

§3 Examples

§3.1 The transfer lemma for embeddings of ZFC-models

We recapitulate and expand upon some facts developed in [J] §5.

Def Let $M = \langle M, \in, \dots \rangle$ be a transitive ZFC-model. Let $\pi : M \hookrightarrow M'$, where M' is transitive. π is cofinal in M' iff $M' = \bigcup_{u \in M} \pi(u)$. *

In the following, we suppose M, N, \dots to be transitive ZFC-models unless otherwise stated.

Fact 1 Let $\pi : M \hookrightarrow M'$ and set $\tilde{M} = M' \upharpoonright \bigcup_{u \in M} \pi(u)$.

Then $\tilde{M} \hookrightarrow M'$ and $\pi : M \hookrightarrow \tilde{M}$ cofinally.

(The proof uses: Let $x_1, \dots, x_n \in \tilde{M}$, $x_i \in \pi(u_i)$.

Then $\tilde{M} \models \varphi(x_1, \dots, x_n) \leftrightarrow \langle x_1, \dots, x_n \rangle \in X$,

where $X = \{ \langle \vec{z} \rangle \in U_1 \times \dots \times U_n \mid M \models \varphi(\vec{z}) \}$.)

Hence:

Fact 2 Let $\sigma > \omega$ be regular in M , where

$\pi : M \hookrightarrow M'$. Set $\bar{H} = H_\sigma^M$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$,

$\bar{\pi} = \pi \upharpoonright \bar{H}$. Then $\bar{\pi} : \bar{H} \hookrightarrow \tilde{H}$ cofinally.

Def Let $\pi : M \hookrightarrow M'$. Let σ be regular in M . π is σ -cofinal iff

$$M' = \bigcup \{ \pi(u) \mid u \in M \wedge \bar{u} < \sigma \text{ in } M \}$$

(Hence σ -cofinality implies cofinality.)

*1) An ZFC-model the axiom of choice holds. Every set is enumerable by an ordinal.

Def Let $\bar{\sigma} > \omega$ be regular in M , $\bar{H} = H_{\bar{\sigma}}^M$.

Let $\bar{\pi} : \bar{H} \prec H$ cofinally. By a liftup of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$ s.t. M' is transitive, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : M \prec M'$ $\bar{\sigma}$ -cofinally.

(We also say: " $\langle M', \pi \rangle$ is a liftup of M by $\bar{\pi}$ ".)

Fact 3 Let $\langle M, \bar{\pi} \rangle$ be as above. There is at most one liftup $\langle M', \pi \rangle$.

Proof.

Clearly, every element of M' has the form $\pi(f)(x)$, where $f \in M$, $f : u \rightarrow M$ for a $u \in \bar{H}$, and $x \in \bar{\pi}(u)$. But

$$M' = \mathcal{P}(\pi(f_1)(x_1), \dots, \pi(f_n)(x_n)) \leftrightarrow$$

$$\leftrightarrow \langle x_1, \dots, x_n \rangle \in \bar{\pi}(X), \text{ where}$$

$$X = \{ \langle z_1, \dots, z_n \rangle \mid \mathcal{P}(f_1(z_1), \dots, f_n(z_n)) \}$$

(hence $X \in \bar{H}$).

This means that if $\langle M'', \pi'' \rangle$ is a second liftup, we can define $\sigma : M' \overset{\sim}{\leftrightarrow} M''$ by

$$\sigma(\pi(f)(x)) = \pi''(f)(x). \text{ Hence } \sigma \cdot \pi = \text{id},$$

$$M' = M''. \text{ But } \pi(z) = \pi(\text{const}_z(0)) =$$

$$\sigma(\pi(\text{const}_z(0))) = \pi''(\text{const}_z(0)) = \pi''(z) \text{ for } z \in M';$$

$$\text{where } \text{const}_z = \{ \langle z, 0 \rangle \}. \quad \text{QED (Fact 3)}$$

Note By this analysis it follows easily that, if $\langle M', \bar{\pi} \rangle$ is the liftup of M by $\bar{\pi}: \bar{H} \prec H$, where $\bar{H} = H_{\bar{z}}^M$, and $z' = \text{Cm} \cap H$, then $\pi(z) = z'$ and $H = H_{z'}^{M'}$. \bar{H} need not be an element of M , but if it is, it follows that $\pi(\bar{H}) = H$.

The proof of Fact 3 suggests a general method of constructing the liftup:

Def Let M be a transitive ZFC-model with predicates A_1, \dots, A_m . Let $\bar{H} = H_{\bar{z}}^M$, where \bar{z} is regular in M , and let $\bar{\pi}: \bar{H} \prec H$ cofinally.

$\mathbb{D} = \text{ID}_{M, \bar{\pi}} = \langle D, E, I, \tilde{A}_1, \dots, \tilde{A}_m \rangle$ is defined by:

$$D = \{ \langle x, f \rangle \mid f \in M \wedge f: u \rightarrow M \text{ for } u \in \bar{H} \wedge x \in \bar{\pi}(u) \}$$

$$\langle x, f \rangle E \langle y, g \rangle \iff \langle x, y \rangle \in \bar{\pi}(\{ \langle z, w \rangle \mid f(z) \in g(w) \})$$

$$\langle x, f \rangle I \langle y, g \rangle \iff \langle x, y \rangle \in \bar{\pi}(\{ \langle z, w \rangle \mid f(z) = g(w) \})$$

$$\tilde{A}_i(\langle x, f \rangle) \iff x \in \bar{\pi}(\{ z \mid A_i(f(z)) \})$$

We then get Loz Theorem in the form

$$\text{ID} \models \varphi(\langle x_1, f_1 \rangle, \dots, \langle x_n, f_n \rangle) \iff$$

$$\langle x_1, \dots, x_n \rangle \in \bar{\pi}(\{ \langle z_1, \dots, z_n \rangle \mid M \models \varphi(f_1(z_1), \dots, f_n(z_n)) \})$$

The proof is by induction on \mathcal{Q} and is just like the proof of Los Theorem for ultrapowers. Then $\mathbb{D} \models ZFC^-$ and \mathbb{D} is an equality model with equality relation I .

