§1 Preliminaries

Let $\mathcal{I} = \langle\langle M_i\rangle, \langle \pi_{ij}\rangle, \langle \nu_i\rangle, T\rangle$ be a Σ^*-iteration of a passive premouse M_0. We assume that $\mathcal{I}$ is of limit length $\lambda = lh(\mathcal{I})$ and that it has two distinct cofinal well founded branches b_0 and b_1. Fix $\alpha < \lambda$ s.t. $(b_0 \setminus \alpha) \cap (b_1 \setminus \alpha) = \emptyset$. (In the sequel we shall occasionally place further requirements on α.)

Define $\langle i_n \mid n < \omega\rangle$ by:

Def i_0 = the least i s.t. $i+1 \in b_0 \setminus \alpha$

i_{2n+1} = the least i s.t. $i+1 \in b_1$ and $i > i_{2n}$

i_{2n+2} = " " " " " b_0 " $i > i_{2n+1}$.

Then $\langle i_n \mid n < \omega\rangle$ is monotone. Moreover $\sup\limits_n i_n = \lambda$, since otherwise $\sup\limits_n i_n \in (b_0 \setminus \alpha) \cap (b_1 \setminus \alpha)$.

Set: $\xi_n = T(i_n + 1)$.

We note that $\xi_{2n+1} \leq i_{2n}$,

$\xi_{2n+1} \neq i'_{2n}+1$, since otherwise $\xi_{2n+1} \in b_0 \cap b_1$, α.

since otherwise $\xi_{2n+1} \in b_1$, $\xi_{2n+1} > i'_{2n}+1$. But then $i'_{2n+1}+1 \le \xi_{2n+1}$, since $i'_{2n+1}+1$ is the least $\eta \in b_1$ s.t. $\eta > i'_{2n}+1$. Contradiction! Similarly we get $\xi_{2n+2} \le i'_{2n+1}+1$. Thus

(1) $\xi_{n+1} \le i'_n$ for $n < \omega$,

But then

(2) $\kappa_{i_{n+1}} < \lambda_{\xi_{n+1}} \le \lambda_{i'_n} < \kappa_{i_{n+2}}$

($\lambda_{i'_n} \le \kappa_{i_{n+2}}$, since $(i'_n+1) . T (i'_{n+2}+1)$)

In particular $\kappa_{i_n} < \lambda_{\xi_n} < \kappa_{i_{n+1}}$ for $n \ge 1$. But we may assume α so chosen that

(3) $\kappa_{i_n} < \lambda_{\xi_n} < \kappa_{i_{n+1}}$ for $n < \omega$,

(If not, replace α by $\alpha' = i_{2p}+1$ for any $p > 0$. Our definition then yeild a new sequence $\langle i''_n \mid n < \omega \rangle$. By induction on n we then get: $i''_n = i_{2p+n}$ for $n < \omega$.)

We have: $(i'_n + 1) \mathrel{T} (i'_{n+2} + 1)$, hence:
hence $(i'_n + 1) \underset{T}{\leq} \xi_{n+2}$.

Thus $\xi_n < \xi_{n+1}$ for $n \geq 1$, since $\xi_{n+1} \leq i'_n < \xi_{n+2}$. Arguing as above we can suppose α to be so chosen that:

(4) $\xi_n < \xi_{n+1}$ for $n < \omega$.

It is occasionally useful to use the convention:

<u>Def</u> $b_n = \begin{cases} b_0 & \text{if } n \text{ is even} \\ b_1 & \text{if } n \text{ is odd} \end{cases}$

Thus:

(5) $i'_n + 1 \in b_n$ for $n < \omega$,

We recall some basic notions.

