On the subcompleteness of some Namba-type forcings

§1 In this paper we show — under the assumption $CH + 2^{\omega_1} = \omega_2$ — that two Namba-like forcings $\mathbb{N}'$ and $\mathbb{N}^*$ are subcomplete.
$\mathbb{N}'$ is the set of Namba trees with a finite stem $s = \mathrm{stem}(T)$ s.t. for all $t \in T$, either $t = s \upharpoonright n$ for an $n < |s|$ or else $t \supset s$ has ω_2 many immediate successors. The salient property of $\mathbb{N}'$-generic sequences $\langle \gamma_i \mid i < \omega \rangle$ is that, whenever $F : \omega_2 \to \omega_2$ is a function in the ground model, then:

$$\bigvee n \bigwedge m \geq n \quad F(\gamma_m) < \gamma_{m+1}.$$

$\mathbb{N}^*$ is defined like $\mathbb{N}'$ except that we impose the stronger requirement that, if $t \in T$ and $t \supset \mathrm{stem}(T)$, then:

$$\{\alpha \mid t^\frown\langle\alpha\rangle \in T\} \text{ is stationary in } \omega_2.$$

The salient property of $\mathbb{N}^*$-generic sequences is that whenever $A \subset \omega_2$ is club in the ground model, then:

$$\bigvee n \bigwedge m \geq n \quad \gamma_m \in A.$$

Both forcings add no reals, assuming CH in the ground model.

IN$'$ has been treated extensively in the literature, especially [PIF]. We are not aware of previous treatments of IN* and would be grateful for any references.

In [DSP] we generalized Shelah's notions of "dee-complete" and "ω_1-proper" forcing to "dee-subcomplete" and "ω_1-subproper". (Unfortunately we inadvertently changed the term "dee-complete" into "dee-proper". We apologize for this and will avoid doing so in the future.) We showed that, under CH, IN$'$ had both properties and, therefore, can be iterated without adding reals. Quite recently we introduced the notion of "almost subcomplete forcing" and proved an iteration theorem for these forcings. Assuming $CH + 2^{\omega_1} = \omega_2$ we showed that IN$'$ and IN* are both almost subcomplete. We then belatedly realized that our proof showed IN$'$, IN* to be, in fact, fully subcomplete. (Embarrassingly, this leaves us with no "real" applications for almost subcomplete forcing.)

Our proof will deal mainly with $\mathbb{N}^*$, though we shall briefly indicate the changes to be made in proving the same result for $\mathbb{N}'$. After that we introduce the basic theory of $\mathcal{L}$-forcing in §3. (For this the reader may also consult [LF], which for present purposes is more suitable than the rather abstruse treatment in [Sing]. In §4 we prove the equivalence of $\mathbb{N}^*$ with an $\mathcal{L}$-forcing. In §5 we then prove the main result. In §6 we discuss further properties of the forcings $\mathbb{N}$, $\mathbb{N}'$, and $\mathbb{N}^*$. Throughout this paper - except in §3 - we assume CH. In §5 we also assume: $2^{\omega_1} = \omega_2$.

Bibliography

[PIF] Proper and Improper Forcing

[LF] $\mathcal{L}$-Forcing

[SPSC] Subproper and Subcomplete Forcing

[Sing] Singapore Notes

[FCH] Forcing Axioms Compatible with CH

[ITSC] Iteration Theorems for Subcomplete and Related Forcings

[EN] The Extended Namba Problem

[DSP] deo Subproper Forcing