This gives:

Fact 4 The liftup of $\langle \bar{M}, \bar{\pi} \rangle$ exists iff \mathbb{D} is well founded.

Proof (sketch)

(\rightarrow) Let $\langle M', \pi \rangle$ be the liftup, then

$a \in b \iff k(a) \in k(b)$ for $a, b \in \mathbb{D}$, where

k is defined by $k(\langle x, f \rangle) = \pi(f)(x)$.

(\leftarrow) Factor \mathbb{D} by I to get $\mathbb{D}^* = \mathbb{D}/I$. Let $[u]$ be the equivalence class of u for $u \in \mathbb{D}$.

Then \mathbb{D}^* satisfies extensionality and has a well founded \in -relation. Hence there is $\sigma^* : \mathbb{D}^* \xrightarrow{\sim} M'$, where M' is transitive

by Mostowski's isomorphism theorem. Set:

$\sigma(u) = \sigma^*([u])$ for $u \in \mathbb{D}$. We can define

$\bar{\pi} : M \rightarrow M'$ by $\bar{\pi}(x) = \sigma(\langle 0, \text{const}_x \rangle)$,

where $\text{const}_x = \{ \langle x, 0 \rangle \}$ = the constant function

x on $\{0\}$. Set:

$\tilde{\mathbb{D}} = \{ \langle x, f \rangle \in \mathbb{D} \mid f \in H \}$; $H' = \{ \sigma(u) \mid u \in \tilde{\mathbb{D}} \}$.

H' is easily seen to be transitive.

But $H = \{ \bar{\pi}(f)(x) \mid \langle x, f \rangle \in \tilde{\mathbb{D}} \}$

Moreover:

$$\begin{aligned} \bar{\pi}(f)(x) \in \bar{\pi}(g)(y) &\iff \langle x, y \rangle \in \bar{\pi}(\{\langle z, w \rangle \mid f(x) = g(y)\}) \\ &\iff \sigma(\langle x, f \rangle) \in \sigma(\langle y, g \rangle) \end{aligned}$$

for $\langle x, f \rangle \in \tilde{D}$. Hence there is an isomorphism $i: H \xrightarrow{\sim} H'$ defined by $i(\bar{\pi}(f)(x)) = \sigma(\langle x, f \rangle)$. Hence $i = \text{id}$, $H = H'$ and $\sigma(\langle x, f \rangle) = \bar{\pi}(f)(x)$ for $\langle x, f \rangle \in \tilde{D}$. In particular,

$$\begin{aligned} \pi(z) &= \sigma(\langle 0, \text{const}_z \rangle) = \bar{\pi}(\text{const}_z)(0) = \\ &= \text{const}_{\bar{\pi}(z)}(0) = \bar{\pi}(z) \text{ for } z \in \bar{H} \end{aligned}$$

Hence $\pi \upharpoonright \bar{H} = \bar{\pi}$. But then for $\langle x, f \rangle, \langle y, g \rangle \in \tilde{D}$, we have:

$$\begin{aligned} \sigma(\langle x, f \rangle) \in \sigma(\langle y, g \rangle) &\iff \langle x, y \rangle \in \bar{\pi}(\{\langle z, w \rangle \mid f(x) = g(y)\}) \\ &\iff \bar{\pi}(f)(x) \in \bar{\pi}(g)(y). \end{aligned}$$

Hence there is $i: M' \xrightarrow{\sim} M'' \subset M''$ defined by $i(\sigma(\langle x, f \rangle)) = \bar{\pi}(f)(x)$. M'' is easily seen to be transitive, however, as

$i = \text{id}$ and each $z \in M'$ has the form $\bar{\pi}(f)(x)$, where $\langle x, f \rangle \in \tilde{D}$. It follows easily that $\langle M', \bar{\pi} \rangle$ is the liftup of $\langle M, \bar{\pi} \rangle$. QED (Fact 4)

This gives us the interpolation lemma:

Fact 5 Let $\pi': \bar{M} \prec M'$. Let $\bar{c} \in \bar{M}$ be regular in \bar{M} and set $\bar{H} = H_{\bar{c}}^{\bar{M}}$. Let $\pi: \bar{H} \prec H \prec \text{cofinally}$. Then:

(a) The lift-up $\langle M, \pi \rangle$ of $\langle \bar{M}, \pi' \rangle$ exists

(b) There is a unique $\sigma: M \prec M'$ s.t.
 $\sigma \pi = \pi'$ and $\sigma \upharpoonright H = \text{id}$.

prf.

To prove (a) we note that E is well founded, since $\langle x, f \rangle E \langle y, g \rangle \iff \pi'(f)(x) \in \pi'(g)(y)$.