Let $M = \langle J_\alpha^A, B\rangle$ be amenable. We say that F is an <u>extender</u> <u>at</u> κ <u>on</u> M of <u>length</u> λ, determining the <u>ultrapower</u> M' with <u>canonical embedding</u> π (in symbols: $\pi : M \xrightarrow[F]{} M'$) iff the following hold:

- $\lambda > \kappa$ is p.r. closed.
- $\pi : M \xrightarrow[\Sigma_0]{} M'$ cofinally, where M' is transitive
- $\kappa = \operatorname{crit}(\pi)$ and $\lambda \leq \pi(\kappa)$
- M' is the Σ_0-closure of $\operatorname{rng}(\pi) \cup \lambda$ in itself
- $F : \mathbb{P}(\kappa) \cap M \to \mathbb{P}(\lambda)$ is defined by
  $E(X) = \pi(X) \cap \lambda$.

π, M' are then uniquely determined by E, M.

If $\kappa < \bar\lambda \leq \lambda$ and $\bar\lambda$ is p.r. closed, we define $F|\bar\lambda$ of length $\bar\lambda$ by:

$$F|\bar\lambda(X) = F(X) \cap \bar\lambda$$

for $X \in \mathbb{P}(\kappa) \cap M$.

$F|\bar\lambda$ is then also an extender on M.
Moreover if : $\tilde\pi : M \xrightarrow[F|\lambda]{} \tilde M$, there is
$k : \tilde M \to M$ defined by:

$$k(\tilde\pi(f)(\xi)) = \pi(f)(\xi)$$

for $\xi < \bar\lambda$ and $f \in M$, $f : \kappa \to M$. It follows that:

$$k : \tilde M \xrightarrow[\Sigma_0]{} M' \text{ cofinally}$$

and:

$$\tilde M|\bar\lambda = M'|\bar\lambda \ , \ k\restriction(\tilde M|\lambda) = id,$$

where: $\langle J_\alpha^A, B\rangle|\lambda =_{Df} \langle J_\lambda^A, B \cap J_\lambda^A\rangle$

Moreover k is the unique map s.t. $k : \tilde M \xrightarrow[\Sigma_0]{} M'$, $k\restriction\bar\lambda = id$, and $k\tilde\pi = \pi$.

(These are well known facts which we state without proof.)

If M is acceptable, then the Σ^*-ultrapower M' and the canonical Σ^*-embedding π are defined as in [MO].
(In symbols $\pi : M \xrightarrow[F]{*} M'$.)
Analogues of the above facts continue to hold.

Note If F is an extender on $M = J_\alpha^A$ and $\langle M, B\rangle$ is amenable, then F is an extender on $\langle M, B\rangle$.

We recall that E is called ω-complete iff whenever $W \subset \mathcal{P}(\kappa) \cap M$ and $Y < \lambda$ are countable, then there is $\sigma : Y \to \kappa$ s.t. $\sigma(\xi) \in X \leftrightarrow \xi \in E(X)$ for all $X \in W$, $\xi \in \lambda$. We know that if E on M is ω-complete and $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap M'$, then E is also an extender on M'.

Hence:

(6) Let $N = \langle J_\delta^E, B\rangle$ be acceptable. Let $F \in N$ be an extender on $M = (J_{\kappa^+}^E)^N$ at κ. If N thinks that F is ω-complete, then F is an extender on N.

proof.

For $\kappa^+ < \eta$ in N we know that $\pi_\eta : (N|\eta) \xrightarrow[F]{} (N|\eta)'$ exists. Set $\pi = \bigcup_\eta \pi_\eta$. Then $\pi : N \xrightarrow[F]{} N'$ where $N' = \bigcup_{\eta < \delta} (N|\eta)'$ QED

Def Let $N = \langle J_\delta^E, B\rangle$ be acceptable.
Let $\kappa \in N$ and let F be an
extender on $M = (J_{\kappa^+}^E)^N$ at κ. Let
$\kappa^{+M} < \lambda \in N$. F is λ-<u>strong</u> wrt. N
iff $lh(F) = \lambda$ and, letting $\pi : M \to_F M'$,
we have: $J_\lambda^{E^N} = J_\lambda^{E^{M'}} \wedge B \cap J_\lambda^{E^N} = B \cap J_\lambda^{E^{M'}}$

Note that if F is λ-strong, $\kappa < \bar\lambda < \lambda$ and $\bar\lambda$ is p.r. closed, then $F|\bar\lambda$ is $\bar\lambda$-strong.

