§4 Improvements

§4.1 Active Mice

For the sake of simplicity we have up till now assumed that $\mathcal{I}$ is a truncation free iteration of a passive premouse. We now drop these assumptions and show how the proof can be modified. Since both branches have at most finitely many truncations, we may w.l.o.g. assume the ordinal α to be so chosen that no $j \in b_h \setminus \xi_h$ is a truncation point ($h = 0, 1$). (Thus $\pi_{i, b_h} : M_i \longrightarrow M_{b_h}$ will still be a total function on M_j.) However, M_i may not be passive, either because M_0 was not passive or because a top extender was introduced by an earlier truncation. This vitiates an important element of our proof: We often used:

(*) If $\kappa_i < \kappa < \lambda_i$ and κ is a cardinal in $J_{\lambda_i}^{E^{M_i}}$, then $E_{\nu_i}^{M_i} | \kappa \in M_i$.

(This was because, in fact, $E_{\nu_i}^{M_i} \in M_i$.)

Thus when we set: $F^0 = E_{\nu_{i_0}} | \kappa_{i_1}$, we knew that $F^0 \in M_{i_0}$. Since $\lambda_{\xi_1} \leq \lambda_{i_0}$, we also knew that $\tau_{i_0} = \kappa_{i_1}^{+M_{i_0}}$ and hence that $F^0 \in J_{\tau_{i_1}}^{E^{M_{i_0}}} \in N$.

(*) can fail, however, if ν_i is the top extender of M_i. This means that §1 (10) and with it §1 Lemma 1 may fail. In [NFS] §6 we developed a method of circumventing this problem, which we can also use here to prove a modified version of §1 Lemma 1.

(Note All of the proofs we have done up till now can easily be redone in Steel mice instead of λ-mice. Steel mice have the advantage that the above problem does not occur: Steel's indices are so chosen that the analogue of (*) holds even at top extenders. For λ-mice, however, and for many possible intermediate indexing schemes, the method of [NFS] §6 is needed.)

In the following we shall state a number of facts without proof, for which we refer the reader to [NFS] §6. We also follow the convention used there of writing E^M_{top} to denote the top extender of an active premouse M. We define:

$ND =$ the set of $n > 0$ s.t. $E^{M_{i_n+1}}_{top}$ exists and $\mathrm{crit}(E^{M_{i_n+1}}_{top}) \in [\kappa_{i_{n-1}}, \kappa_{i_n})$.

We then observe that in each b_h there can be at most one n with $i_n \in ND$. Hence we can w.l.o.g. suppose α to be so chosen that $b_h \cap ND = \emptyset$ $(h=0,1)$.

In place of the sequence F^n_{m} of extenders on κ_{i_m} we define a new sequence G^n_m ($n<\omega$) of extenders on a κ^*_m s.t. $\kappa_{i_m} < \kappa^*_m < \kappa_{i_{m+1}}$. We display this procedure for $m=0$.

At $E_{\nu_{i_0}} | \kappa_{i_1} \in M_{i_0}$ we set: $G^n = F^n$ exactly as before. Now let $E_{\nu_{i_0}} | \kappa_{i_1} \notin M_{i_0}$. Then $E_{\nu_{i_0}}$ is the top extender of M_{i_0}. Moreover $\rho^1_{M_{i_0}} \leq \kappa_{i_1}$, since $E_{\nu_{i_0}} | \kappa_{i_1}$ can be coded as a subset of $J^{E^{M_{i_0}}}_{\kappa_{i_1}}$. But then $i_0 > \xi_1$, since otherwise $\kappa_{i_1} < \lambda_{\xi_{i_0}}$, hence $\xi_{i_1} \leq \xi_{i_0}$. Contradiction! Thus i_0 is either a successor or limit ordinal. But then there is a $\gamma+1 \leq_T i_0$ s.t. $\kappa_{i_1} < \lambda_\gamma$. (Otherwise $i_0 = T(i_1+1) = \xi_1$. But, since $\kappa_{i_0} < \kappa_{i_1}$, we then have $\mathrm{crit}(E^{M_{i_1}+1}_{top}) = \kappa_{i_0}$. Hence $1 \in ND$. Contradiction!)