But for $\langle x_1, f_1 \rangle, \dots, \langle x_n, f_n \rangle \in \mathbb{D}$ we have:

$$M \models \varphi(\pi(f_1)(x_1), \dots, \pi(f_n)(x_n)) \iff$$

$$\iff M' \models \varphi(\pi'(f_1)(x_1), \dots, \pi'(f_n)(x_n))$$

$$\iff \langle x_1, \dots, x_n \rangle \in \pi \left(\left\{ \bar{z} \mid \bar{M} \models \varphi(f_1(\bar{z}_1), \dots, f_n(\bar{z}_n)) \right\} \right).$$

Hence there is $\sigma: M \prec M'$ defined by $\sigma(\pi(f)(x)) = \pi'(f)(x)$ for $\langle x, f \rangle \in \mathbb{D}$. But this σ is characterized by the above conditions. \square ED (Fact 5)

The structure \mathbb{D}^* will be of interest to us, however, even if it is ill founded.

An embedding $\tilde{\pi}: M \prec \mathbb{D}^*$ is definable

by $\tilde{\pi}(x) = [\langle 0, \text{cut}_x^* \rangle]$. This embedding

is cofinal in the sense that for every $z \in \mathbb{D}^*$ there is $u \in M$ s.t.

$$\mathbb{D}^* \models z \in \tilde{\pi}(u).$$

When dealing with ill founded models of set theory it is useful to work with solid structures in the following sense:

Def Let $M = \langle A, \in_M, \dots \rangle$ model the extensionality axiom. M is solid iff the well founded core $wfc(M)$ is transitive and $\in_M \cap wfc(M)^2 = \in_M \cap wfc(M)^2$, ($wfc(M)$ is the set of $x \in M$ s.t. $\in_M \cap X^2$ is well founded, where X is the closure of $\{x\}$ under \in_M).

Clearly, every model is isomorphic to a solid model.

We note the following facts about solid models of ZFC:

Fact 6 Let M be a solid model of ZFC. Let $H = wfc(M)$. Then

(a) $\omega \subset H$; $\alpha \in H \rightarrow \alpha + 1 \in H$

(b) $\forall \alpha \in H, x \in M$ and $|M \cap \{x\}^{\in_M} \cap \alpha| \leq \alpha$,

then $x \in H$

(c) H is admissible

prf.

(a), (b) are trivial. We prove (c).

(Note We take the replacement axiom of ZFC - as reading:

$$\wedge x \forall y \varphi(x, y, \vec{z}) \rightarrow \wedge u \forall v \wedge x \in u \forall y \in v \varphi(x, y, \vec{z})$$

for arbitrary formulae φ . The theory KP ("Kripke-Platek set theory") is obtained by restricting the formula φ in this schema - and in the separation schema - to Σ_0 formulae. A transitive structure is called admissible iff it satisfies KP.)

By (b), H is easily seen to satisfy Σ_0 -separation, as well as the trivial existence axioms: " \emptyset is a set", " $\{x, y\}$ is a set", " $\cup x$ is a set". We prove Σ_0 replacement,

Let $H \models \wedge x \forall y \varphi(x, y, \vec{z})$. Let $u \in H$.

Let $R(x, y)$ mean " $\varphi(x, y, \vec{z})$ and y is of minimal rank." Then there is $v \in \mathcal{U}$ such that $\mathcal{U} \models \wedge x \in u \forall y \in v R(x, y)$.

But if we take v as being of minimal rank in \mathcal{U} , it must have rank $\in H$. Hence, $v \in H$. QED (Fact 6)

Note It follows that if $u \in H$ is transitive and $\mathcal{U} = \text{On} \cap H$, then $L_{\mathcal{U}}(u)$ is admissible.

We now extend some of our definitions to solid models of ZFC⁻,

Def Let \mathcal{M} be a solid model of ZFC⁻,

Let $\tau \in \text{wfc}(\mathcal{M})$ be regular in \mathcal{M}
and let $\bar{H} = H_{\tau}^{\mathcal{M}}$ (hence $\bar{H} \subset \text{wfc}(\mathcal{M})$).

Let $\pi: \bar{H} \prec H$ cofinally, where H is transitive. $\langle \mathcal{M}', \pi \rangle$ is a liftup of $\langle \mathcal{M}, \bar{\pi} \rangle$ iff \mathcal{M}' is solid, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi: \mathcal{M} \prec \mathcal{M}'$ is τ -cofinal (i.e., for each $x \in \mathcal{M}'$ there is $u \in \mathcal{M}$ s.t., $\bar{u} \prec \tau$ in \mathcal{M} and $\mathcal{M}' \models x \in \pi(u)$).

A virtual repetition of the proof of Fact 3 gives:

Fact 7 Let $\langle \mathcal{M}, \bar{\pi} \rangle$ be as above. Up to isomorphism there is at most one liftup $\langle \mathcal{M}', \pi \rangle$.

Note As before, $\pi(\tau) = \sup \{ \bar{\pi}(v) \mid v \prec \tau \}$ in \mathcal{M}' ; hence $\pi(\tau) \in \text{wfc}(\mathcal{M}')$ and $H = H_{\pi(\tau)}^{\mathcal{M}'}$ \subset $\text{wfc}(\mathcal{M}')$.

If $\bar{H} \in \text{wfc}(\mathcal{M})$, then $H = \pi(\bar{H}) \in \text{wfc}(\mathcal{M}')$.

But we can then form \mathbb{D} as before [taking $\langle x, f \rangle \in \mathbb{D}$ iff $(\mathcal{M} \models f: u \rightarrow v)$ for a $u \in \bar{H}$ and $x \in \bar{\pi}(u)$].

Repeating the proof of Fact 4 we get:

Fact 8 Let $\langle \mathcal{M}, \pi \rangle$ be as above. Then the liftup exists.

(Note The liftup $\langle \mathcal{M}', \pi \rangle$ is unique only up to isomorphism. But then $wfc(\mathcal{M}')$ is unique, by solidity.)