By (6) we get:

(7) Let N be as above. Let $F \in N$ be an extender on $M = (J_{\kappa^+}^E)^N$ at κ which is λ-strong where $\lambda > \kappa$ is the length of F. If λ is regular in N, then F is an extender on N.

prf.
We show that F is ω-complete in N.
Let $W \subset \mathbb{P}(\kappa)$, $X \subset \lambda$ be countable in N. Let $\langle \xi_i \mid i < \omega\rangle$ enumerate W
and $\langle Y_i \mid i < \omega\rangle$ enumerate X in N.
Set $u = \{\langle i, j\rangle \mid \xi_i \in Y_j\}$. Then
$u \in \mathbb{P}(\omega) \subset M$ and $\langle Y_i \mid i < \omega\rangle \in M$
in N. It suffices to show
that there is $\sigma : \omega \to \kappa$ in M with
$\sigma(i) \in Y_j \longleftrightarrow \langle i, j\rangle \in u$ for $i, j < \omega$

Since $\pi : M \xrightarrow[F]{} M'$ is an elementary embedding, it suffices to show that the same statement holds of:

$$\pi(u) = u \quad , \quad \pi(\langle Y_i \mid i<\omega\rangle) = \langle \pi(Y_i) \mid i<\omega\rangle$$

in M'. But that does hold if we take $\sigma = \langle \xi_i \mid i<\omega\rangle$, where $\sigma \in H_\lambda^N = J_\lambda^E \in M'$.

QED (7)

<u>Def</u> Let $N = \langle J_\delta^E, B\rangle$ be acceptable. Let $\kappa \in N$. κ is <u>strong</u> in N iff for arbitrarily large $\lambda \in N$ there is an extender $F \in N$ at κ on $(J_{\kappa^+}^E)^N$ which is λ-strong wrt. N.

(We call κ strong in $N = J_\delta^E$ iff it is strong in $\langle N, \emptyset\rangle$.)

Trivially we have:

(8) Let κ be strong in N. Then for <u>every</u> p.r. closed $\lambda > \kappa$ in N there is an $F \in N$ at κ which is λ-strong wrt. N

prf: Let $F' \in N$ be λ'-strong where $\lambda \leq \lambda'$. Set $F = F' | \lambda$. QED (8)

(9). Let N be acceptable s.t. $N \models$ There are arbitrarily large cardinals. Let κ be strong in N. Then for every p.r. closed $\lambda > \kappa$ in N there is a λ-strong $F \in N$ at κ which is an extender on N.

proof.

Let $\tau \geq \kappa$ be regular in N. Let $F' \in N$ be τ-strong. Then F' is an extender on N by (7). Hence so is $F = F' | \lambda$.

QED (9)

We now return to our iteration $\mathcal{I}$ and the set $N = J_\delta^E$ defined at the outset. *)

We note:

(10) $(E_{\nu_{i_n}}^{M_{i_n}} | \kappa_{i_{n+1}}) \in N$ for $n < \omega$.

prf.

Since there is no truncation we know that $\tau_{i_{n+1}} = \kappa_{i_{n+1}}^+$ and $H_{\tau_{i_{n+1}}} = J_{\tau_{i_{n+1}}}^E$ **)

in M_j for all $j \geq \xi_{n+1}$. But $\xi_{n+1} \leq i_n$. Hence, taking $j = i_n$, we conclude:

$$(E_{\nu_{i_n}}^{M_{i_n}} | \kappa_{i_{n+1}}) \in J_{\tau_{i_{n+1}}}^E \subset N. \quad \text{QED (8)}$$

*) $\delta = \sup_{i<\lambda} \kappa_i = \sup_{i<\lambda} \nu_i$; $N = J_\delta^E =$

$$= \bigcup_{i<\lambda} J_{\kappa_i}^{E^{M_i}} = \bigcup_{i<\lambda} J_{\lambda_i}^{E^{M_i}}$$

