>Case 2</u> $\nu < \xi_j$ (hence $j > 1$ since $a_1^{i\nu} = 0$ for $\nu < \xi_1$). Moreover

Then $b_k \cap a_{kl}^{i\nu} = b_j \cap a_{jl}^{i\nu}$ by (VI).

$d_k^{i\nu} = d_j^{i\nu}$. Hence

$h_{\xi_j \xi_k}(b_k \cap a_{kl}^{i\nu} \cap h_{\xi_k \xi_l}(d_{kl}^{i\nu})) =$

$h_{\xi_j \xi_k}(b_j \cap a_{jl}^{i\nu} \cap h_{\xi_k \xi_l}(d_{jl}^{i\nu})) =$

$= b_j \cap a_{jl}^{i\nu} \cap h_{\xi_j \xi_l}(d_{jl}^{i\nu}) = b_j \cap a_{jl}^{i\nu} \cap c_j$

(by the def. of c_j) $= b_k \cap a_{kl}^{i\nu} \cap c_j$

We must verify $(\overline{IV})-(\overline{VII})$ for $k+1$ in place of k. We first verify $(\overline{IV})$:

(2) (a)–(g) hold (with $k+1, k$ in place of h, j) whenever $G \ni b_{k+1}$ is $\mathbb{B}_{\bar{\xi}_{k+1}}$ - generic.

prf.
If $e_k^{iv} \cap b_k \in G$, then $\sigma_{k+1} = \sigma_k$ and (a)–(g) hold by the ind. hypothesis.
If $b_k \cap a_{k,k+1}^{iv} \cap d_{k,k+1}^{iv} \in G$, then $\sigma_{k+1} = \sigma_{k,k+1}^{iv}$ and the conclusion follows by $(\overline{IX})$. QED (2)

The verification of $\overline{V}$ is immediate by the definitions:

(3) $h_{\bar{\xi}_k \bar{\xi}_{k+1}}(b_{k+1}) \supset c_k$

We verify $\overline{VI}$:

(4) If $\mu < \bar{\xi}_k$, then
$b_{k+1} \cap a_{k+1}^{j\mu} = b_k \cap a_k^{j\mu} \cap [\![\dot{\sigma}_k(\check{\bar{\xi}}_{i+1}) > \check{\bar{\xi}}_{k+1}]\!]_{\bar{\xi}_k}$

proof.
Note that $a_k^{j\mu} \cap [\![\dot{\sigma}_k(\check{\bar{\xi}}_{i+1}) > \check{\bar{\xi}}_{k+1}]\!]_{\bar{\xi}_k} = e_k^{j\mu}$.

It suffices to show:
Claim Let G be $\mathbb{B}_{\bar{\xi}_{k+1}}$ - generic. Then
$b_{k+1} \cap a_{k+1}^{j\mu} \in G \longleftrightarrow b_k \cap e_k^{j\mu} \in G$,

($\rightarrow$) It is easily seen that $a^{i\nu}_{k,k+1} \cap a^{j\mu}_{k+1} = 0$ for $i<\nu$, $\nu \leq \xi_k$, since if $a^{i\nu}_{k,k+1} \cap a^{j\mu}_{k+1} \in G$, then $\mu \geq \sigma_k(\bar{\xi}_{i+1}) \geq \xi_k$. Contr! Hence $b_k \cap e^{i\nu}_k \cap a^{j\mu}_{k+1} \in G$, for some $i<\omega$, $\nu \leq \xi_k$.

Since $\sigma_k(\bar{\xi}_j) = \mu \wedge \sigma_k(\bar{\xi}_{j+1}) > \xi_{k+1}$, we conclude: $\nu = \mu$, $i = j$. Hence $b_k \cap e^{j\mu}_k \in G$.

($\leftarrow$) If $b_k \cap e^{j\mu}_k \in G$, then $\sigma_k(\bar{\xi}_j) = \mu < \xi_k$ and $\sigma_k(\bar{\xi}_{j+1}) > \xi_{k+1}$.

Hence $a^{j\mu}_{k+1} \in G$. But $b_{k+1} \in G$ since $b_{k+1} \supset b_k \cap e^{j\mu}$. QED (4)

It remains to verify ($\underline{\text{VII}}$):

(5) $b_{k+1} \cap a^{j\mu}_{k+1} \cap [\![\dot{\sigma}_{k+1}(\check{x}) = \check{y}]\!] \in \mathbb{B}_{\xi_{k+1}\mu}$

for $\mu \leq \xi_{k+1}$, $j<\omega$.

proof.