We now weaken our earlier definition of fullness to:

Def Let N be a transitive ZFC-model s.t. $N = \langle L_z[A], \epsilon, A, in \rangle$. N is almost full iff there is a solid model \mathcal{M} of ZFC- s.t. $N \in wfc(\mathcal{M})$, N is regular in \mathcal{M} , and $\mathcal{M} \models V = L(N)$.

Then by the above we have:

Fact 9 Let $N = \langle L_z[A], \epsilon, A, in \rangle$ be almost full. Let $\pi: N \prec N'$ cofinally. Then N' is almost full. (Moreover, if \mathcal{M} verifies the almost fullness of N and $\langle \mathcal{M}', \pi' \rangle$ is the liftup of $\langle \mathcal{M}, \pi \rangle$, then \mathcal{M}' verifies the almost fullness of N' .)

By Fact 6:

Fact 10 Let N be almost full. There is δ s.t. $L_\delta(N)$ is admissible and N is regular in $L_\delta(N)$.

Def $\delta_N =$ the least δ s.t. $L_\delta(N)$ is admissible.

A major tool will be the following transfer lemma:

Fact 11 Let \bar{N} be almost full. Let

$\pi: \bar{N} \rightarrow N$ cofinally. Let $x_1, \dots, x_n \in \bar{N}$ and let φ be a Π_1 formula. Then

$$L_{\delta_{\bar{N}}}(\bar{N}) \models \varphi(\bar{N}, \vec{x}) \rightarrow L_{\delta_N}(N) \models \varphi(N, \pi(\vec{x})).$$

proof.

Let $\bar{\alpha}$ witness the almost fullness of \bar{N} and let $\pi': \bar{\alpha} \rightarrow \alpha$ be the liftup of $\langle \bar{\alpha}, \pi \rangle$. Obviously:

$$(1) \alpha \notin \text{wfc}(\bar{\alpha}) \rightarrow \pi'(a) \notin \text{wfc}(\alpha)$$

$$(2) L_{\delta_{\bar{N}}}(\bar{N}) \subset \text{wfc}(\bar{\alpha}), L_\delta(N) \subset \text{wfc}(\alpha) \text{ by Fact 6}$$

Suppose not. Then there is a least $d < \delta_N$ s.t.

$$L_d(N) \models \neg \varphi(N, \pi(\vec{x})).$$

Since $L_{\delta_N}(N)$ is an initial segment of α ,

we have:

$$(3) \alpha \models d \text{ is least s.t. } L_d(N) \models \neg \varphi(N, \pi(\vec{x})).$$

$$(4) \alpha \models \exists \nu \leq d \cdot L_\nu(N) \text{ is not admissible.}$$

But then $\alpha = \pi(\bar{\alpha})$, where in \bar{M} :

(5) $\bar{\alpha}$ is least s.t. $L_{\bar{\alpha}}(\bar{N}) \models \exists \varphi(N, \bar{\alpha})$

(6) $\forall \gamma \leq \bar{\alpha} \ L_{\gamma}(\bar{N})$ is not admissible.

But $\bar{\alpha} \in \text{wfc}(\bar{M})$ by (1). Hence

$\bar{\alpha} < \beta = 0$ on $\text{wfc}(\bar{M})$ and $L_{\beta}(\bar{N})$ is

admissible. Thus (5), (6) hold in

$L_{\beta}(\bar{N})$, since $L_{\beta}(\bar{N})$ is an initial

segment of \bar{M} . Hence (5), (6) hold

outright and $\alpha < \delta_N$. Contr!

QED (Fact 11)

Note Fact 11 is actually a special case of a more general theorem:

Let $\bar{N} = \langle L_{\bar{N}}[A], \epsilon, A, \dots \rangle$ is a ZFC-model,

$\pi: \bar{N} \prec N$ cofinally, and N is regular

in $L_{\delta_N}(N)$, then the conclusion of

Fact 11 holds (even if \bar{N} is not regular in $L_{\delta_{\bar{N}}}(\bar{N})$).

We shall not need this, however,

and do not prove it here, since our

proof involves a modest application

of fine structure theory.

§ 3.2 Barwise Theory

An addition to the transfer lemma we shall make use of Barwise' theory of infinitary languages on admissible structures. In the following let M be an admissible structure satisfying choice in the form: Every set is enumerable by an ordinal. In admissibility theory the basic three notions of recursion theory are redefined as follows:

$$M\text{-recursive} = \underline{\Delta}_1(M)$$

$$M\text{-recursively enumerable} = \underline{\Sigma}_1(M)$$

$$M\text{-finite} = \text{element of } M.$$

Barwise then developed an extension of first order logic involving formulae which are infinitely long but still M -finite. Thus a Barwise language on M is like predicate logic except that, whenever $\langle \varphi_i \mid i \in \alpha \rangle \in M$ is a sequence of formulae, then $\bigwedge_{i \in \alpha} \varphi_i$ and $\bigvee_{i \in \alpha} \varphi_i$ are formulae. (A finite blocks of quantifiers are not allowed, however.) The set of variables is M -infinite (i.e. we could have a variable v_ξ for each $\xi \in \text{On} \cap M$). A language is then specified by fixing its predicates, constants, and function symbols.