**) $\tau_i =_{\text{df}} (\kappa_i^+)^{J_{\nu_i}^{E^{M_i}}}$

Clearly we can choose α large enough that if $\delta \in M_{b_m}$, then $\delta \in \operatorname{rng}(\pi_{\xi_m, b_m})$ (where, as before, $N = J_\delta^E$). From now on suppose α to be so chosen. For $m < \omega$ set: $b_m^* = b_m \setminus \xi_m$. Then $b_m^* \cap b_{m+1}^* = \emptyset$ and $b_{m+2}^* \subset b_m^*$.

Def Let $i \in b_h^*$ $(h = 0, 1)$

- If $\delta \in M_{b_h}$ set:

$$N_i = \pi_{i b_h}^{-1}(N) = J_{\delta_i}^{E M_i}$$

$$\bar{\pi}_{i b_h} = \pi_{i b_h} \restriction N_i$$

$$\bar{\pi}_{i l} = \pi_{i l} \restriction N_i \quad \text{for } i \leq l \in b_h.$$

- If $\delta \notin M_{b_h}$, then $N = M_{b_h}$ and we set: $N_i = M_i$, $\bar{\pi}_{i b_h} = \pi_{i b_h}$, and $\bar{\pi}_{i l} = \pi_{i l}$ for $i \leq l \in b_h$.

If $B \subset N$, $i \in b_h$, we also set:

$B_i = \bar{\pi}_{i b_h}^{-1} {}'' B$.

We now introduce an important concept:

<u>Def</u> Let $B \subset N$. B is <u>captured</u> <u>at</u> $n < \omega$ iff $\langle N, B\rangle$ is amenable and for all $j \in b_n^* \cup b_{n+1}^*$ we have:

- $\langle N_j, B_j\rangle$ is amenable
- $\bar{\pi}_{j\, b_m} : \langle N_j, B_j\rangle \to_{\Sigma_0} \langle N, B\rangle$ if $j \in b_m^*$, $m \geq n$

It is easily seen that:

(11) If B is captured at n, then it is captured at every $m \geq n$.

(12) $\emptyset$ is captured at 0. Moreover, if $B \subset N$ and $B \in M_{b_0} \cap M_{b_1}$, then B is captured at some n.

(13) If B is captured at n and B' is $\Sigma_0(\langle N, B\rangle)$ in a parameter $p \in J^E_{\kappa_{i n}}$, then B' is captured at n.

<u>Lemma 1</u> If $B \subset N$ is captured at n, then κ_{i_n} is strong in $\langle N, B\rangle$.

proof.

We display the proof for $n=0$.

We construct a sequence F^m $(m<\omega)$ of extenders on κ_{i_0} s.t. $lh(F^m) = \kappa_{i_{m+1}}$.

We show that $F^m \in N$ is $\kappa_{i_{m+1}}$ - strong for $\langle N, B\rangle$.

<u>Case 1</u> $m=0$

$F^0 =_{\mathrm{Df}} E_{\nu_{i_0}} | \kappa_{i_1}$. Then $F^0 \in N$ by (10).

$B \cap \kappa_{i_0} = B_{\xi_0} \cap \kappa_{i_0}$, since $crit(\pi_{\xi_0, b_0}) = \kappa_{i_0}$

Let $\pi : \langle J^E_{\tau_{i_0}}, B_{\xi_0} \cap J^E_{\tau_{i_0}}\rangle \xrightarrow[E_{\nu_{i_0}}]{} \langle J^E_{\nu_{i_0}}, B'\rangle$

Then $\pi = \pi_{\xi_0, i_0+1} \upharpoonright J^E_{\tau_{i_0}}$, Hence:

$$B' = \bigcup_{x \in J^E_{\tau_{i_0}}} \pi(x \cap B_{\xi_0}) = B_{i_0+1} \cap J^E_{\nu_{i_0}}.$$