<u>Case 1</u> $\mu < \xi_k$

Then $b_{k+1} \cap a^{i\mu}_{k+1} = b_k \cap a^{i\mu}_k \cap [\![\dot\sigma_k(\check{\bar\xi}_{j+1}) > \check\xi_{k+1}]\!]_{\xi_k}$

$= b_k \cap e^{i\mu}_k$

But $b_k \cap e^{i\mu}_k \Vdash \dot\sigma_k = \dot\sigma_{k+1}$ since $\mu < \xi_k$. Hence:

$b_{k+1} \cap a^{i\mu}_{k+1} \cap [\![\dot\sigma_{k+1}(\check x) = \check y]\!]_{\xi_{k+1}} =$

$= b_k \cap e^{i\mu}_k \cap [\![\dot\sigma_k(\check x) = \check y]\!]_{\xi_k} \in \mathbb{B}_\mu$.

<u>Case 2</u> $\mu \geq \xi_k$.

Let $a = a^{i\mu}_{k+1}$. Then $b_k \cap e^{iv}_k \cap a = 0$ for all $i < \omega$, $v < \xi_k$. Hence

$b_{k+1} \cap a \cap [\![\dot\sigma(\check x) = \check y]\!]_{\xi_{k+1}} =$

$= \bigcup_{a^{iv}_{k,k+1} \neq 0} (b_k \cap a^{iv}_{k,k+1} \cap a \cap d^{iv}_{k,k+1} \cap [\![\dot\sigma(\check x) = \check y]\!]_{\xi_{k+1}})$.

Set: $u = d^{iv}_{k,k+1} \cap [\![\dot\sigma(\check x) = \check y]\!]_{\xi_{k+1}}$.

Clearly it suffices to show:

<u>Claim</u> $b_k \cap a^{iv} \cap a \cap u \in \mathbb{B}_\mu$

for $a^{iv} = a^{iv}_{k,k+1} \neq 0$.

Now set $w = h_{\xi_k \xi_{k+1}}(b_k \cap a^{iv}_{k,k+1} \cap a \cap u) =$

$= b_k \cap a^{iv}_{k,k+1} \cap a \cap h_{\xi_k \xi_{k+1}}(u)$.

Suppose that $G \ni w$ is $\mathbb{B}_{\overline{\xi}_k}$ - generic. Then:

(A) $\sigma_k(\overline{\xi}_m) = \mu \leq \xi_k < \sigma_k(\overline{\xi}_{m+1})$, since $a \in G$

(B) G extends to a $G' \ni u$ which is $\mathbb{B}_{\overline{\xi}_{k+1}}$ - generic.

By (IX)(h) we conclude:

(C) $u/G_\nu \in \mathbb{B}_\mu$ whenever $G \ni w$ is $\mathbb{B}_{\overline{\xi}_k}$ - generic.

But then the set:

$\Delta = \{ e \subset w \mid \bigvee v \in \mathbb{B}_\mu \; e \Vdash_{\overline{\xi}_k} \check{u}/\dot{G}_\nu = \check{v}/\dot{G}_\nu \}$

is dense above w in $\mathbb{B}_{\overline{\xi}_k} \setminus \{0\}$.

Let A be a maximal antichain in Δ. Let $e \Vdash_{\overline{\xi}_k} \check{u}/\dot{G}_\nu = \check{v}_e/\dot{G}_\nu$ for $e \in A$, where $v_e \in \mathbb{B}_\mu$. Set: $v = \bigcup_{e \in A} e \wedge v_e$.

Then $v \in \mathbb{B}_\mu$ and:

(D) $u/G_\nu = v/G_\nu$ whenever $G \ni w$ is $\mathbb{B}_{\overline{\xi}}$ - generic.

Hence:

$$b_k \wedge a^{i\nu}_{k,k+1} \wedge a \wedge u = w \wedge u = w \wedge v \in \mathbb{B}_\mu .$$

(Suppose not. Let e.g. $w \wedge (u \setminus v) \neq 0$. Let $G \ni w \wedge (u \setminus v)$ be $\mathbb{B}_{\xi_{k+1}}$ - generic. Then $u/G_\nu \in G/G_\nu$ and $v/G_\nu \notin G/G_\nu$, contradicting (D).)

This completes the construction and proves the theorem.