The syntax is developed internally in such a way that the basic syntactical notions (e.g. "formula", "term", "sentence") are $\Delta_1(M)$. A mathematical theory

$\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ then consists of a language \mathcal{L}_0 and a set \mathcal{L}_1 of axioms (all of which are sentences). \mathcal{L}_1 should be $\Sigma_1(M)$, if we wish to make use of the admissibility of M . We augment the usual predicate logical rules of inference by two infinitary rules:

$$\frac{\psi \rightarrow \varphi_i \quad (i \in x)}{\psi \rightarrow \bigwedge_{i \in x} \varphi_i}$$

$$\frac{\varphi_i \rightarrow \psi \quad (i \in x)}{\bigvee_{i \in x} \varphi_i \rightarrow \psi}$$

$$\psi \rightarrow \bigwedge_{i \in x} \varphi_i$$

$$\bigvee_{i \in x} \varphi_i \rightarrow \psi$$

for $\langle \varphi_i \mid i \in x \rangle \in M$.

A proof is then a (possibly infinite) sequence of formulae, each of which is an axiom or follows from the previous formulae by a rule of inference. At the axiom set \mathcal{L}_1 is $\Sigma_1(M)$, it turns out that every provable formula has a proof p which is M -finite (i.e. $p \in M$). From this we get the

M-finiteness lemma: If φ is provable in \mathcal{L} , then it is provable from an M-finite $u \in \mathcal{L}_\varphi$.

A model \mathcal{M} of the language \mathcal{L}_0 is described by fixing its domain of individuals $|M|$ and the interpretation $S^{\mathcal{M}}$ of each predicate symbol, constant, or function symbol s , just as in finitary predicate logic. We can then straightforwardly define $\text{truth}(\mathcal{M} \models \varphi)$ for \mathcal{L}_0 -sentences φ and satisfaction

$(\mathcal{M} \models \varphi [a_1, \dots, a_m])$ for \mathcal{L}_0 + formulae containing only finitely many free variables. We say that \mathcal{M} models the

theory $\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ iff all axioms in \mathcal{L}_1 are true in \mathcal{M} . The notion of proof is correct in the sense that, if \mathcal{M} models \mathcal{L} , then sentence provable in \mathcal{L} is true in \mathcal{M} .

The final stone in this mosaic is the completeness theorem for countable \mathcal{M} :

If \mathcal{M} is countable, then \mathcal{L} is consistent iff \mathcal{L} has a model.

This means that for any admissible M , we can make the completeness theorem true in a generic extension of V simply by collapsing M to ω . In many cases we can then use this to prove properties of V .

We note that if L_1 is $\Sigma_1(M)$ in parameters \vec{P} , then the statement " L is consistent" is uniformly $\Pi_1(M)$ in \vec{P} , since it says that M contains no proof of a contradiction. (But by the foregoing, " L is consistent" is equivalent to:

\Vdash_{IP} " L has a model", where IP is any set of conditions which collapses

M to ω .) At this point we will apply the transfer lemma: Let

N be almost full and let $\pi: N \prec N'$ cofinally. Let L be a theory on

$L_{\delta_N}(N)$ s.t. the set of axioms L_1 is

$\Sigma_1(N)$ in parameters N and $\vec{P} \in N$.

Let L' have the same definition

in N' , $\pi(\vec{P})$ over $L_{\delta_{N'}}(N')$. Then:

L is consistent $\rightarrow L'$ is consistent.

In this paper we shall deal only with languages \mathcal{L} on M which contain a binary " \in -predicate" $\dot{\in}$ and a designated constant \underline{x} for each $x \in M$.

(We suppose $\dot{\in}$ and $\langle \underline{x} \mid x \in M \rangle$ to have a uniform $\Delta_1(M)$ definition over any admissible M .) We also suppose that the set of axioms \mathcal{L}_0 contains a base theory consisting of:

- ZFC⁻ (including the schemata of separation and replacement for all finite formulae of \mathcal{L}_0)
- the "defining" axioms for the constants \underline{x} ($x \in M$): $\bigwedge \sigma (\sigma \in \underline{x} \leftrightarrow \bigvee_{z \in x} \sigma = \underline{z})$.

We note that if \mathcal{M} is a solid model of \mathcal{L} , we then have $\underline{x}^{\mathcal{M}} = x \in \text{wfc}(\mathcal{M})$ for all $x \in M$.

§ 3.4 The forcing \mathbb{P}_A .

Let $A \subset \omega_2$ be a stationary set of points of cofinality ω . We define:

Def $\mathbb{P}_A =$ the set of $p: \alpha+1 \rightarrow A$ s.t., $\alpha < \omega_1$ and p is a normal function.

$$p \leq q \text{ in } \mathbb{P}_A \iff p \supset q.$$

Hence if G is \mathbb{P}_A -generic, $f = \bigcup G$ is a cofinal normal function

$f: \omega_1 \rightarrow A$. It is easily established that f adds no reals.

Lemma 1 \mathbb{P}_A is subcomplete.

Proof.

Let $\mathbb{P}_A \in H_\theta$. Let $\bar{\sigma} > \theta$ be regular. Let

$N = \langle L_{\bar{\sigma}}[A], A, \dots \rangle$ where $H_{\bar{\sigma}} \subset N$. Let

$\sigma: \bar{N} \prec N$ s.t., $\sigma(\bar{\theta}, \bar{\mathbb{P}}) = \theta, \mathbb{P}_A$ and \bar{N} is countable and full.

Claim σ witnesses the subcompleteness of \mathbb{P}_A .

Let $\bar{\lambda}_0 = 0$ in \bar{N} , $\bar{\lambda}_i = \sigma^{-1}(\lambda_i)$ ($i=1, m, m$)

where $\mathbb{P}_A \in H_{\lambda_i}$ & $\lambda_i \in (\omega_1, \theta)$ is regular

($i=1, m, m$).