Hence $B' \cap \lambda_{i_0} = B \cap \lambda_{i_0}$, since $crit(\pi_{i_0+1, b_0}) \geq \lambda_{i_0}$. Hence

if $\pi : \langle J^E_{\tau_{i_0}}, B_{\xi_0} \cap J^E_{\tau_{i_0}}\rangle \xrightarrow[F^0]{} \langle J^{E'}_{\nu'}, B''\rangle$

we have: $J^{E'}_{\kappa_{i_1}} = J^E_{\kappa_{i_1}}$ and:

$B'' \cap J^E_{\kappa'_{i_1}} = B' \cap J^E_{\kappa'_{i_1}} = B \cap J^E_{\kappa'_{i_1}}$, since $\kappa'_{i_1} < \lambda_{i_0}$. QED (Case 1)

Case 2 $n = m + 1$.

Let $\pi : \langle J^E_{\tau_{i_0}}, B \cap J^E_{\tau_{i_0}} \rangle \longrightarrow_{F^m} \langle J^{E'}_{\nu'}, B' \rangle$.

We know: $J^{E'}_{\kappa'_{i_m}} = J^E_{\kappa_{i_m}}$ and $B' \cap J^{E'}_{\kappa'_{i_m}} = B \cap J^{E'}_{\kappa'_{i_m}}$.

Since $F^m \in N$, we have:

$F^m \in J^E_{\tau_{i_m}}$, since $\tau_{i_m} = (\kappa^+)^N$.

Arguing as above with n in place of 0 we have:

$\tilde{\pi} : \langle J^E_{\tau_{i_m}}, B \cap J^E_{\tau_{i_m}} \rangle \longrightarrow_{E_{\nu_{i_m}}} \langle J^E_{\nu_{i_m}}, \tilde{B} \rangle$

with $\tilde{B} = B_{i_m+1} \cap J^E_{\nu_{i_m}}$, where

$\tilde{\pi} = \pi_{\xi_n, b_n} \upharpoonright J^E_{\tau_{i_m}}$. Hence

$\tilde{B} \cap J^E_{\lambda_{i_m}} = B \cap J^E_{\lambda_{i_m}}$, since

$\mathrm{crit}(\pi_{i_m+1, b_n}) \geq \lambda_{i_m}$.

Set:

$F'^m =_{Df} \tilde{\pi}(F^m) | \kappa_{i_{m+1}}$

Clearly $\tilde{\pi}(F^m) \in J^E_{\nu_{i_m}} \subset N$; hence $F^{m+1} \in N$. Clearly $\pi, \langle J^{E'}_{\nu'}, B' \rangle \in J^E_{\tau_{i_m}}$, since $F^m \in J^E_{\tau_{i_m}}$ and $J^E_{\tau_{i_m}}$ is a ZFC$^-$ model. Let $\pi' = \tilde{\pi}(\pi)$ and $\langle J^{E''}_{\nu''}, B'' \rangle = \tilde{\pi}(\langle J^{E'}_{\nu'}; B' \rangle)$. Then

$$\pi' : \langle J^E_{\tau_{i_0}}, B \cap J^E_{\tau_{i_0}} \rangle \xrightarrow[\tilde{\pi}(F^m)]{} \langle J^{E''}_{\nu''}, B'' \rangle .$$

Since $J^{E'}_{\kappa_{i_m}} = J^E_{\kappa_{i_m}}$, $B' \cap J^E_{\kappa_{i_m}} = B \cap J^{E'}_{\kappa_{i_m}}$ and $\tilde{\pi} : \langle J^E_{\tau_{i_m}}, B \cap J^E_{\tau_{i_m}} \rangle \prec \langle J^E_{\nu_{i_m}}, \tilde{B} \rangle$, where $\tilde{\pi}(\kappa_{i_m}) = \lambda_{i_m}$, we conclude:

$J^{E''}_{\lambda_{i_m}} = J^E_{\lambda_{i_m}}$ and:

$$B'' \cap J^E_{\lambda_{i_m}} = \tilde{B} \cap J^E_{\lambda_{i_m}} = B \cap J^m_{\lambda_i} .$$

But $\kappa_{i_{m+1}} < \lambda_{i_m}$. Hence, if

$$\bar{\pi} : \langle J^E_{\tau_{i_0}}, B \cap J^E_{\tau_{i_0}} \rangle \xrightarrow[F^m]{} \langle J^{\bar{E}}_{\bar{\nu}}, \bar{B} \rangle ,$$

we have $\bar{B} \cap \kappa_{i_{m+1}} = \tilde{B} \cap \kappa_{i_{m+1}} = B \cap \kappa_{i_{m+1}}$.