Set: $\gamma = \gamma^1 = \gamma^1_0$ = the least such γ. γ can then be shown to have the following properties:

(a) $\kappa_{i_0} < \kappa_\gamma < \kappa_{i_1}$

(b) $\pi_{\gamma+1, i_0} : M_{\gamma+1} \to M_{i_0}$ is a total map

(i.e. there is no truncation between $\gamma+1$ and i_0)

If $E^{M_\gamma}_{\nu_\gamma} | \kappa_{i_1} \in M_\gamma$, we set: $G^0 = E^{M_\gamma}_{\nu_\gamma} | \kappa_{i_1}$.
Then $\kappa_{i_1}^{+M_\gamma} = \kappa_{i_1}^{+M_{i_0}} = \kappa_{i_1}^{+N}$ and we conclude that $G^0 \in N$. If $E^{M_\gamma}_{\nu_\gamma} | \kappa_{i_1} \notin M_\gamma$, then $E^{M_\gamma}_{\nu_\gamma}$ is the top extender of M_γ, and we repeat the process with γ in place of i_0. In this way we obtain a descending sequence: $i_0 = \gamma^0 > \gamma^1 > \gamma^2 > \ldots$.
The sequence must terminate at some integer p. We then set:

$$\bar{\gamma} = \gamma^p , \quad G^0 = E^{M_{\bar{\gamma}}}_{\nu_{\bar{\gamma}}} | \kappa_{i_1} .$$

Then $G^0 \in M_{\bar{\gamma}}$ and, in fact, $G^0 \in N$. We also have $\kappa_{i_0} \leq \bar{\kappa} < \kappa_{i_1}$, where $\bar{\kappa} = \kappa_{\bar{\gamma}}$. Moreover each of the maps π_h is total on M_{γ^h}, where $\pi_0 = \mathrm{id} \restriction M_{i_0}$, $\pi_{h+1} = \pi_h \pi_{\gamma^{h+1}, \gamma^h}$ for $h < p$.

Just as before we define:

$$G^{n+1} = \overline{\pi}_{\xi_{n+1}, i'_{n+1}+1} (G^n) \mid \kappa_{i'_{n+2}}$$

for $n<\omega$. Hence $G^n \in N$ for $n<\omega$, verifying the strongness of $\overline{\kappa} = \overline{\kappa}_0$ in N.

In place of §1 Lemma 1 we then get:

<u>Lemma 1</u> If $B \subset N$ is captured at m, then $\overline{\kappa}_m$ is strong in $\langle N, B \rangle$.

proof (sketch).

B can be easily coded as a subset of δ, so suppose $B \subset \delta$. We display the proof for $m=0$, showing:

<u>Claim</u> $\pi_m(B \cap \overline{\kappa}) = B \cap \kappa_{i'_{n+1}}$ $(n<\omega)$,

where $\pi_m : N \xrightarrow[G^m]{} N'$.

The case $m>0$ is exactly as before, so let $m=0$. If $\overline{\gamma} = i_0$, then $G^0 = F^0$ and the proof is exactly as before. Now let e.g. $\overline{\gamma} = \gamma^1$. Then $\overline{\kappa} = \kappa_\gamma$ where $\gamma = \gamma^1$.

Let $\xi = T(\gamma+1)$. Then

$$\pi_{\xi, i_0}(B \cap \kappa_\gamma) = \pi_{\xi, \beta+1}(B \cap \kappa_\gamma),$$

since $\pi_{\beta+1, i_0} \upharpoonright \lambda_\gamma = \mathrm{id}$. Hence:

$$\pi^0(B \cap \kappa_\gamma) = \pi_{\xi, i_0}(B \cap \kappa_\gamma) \cap \kappa_{i_1}.$$

Let $\pi_{\xi, i_0} : M^*_\xi \to M_{i_0}$.

Then π_{ξ, i_0} takes $E^{M^*_\xi}_{top}$ to $E^{M_{i_0}}_{top}$.

Thus, if we let:

$$\sigma : M_{i_0} \xrightarrow[E_{top}]{} M, \quad \bar\sigma : M^*_\xi \xrightarrow[E_{top}]{} \bar M,$$

Then $\pi_{\xi, i_0} \bar\sigma = \sigma \pi_{\xi, i_0}$. Thus:

$$\pi_{\xi, i_0}(B \cap \kappa_\gamma) = \pi_{\xi, i_0}(\bar\sigma(B \cap \kappa_{i_0}) \cap \kappa_\gamma) =$$

$$= \sigma(B \cap \kappa_{i_0}) \cap \lambda_\gamma = B \cap \lambda_\gamma$$

(since $\kappa_{i_0} < \kappa_\gamma$).