Let $\tilde{\lambda}_i = \sup \sigma \text{ " } \bar{\lambda}_i \quad (i=0, m, m)$

Let $\sigma(\bar{\alpha}) = \alpha$

Claim 1 There is $\sigma_0 : \bar{N} \prec N$ with

(a) $\sup \sigma_0 \text{ " } \omega_2^{\bar{N}} \in A$

(b) $\sigma_0(\bar{\alpha}, \bar{\lambda}_i, \bar{P}) = \alpha, \lambda_i, P$

(c) $\sup \sigma_0 \text{ " } \bar{\lambda}_i = \tilde{\lambda}_i \quad (i=0, m, m)$

pf.

For $\alpha < \omega_2$ set $X_\alpha =$ the smallest $X \prec N$ s.t.

$\alpha \cup \text{rng}(\sigma) \subset X$. Set

$C = \{ \alpha < \omega_2 \mid \alpha = \omega_2 \cap X \}$. Then C is club in ω_2 .

For $\alpha \in C$ set $\pi_\alpha : N_\alpha \xrightarrow{\sim} X_\alpha$. Then

(1) $\alpha = \text{crit}(\pi_\alpha), \pi_\alpha(\alpha) = \omega_2$

(2) Set $\sigma_\alpha = \pi_\alpha^{-1} \sigma$. Then

$\langle N_\alpha, \sigma_\alpha \rangle =$ the liftup of $\langle \bar{N}, \sigma \upharpoonright H_{\omega_3}^{\bar{N}} \rangle$

pf.

Form $\langle N', \sigma' \rangle =$ the liftup of $\langle \bar{N}, \sigma \upharpoonright H_{\omega_3}^{\bar{N}} \rangle$

Then there is $\pi' : N' \prec N_\alpha$ s.t. $\pi' \sigma' = \sigma_\alpha$

and $\pi' \upharpoonright H_{\omega_3}^{N'} = \text{id}$. But then

$\pi' \upharpoonright \alpha = \text{id}$, since $\alpha < \omega_3^{N_\alpha}$. Hence

$\alpha \cup \text{rng}(\sigma) \subset \text{rng}(\pi')$. Hence

$\text{rng}(\pi') = \text{rng}(\pi_\alpha), \pi' = \pi_\alpha$, QED (2)

Now let $\langle N', \sigma' \rangle =$ the liftup

of $\langle \bar{N}, \sigma \upharpoonright H_{\omega_2}^{\bar{N}} \rangle$. Since

$\pi_\alpha \upharpoonright H_{\omega_2}^{M_\alpha} = \text{id}$ and $\pi_\alpha \sigma_\alpha = \sigma$, we have

$\sigma_\alpha \upharpoonright H_{\omega_2}^{\bar{N}} = \sigma \upharpoonright H_{\omega_2}^{\bar{N}}$. Hence:

(3) $\langle N', \sigma' \rangle = \text{the liftup of } \langle \bar{N}, \sigma_\alpha \upharpoonright H_{\omega_2}^{\bar{N}} \rangle$.

Hence there is $\pi' : N' \prec N_\alpha$ s.t. $\pi' \sigma' = \sigma_\alpha$ and $\pi' \upharpoonright H_{\omega_2}^{N'} = \text{id}$.

(Note It is in fact easily seen that if $\alpha_0 = \min C$, then $\alpha_0 = \omega_2^{N'}$, $N' \equiv N_{\alpha_0}$ and $\pi' = \pi_\alpha^{-1} \pi_{\alpha_0}$.)

Clearly $\pi' : N' \prec N_\alpha$ cofinally, since $\sigma' : \bar{N} \prec N'$ cofinally and $\sigma_\alpha : \bar{N} \prec N_\alpha$ cofinally.

Since $\sigma' : \bar{N} \prec N$ is the liftup of $\langle \bar{N}, \sigma \upharpoonright H_{\omega_2}^{\bar{N}} \rangle$, we have:

(4) $\sigma'(\tau) = \sup \sigma' \upharpoonright \tau$ whenever $\tau \geq \omega_2^{N'}$ is regular in \bar{N} .

Similarly:

(5) $\sigma_\alpha(\tau) = \sup \sigma_\alpha \upharpoonright \tau$ whenever $\tau \geq \omega_3^{N'}$ is regular in \bar{N} .

Now let $\delta' \equiv \delta_{N'}$. Let L' be the language on $L_{\delta'}(N')$ containing the base theory and with a new constant σ' and the axioms:

- $\sigma' : \bar{N} \prec N'$ cofinally
- $\sigma'(\bar{\alpha}, \bar{P}, \bar{\lambda}_i) = \langle \sigma'(\bar{\alpha}), \sigma'(\bar{P}), \sigma'(\bar{\lambda}_i) \rangle$ ($i=1, \dots, m$)
- $\sup \sigma''\tau = \sigma'(\tau)$ whenever τ is regular in \bar{N} .

\mathcal{L}' is consistent, since it is modeled by $\langle H_{\omega_1}, \sigma' \rangle$. Moreover the theory \mathcal{L}' is

$\Sigma_1(L_{\sigma'}(N'))$ in the parameters

$N', \bar{N}, \bar{\alpha}, \bar{\lambda}_i, \sigma'(\bar{\alpha}), \sigma'(\bar{\lambda}_i)$ ($i=1, \dots, m$).

Now let $\sigma_2 = \sigma_{N_2}$ and let \mathcal{L}^2 be

$\Sigma_1(L_{\sigma_2}(N_2))$ by the same definition

in the parameters:

$N_2, \bar{N}, \bar{\alpha}, \bar{\lambda}_i, \sigma_2(\bar{\alpha}), \sigma_2(\bar{\lambda}_i)$ ($i=1, \dots, m$)

Then \mathcal{L}^2 is consistent by the transfer

lemma, since $\pi' : N' \prec N_2$ is cofinal

and $\pi'\sigma' = \sigma_2$ and $\pi' \upharpoonright H_{\omega_1} = \text{id}$.