QED (Lemma 1)

By Lemma 1 and (13) we get:

Lemma 2 Let B be captured at n. Set $\tilde{N} = \langle N, B \rangle$. Then $\tilde{N} | \varkappa_{i_n} \prec_{\Sigma_1} \tilde{N}$.

(Where $\tilde{N} | \varkappa =_{\mathrm{nf}} \langle J^E_\varkappa, B \cap J^E_\varkappa \rangle$).

proof.

It suffices to show:

$\tilde{N} \models \bigvee y\, \varphi(y, x) \rightarrow \tilde{N} \models \bigvee y \in J^E_{\varkappa_{i_n}} \varphi(y, x)$

if $x \in J^E_{\varkappa_{i_n}}$ and φ is Σ_0.

Suppose not. Let φ, x be a counterexample. Set:

$B' = \{ y \mid \tilde{N} \models \varphi(y, x) \}$.

Then B' is captured at n. Hence $\varkappa_{i_n}$ is strong wrt. $\tilde{N}' = \langle N, B' \rangle$.

Let $y \in B'$, $y \in J^E_\lambda$. (Hence $\lambda > \varkappa_{i_n}$.) By Lemma 1 there is an extender F on $\varkappa_{i_n}$ in N s.t.

$\pi_F(B' \cap J^E_{\varkappa_{i_n}}) \cap J^E_\lambda = B' \cap J^E_\lambda$.

But $B' \cap J^E_{\varkappa_{i_n}} = \emptyset$, hence $\pi_F(B' \cap J^E_{\varkappa_{i_n}}) = \emptyset$, whereas $B' \cap J^E_\lambda \neq \emptyset$. Contr!

QED (Lemma 2)

Before proceeding further, we place another requirement on the ordinal α.

Def b_h is of type A $(h = 0, 1)$ iff for all $j \in b_h \setminus 3_h$ we have: $\sup \pi_{j b_h}{}'' \delta_j < \delta$.

Otherwise it is of type B.

From now on assume w.l.o.g. that α is large enough that $\sup \pi_{j, b_h}{}'' \delta_j = \delta$ whenever b_h is of type B and $j \in b_h \setminus 3_h$.

(14) If b_h is of type A, then:

(a) δ is Σ^*-regular in M_{b_h} (i.e. if $n > \omega$ and f is a good partial $\Sigma_1^{(n)}(M_{b_h})$ map of a $\bar{\tau} < \delta$ to δ, then $\sup f'' \tau < \delta$).

(b) $\delta \in M_{b_h}$ and there is $\nu > \delta$ s.t. $E_\nu \neq \emptyset$ in M_{b_h}.

proof

(a) Suppose not. Let $f, \tau < \delta$ be a counter-example. Let f be a good $\Sigma(M_{b_h})$ map in parameter p. Let $j \in b_h \setminus 3_h$ s.t. $\pi_{j b_h}(\bar{\tau}) = \tau$ and $\pi_{j b_h}(\bar{p}) = p$. Let $\bar{f}$ be $\Sigma_1^{(n)}(M_j)$ in $\bar{p}$ by the same functionally absolute definition. Then $\pi_{j, b_h}(\bar{f}(\xi)) \simeq f(\xi)$ for $\xi < \bar{\tau}$. Hence $\sup \pi_{j, b_h}{}'' \delta_j = \delta_j$. Contr!