Hence:

$$G^0(B \cap \kappa_\gamma) = \pi_{\xi, \gamma+1}(B \cap \kappa_\gamma) \cap \kappa_{i_1} =$$

$$= \pi_{\xi, i_0}(B \cap \kappa_\gamma) \cap \kappa_{i_1} = B \cap \lambda_\gamma \cap \kappa_{i_1} = B \cap \kappa_{i_1}.$$

This proves the case $\bar\gamma = \gamma^1$. If $\bar\gamma < \gamma^1$, we iterate this argument, showing:

$$\overline{\pi}_{\xi,\gamma^{h+1}}(B \cap \kappa_{\gamma^h}) \cap \kappa_{i_1} = B \cap \kappa_{i_1}$$

for $h \leq p$, where $\bar{\gamma} = \gamma^p$, by induction on h. QED (Lemma 1)

(<u>Note</u> The full details can be read in [NFS] §6. The assumption on $B \subset N$ is weaker than here, but the proof is exactly the same.)

Lemma 2 holds as stated in §1, but its proof must be amended:

<u>Lemma 2</u> Let $B \subset N$ be captured at m. Set $\tilde{N} = \langle N, B \rangle$. Then $\tilde{N} | \kappa_{i_m} \prec_{\Sigma_1} \tilde{N}$.

proof.

Let $x \in J^E_{\kappa_{i_m}}$ and $\tilde{N} \models \varphi(x)$, where φ is Σ_1

<u>Claim</u> $\tilde{N} | \kappa_{i_m} \models \varphi(x)$.

Arguing as in §1 but using the amended form of Lemma 1, we first get:

(1) $\tilde{N}|\bar{\kappa}_m \models \varphi(x)$.

But $\bar{\kappa}_m < \kappa_{i_{m+1}} < \lambda_{i_m}$. Since $\tilde{N}|\bar{\kappa}_m \prec_{\Sigma_0} \tilde{N}|\lambda_{i_m}$, we conclude;

(2) $\tilde{N}|\lambda_{i_m} \models \varphi(x)$.

But $\pi_{\xi_m i_{m+1}}(\tilde{N}|\kappa_{i_m}) = \tilde{N}|\lambda_{i_m}$, since $\mathrm{crit}(\pi_{i_m+1, b_m}) \geq \lambda_{i_m}$. Hence $\tilde{N}|\kappa_{i_m} \prec \tilde{N}|\lambda_{i_m}$ and

(3) $\tilde{N}|\kappa_{i_m} \models \varphi(x)$. QED (Lemma 2)

Lemma 3 and Cor 4 of §1 then follow exactly as before.
The proofs in §3 go through virtually unchanged.

§ 4.2 E-Woodinness

<u>Def</u> Let $M = J_\rho(N) = J^{EN}_{\theta+\rho}$, where $N = J^E_\theta$ is a premouse. θ is <u>E-Woodin</u> in M iff it is Woodin as instanced by extenders lying on the sequence given by E.

More precisely:

<u>Def</u> Let $F \in N$ be an extender of length μ. F <u>conforms</u> to N iff there is $\nu \in N$ s.t.

- $F = E_\nu | \mu$ where $\kappa < \mu < \lambda$, $\kappa = \mathrm{crit}(E_\nu)$, $\lambda = E_\nu(\kappa)$
- F generates E_ν – i.e. $E_\nu = \pi \upharpoonright \mathbb{P}(\kappa) \cap J^E_\tau$ where $\tau = \kappa^{+J^E_\nu}$ and $\pi : J^E_\tau \xrightarrow[F]{} J^E_\nu$.

<u>Def</u> κ is E-<u>strong</u> in $\tilde{N} = \langle N, B \rangle$ iff for arbitrarily large $\lambda \in N$ there is $F \in N$ which is strong wrt. $\tilde{N}$ and conforms to N.

<u>Def</u> θ is E-<u>Woodin</u> in $M = J_\rho(N)$ iff for every $B \subset N$ s.t. $B \in M$ there is $\kappa \in N$ which is E-strong wrt $\langle N, B \rangle$

We can strengthen the conclusion of Thm 1 § 3 from "Woodin" to "E-Woodin" if we assume that the M_0 which we are iterating is not only a premouse, but also "mouselike" in the sense that it internally satisfies the <u>condensation</u> <u>lemmas</u> of [CR] § 8 Lemma 4'. In fact, the only consequence of those lemmas we need is:

(*) Let $E_\nu \neq \emptyset$. Let $\kappa = \mathrm{crit}(E_\nu)$, $\kappa < \eta < \lambda = E_\nu(\kappa)$ s.t. η is a limit cardinal in J_λ^E. Then $E_\nu | \eta$ conforms to M_0.

Since (*) holds in M_0, it also holds in N. But that is enough to tell us that the F^n $(n < \omega)$ which verified the $\langle N, B \rangle$-strongness of κ_{i_0} in the proof of § 1 Lemma 1 are all N-conforming. Similarly it tells us that the extenders G^n ($n<\omega$) used in the revised version of in §4.1 are N-conforming.