By this we get:

(6) Let $\text{cf}(\alpha) = \omega$. Then in V there is

a map σ_1 s.t.

- $\sigma_1 : \bar{N} \prec N'$ cofinally

- $\sigma_1(\bar{\alpha}, \bar{P}, \bar{\lambda}_i) = \langle \sigma_1(\bar{\alpha}), \sigma_1(\bar{P}), \sigma_1(\bar{\lambda}_i) \rangle$ ($i=1, \dots, m$)

- $\sup \sigma_1''\tau = \sigma_1(\tau)$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in \bar{N} .

Note If $\alpha > \sup \sigma'' \omega_2 \bar{N}$, then we cannot have $\sigma_1 = \sigma_\alpha$, since $\sigma_\alpha \upharpoonright \omega_2 \bar{N} = \sigma \upharpoonright \omega_2 \bar{N}$.

proof of (6)

Let $\gamma < H_{\omega_2}$ be countable s.t.,

$N_\alpha, \sigma_\alpha \in \gamma$, $\alpha = \sigma_\alpha(\omega_2 \bar{N})$ is ω -cofinal and $\sigma_\alpha(\tau)$ is ω -cofinal whenever $\tau > \omega_2 \bar{N}$ is regular in \bar{N}

by (5). Hence $\gamma \cap \sigma_\alpha(\tau)$ is cofinal in $\sigma_\alpha(\tau)$ whenever $\tau \geq \omega_2 \bar{N}$ is regular in \bar{N} .

Let $k: \bar{H} \xrightarrow{\sim} \gamma$, $k(\bar{N}_\alpha) = N_\alpha$, $k(\bar{\sigma}_\alpha) = \sigma_\alpha$,

$k(\bar{\sigma}_\alpha) = \sigma_\alpha$, $k(\bar{L}^\alpha) = L^\alpha$. Then

$k \upharpoonright \bar{N}_\alpha: \bar{N}_\alpha \prec N_\alpha$ cofinally (since

$\alpha \cap N_\alpha$ has cofinality ω) and

$k \upharpoonright \bar{\sigma}_\alpha(\tau)$ is cofinal in $\sigma_\alpha(\tau)$ whenever

$\tau \geq \omega_2 \bar{N}$ is regular in \bar{N} , \bar{L}^α is

consistent and therefore, by countability, has a solid model \mathcal{M} .

Let $\bar{\sigma}_1 = \dot{\sigma}^{\mathcal{M}}$. Then $\bar{\sigma}_1 \in \text{wfc}(\mathcal{M})$

and:

- $\bar{\sigma}_1 : \bar{N} \rightarrow \bar{N}_\alpha$ cofinally
- $\bar{\sigma}_1(\bar{x}, \bar{P}, \bar{\lambda}_i) = \bar{\sigma}_\alpha(\bar{x}), \bar{\sigma}_\alpha(\bar{P}), \bar{\sigma}_\alpha(\bar{\lambda}_i)$
- $\sup \bar{\sigma}_1 " \bar{\sigma} = \bar{\sigma}_1(\bar{\sigma})$ whenever $\bar{\sigma} \geq \omega_2 \bar{N}$ is regular in \bar{N} .

But then $\sigma_1 = k\bar{\sigma}_1$ has the desired properties,

QED (6).

Now let $\alpha \in A \cap C$. Then $cf(\alpha) = \omega$, let σ_1 be as in (6) and set $\sigma_0 = \pi_\alpha \sigma_1$.

Then $\alpha = \sup \sigma_0 " \omega_2 \bar{N} \in A$, $\text{ran } \pi_\alpha \upharpoonright \alpha = \text{id}$,

$$\sigma_0(\bar{x}, \bar{P}, \bar{\lambda}_i) = \pi_\alpha \sigma_1(\bar{x}, \bar{P}, \bar{\lambda}_i) = \bar{x}, \bar{P}_A, \bar{\lambda}_i$$

$$\text{But } \sup \sigma_1 " \bar{\lambda}_i = \sigma_1(\bar{\lambda}_i) = \bar{\sigma}_\alpha(\bar{\lambda}_i) = \sup \bar{\sigma}_\alpha " \bar{\lambda}_i$$

$$\begin{aligned} \text{Hence } \sup \sigma_0 " \bar{\lambda}_i &= \sup \pi " \sigma_1(\bar{\lambda}_i) = \\ &= \sup \pi " \bar{\sigma}_\alpha(\bar{\lambda}_i) = \sup \bar{\sigma} " \bar{\lambda}_i = \bar{\lambda}_i \end{aligned}$$

QED (Claim 1)

Now let σ_0 be as in Claim 1 and let $\alpha = \sup \sigma_0 " \omega_2 \bar{N} \in A$. Let \bar{G} be \bar{P} -generic over \bar{N} . Set $\bar{g} = \bigcup \bar{G}$.