(b) Suppose not. Then $\nu \leq \delta$ for $E_\nu \neq \emptyset$ in M_{b_h}. Hence $\nu < \delta$ for such ν, since otherwise N would have a largest cardinal. But then if $j \in b_h$, $\nu < \delta_j$ for $E_\nu \neq \emptyset$ in M_j, since $\pi_{j b_h} : M_j \to_{\Sigma^*} M_{b_h}$. Let $j = T(i+1)$ where $i+1 \in b_h$. Then $\kappa_i < \delta_j$, since otherwise $\delta_j < \kappa_i < \lambda_i < \nu_j$ where $E_{\nu_j} \neq \emptyset$ in M_j. But δ_j is Σ^*-regular in M_j. Hence $\pi_{j, i+1}$ takes δ_j cofinally to δ_{i+1}. Since this happens at all successor points of b_h, it follows that $\pi_{j b_h}$ takes δ_j cofinally to δ for $j \in b_h \setminus \xi_h$. Contradiction!

QED (8)

Def $B \subset N$ is strongly captured at n iff

(a) B is captured at n

(b) If b_h is of type A ($h = 0, 1$) and $j \in b_h \setminus \xi_m$, then $B \in \mathrm{rng}(\pi_{j, b_h})$.

Lemma 3 Let B be strongly captured at n. Let B' be $\Sigma_1(\langle N, B\rangle)$ in $p \in J_{\kappa_{i_n}}$. Then B' is strongly captured at n.

proof.

Claim 1 If b_h is of type A and $j \in b_h \setminus \xi_m$, then $B' \in \mathrm{rng}(\pi_{j, b_h})$

prf.

Let $\pi_{j, b_h}(\langle N_j, B_j\rangle) = \langle N, B\rangle$. Let B_j' be $\Sigma_1(\langle N_j, B_j\rangle)$ in p by the same definition. Then $\pi_{j, b_h}(B_j') = B'$.

QED (Claim 1)

This is all we need to prove about b_h of type A. Now let b_h be of type B.

Claim 2 $\langle N, B'\rangle$ is amenable.

proof.

Let $u \in N$, $u \in J^E_{\kappa_{i_m}}$, where $n \le m$.

Then $B' \cap J^E_{\kappa_{i_m}}$ is $\tilde{N}|\kappa_{i_m}$ - definable by Lemma 2, where $\tilde{N} = \langle N, B \rangle$. Hence $B' \cap J^E_{\kappa_{i_m}} \in N$ and $u \cap B' = (B' \cap J^E_{\kappa_{i_m}}) \cap u \in N$.

QED (Claim 2)

Now let $j \in b_h \setminus \xi_n$. Let $B'_j = (\bar{\pi}_{j, b_h}^{-1})^{-1} {}'' B'$. Since $\bar{\pi}_{j, b_h} : \tilde{N}_j \to_{\Sigma_1} \tilde{N}$, where $\tilde{N}_j = \langle N_j, B_j \rangle$ and $B_j = \bar{\pi}_{j, b_h}^{-1} {}'' B$, we know that B'_j is $\Sigma_1(\tilde{N}_j)$ in p by the same definition.

<u>Claim 3</u> $\langle N_j, B'_j \rangle$ is amenable.

proof.

Let $u \in N_j$. We claim that $B'_j \cap u \in N_j$. Let $\bar{\pi}_{j b_h}(u) \in J^{E^N}_{\kappa_{i_n}}$, where $n \leq m$. Let $\bar{\kappa}$ be least s.t. $\bar{\pi}_{j, b_h}(\bar{\kappa}) \geq \kappa_{i_m}$.

<u>Subclaim 3.1</u> $\tilde{N}_j | \bar{\kappa} \prec_{\Sigma_1} \tilde{N}_j$, where $\tilde{N}_j = \langle N_j, B_j \rangle$.

proof.
Let φ be Σ_1. Let $x \in \tilde{N}_j | \bar{\kappa}$. Then Then $\bar{\pi}(x) \in \tilde{N} | \kappa_{i_m}$, where $\bar{\pi} = \bar{\pi}_{j, b_h}$. Thus:

$$\tilde{N}_i \models \varphi(x) \to \tilde{N} \models \varphi(\bar{\pi}(x))$$

$$\to (\tilde{N} | \kappa_{i_m}) \models \varphi(\bar{\pi}(x))$$

$$\to (\tilde{N} | \bar{\pi}(\bar{\kappa})) \models \varphi(\bar{\pi}(x))$$

(since $\tilde{N} | \kappa_{i_m} \prec_{\Sigma_0} (\tilde{N} | \bar{\pi}(\bar{\kappa}))$

$$\to (\tilde{N}_i | \tilde{\kappa}) \models \varphi(x)$$

since $\bar{\pi}(\tilde{N}_i | \bar{\kappa}) = \tilde{N} | \bar{\pi}(\bar{\kappa})$.

QED (Subclaim 3.1)

But then if $\bar{\kappa}$ is as above, then $B_i' \cap J_{\bar{\kappa}}^{EN_i}$ is $\tilde{N}_i | \bar{\kappa}$ - definable. Hence $B_i' \cap J_{\bar{\kappa}}^{EN_i} \in \tilde{N}_i$. Hence $u \cap B_i' = u \cap (B_i' \cap J_{\bar{\kappa}}^{EN_i}) \in \tilde{N}_i$.

QED (Claim 3)

<u>Claim 4</u> $\bar{\pi}_{i, b_n} : \langle N_i, B_i' \rangle \to_{\Sigma_0} \langle N, B' \rangle$.

proof.

This follows by:

<u>Claim 4.1</u> $\bar{\pi}(B_i' \cap u) = B_i \cap \bar{\pi}(u)$ for $u \in N_i$, $\bar{\pi} = \bar{\pi}_{i, b_n}$.

proof.

Let $u, \bar{\kappa}, i_m$ be as in the proof of Claim 3.

We first note that $\tilde{N}|\kappa \prec_{\Sigma_1} \tilde{N}$, where $\kappa = \bar{\pi}(\bar{\kappa})$. To see this, let $\varphi(x)$ be Σ_1. In $\tilde{N}_j$ we have

$$\bigwedge x \in J_{\bar{\kappa}}^{E}\,(\varphi(x) \to \varphi(x)_{\tilde{N}_j|\bar{\kappa}})$$

which is a Π_1 statement about $\bar{\kappa}$. But then the same Π_1 statement holds of κ in $\tilde{N}$.

Thus $B_j' \cap J_{\bar{\kappa}}^{E^{N_j}}$ is $\tilde{N}_j|\bar{\kappa}$ definable in $\bar{p} = \bar{\pi}^{-1}(p)$ and $B' \cap J_{\kappa}^{E^N}$ is $\tilde{N}|\kappa$-definable in p, where $\bar{\pi}(\tilde{N}_j|\bar{\kappa}) = \tilde{N}|\kappa$. Hence:

$$\bar{\pi}(u \cap B_j') = \bar{\pi}(u \cap B_j' \cap J_{\bar{\kappa}}^{E^{N_j}}) =$$
$$= \bar{\pi}(u) \cap (B' \cap J_{\kappa}^{E^N}) = \bar{\pi}(u) \cap B',$$

QED (~~Lemma~~ Lemma 3)

Iterated application of Lemma 2 and Lemma 3 then gives:

<u>Corollary 4</u> Let B be strongly captured at n. Set: $\tilde{N} = \langle N, B \rangle$. Then

(a) $\tilde{N} | \kappa_{i_n} \prec \tilde{N}$

(b) If B' is $\tilde{N}$-definable in a $p \in J^E_{\kappa_{i_n}}$, then B' is strongly captured at n.

<u>Note</u> Since $\tilde{N} | \kappa_{i_n} \in N$ and κ_{i_n} is inaccessible in N, it follows that $\tilde{N}$ is a ZFC$^-$ model. In particular, N is a ZFC$^-$ model, since $\emptyset$ is strongly captured at every n.