Then $\bar{g} : \omega_1 \bar{N} \rightarrow \bar{A}$ is normal and cofinal,

where $\bar{\sigma}(\bar{A}) = A$. But $\sigma_0(\bar{A}) = A$,

since $A = \bigcup_{P \in \bar{P}} \text{dom}(P)$. Set $g = \sigma_0 \circ \bar{g}$

Then $g: \omega_1^{\bar{N}} \rightarrow A$ is normal with

$\sup g''\omega_1^{\bar{N}} = \alpha \in A$, Set $p = g \cup \{(\alpha, \omega_1^{\bar{N}})\}$

Then $p \in \mathbb{P}_A$ and $\sigma_0(p)$

$$p \leq \sigma_0(q) \iff q \subset \bar{q} \iff q \in \bar{G}$$

for $q \in \mathbb{P}$. Hence, if $G \ni p$ is \mathbb{P}_A -generic, then $\bar{G} = \sigma_0^{-1}''G$.

QED (Lemma 1)

Note \mathbb{P}_A is provably not semiproper.
Hence we have shown that not every subcomplete forcing is semiproper.

Note This example is rather special in the sense that σ_0 lies in V . That will not hold if - as in the next example - a regular cardinal becomes ω -cofinal.

§3.5 Prüfer forcing

Let U be a normal measure on κ . Let $\mathbb{P} = \mathbb{P}_U$ be the set of Prüfer conditions for adding a cofinal ω -sequence to κ .

Lemma 2 \mathbb{P} is subcomplete.

We remember that \mathbb{P} consists of all pairs $\langle \alpha, X \rangle$ s.t. $X \in U$ and $\alpha: \omega \rightarrow \kappa$ is monotone for some n .

$$\langle \alpha', X' \rangle \leq \langle \alpha, X \rangle \iff \begin{array}{l} \alpha' \restriction \alpha \cap X' \subset X \cap \\ \cap \text{rng}(\alpha') \setminus \text{rng}(\alpha) \subset X \end{array}$$

If G is \mathbb{P} -generic, then

$$S = \bigcup \{ \alpha \mid \forall X \langle \alpha, X \rangle \in G \}$$

is called a Prüfer sequence. G is then recoverable from S by:

$$G = \{ \langle \alpha, X \rangle \mid \alpha \in S \wedge \text{rng}(S) \setminus \text{rng}(\alpha) \subset X \}$$

It can be shown that $S: \omega \rightarrow \kappa$ is a Prüfer sequence iff $\text{rng}(S)$ is almost contained in every $X \in U$,

Now let $IP \in H_\theta$. Let $\tau > \theta$ be regular and $N = \langle L_\tau[A], A, \in \rangle$ s.t. $H_\theta \subset N$. Let $\sigma: \bar{N} \prec N$ be countable and full s.t. $\sigma(\bar{IP}) = IP$.

Claim σ witnesses the subcompleteness of IP .

Let $\lambda_i \in \text{rng}(\sigma)$ s.t. $IP \in H_{\lambda_i}$ and $\lambda_i \in (\omega_1, \theta)$ is regular ($i=1, \dots, m$).
 Set: $\bar{\lambda}_i = \sigma^{-1}(\lambda_i)$.

Let $\sigma(\bar{\pi}) = \pi$. Let \bar{G} be \bar{IP} -generic over \bar{N} . We must show:

Claim There is $p \in IP$ which forces that whenever $G \ni p$ is IP -generic, then there is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0: \bar{N} \prec N$ cofinally

(b) $\sigma_0(\bar{\pi}, \bar{IP}, \bar{\lambda}_i) = \pi, IP, \lambda_i$ ($i=1, \dots, m$)

(c) $\text{sup } \sigma_0 \text{'' } \bar{\lambda}_i = \tilde{\lambda}_i =_{\text{HF}} \text{sup } \sigma \text{'' } \bar{\lambda}_i$

for $i=1, \dots, m$.

(d) $\bar{G} = \sigma_0^{-1} \text{'' } G$.

Let $\langle N', \sigma' \rangle =$ the liftup of $\langle \bar{N}, \sigma \upharpoonright H_{\bar{\kappa}}^{\bar{N}} \rangle$

where $\sigma(\bar{\kappa}) = \kappa$. Then $\sigma': \bar{N} \prec N'$

cofinally and $\sup \sigma'' \alpha = \sigma'(\alpha)$ for all $\alpha \geq \bar{\kappa}$ i.e., α is regular in \bar{N} .

Let $\bar{g}: \omega \rightarrow \bar{\kappa}$ be the Prikry sequence engendered by \bar{G} . Set $g' = \sigma' \circ \bar{g}$.

Then $g': \omega \rightarrow \kappa' = \sigma'(\bar{\kappa})$ cofinally,

(1) g' is a Prikry sequence for N'
(wrt. $U' = \sigma'(U)$)

prf.

We must show that $\text{rng}(g')$ is almost contained in X for every $X \in U'$. But

$X = \sigma'(f)(\bar{\zeta})$, where $f \in \bar{N}$, $f: \alpha \rightarrow \bar{\kappa}$

for an $\alpha < \kappa$, and $\bar{\zeta} < \sigma(\alpha) = \sigma'(\alpha)$.

Hence $\bar{Y} = \bigcap f'' \alpha \in \bar{U}$ and

$Y = \sigma'(\bar{Y}) = \bigcap \sigma'(f)'' \sigma(\alpha) \in U'$.

Hence \bar{g} is almost contained in \bar{Y}

and g' is almost contained in $Y \subset X$.

QED(1)

Now let $\langle N'', \sigma'' \rangle$ be the liftup

of $\langle \bar{N}, \sigma \upharpoonright H_{\bar{\mu}}^{\bar{N}} \rangle$, where $\bar{\mu} = \bar{\kappa}^{++ \bar{N}}$.

Then $\sigma''(\bar{\kappa} + \bar{N}) = \kappa^+$, $\sigma''(H_{\bar{\kappa} + \bar{N}}^{\bar{N}}) = H_{\kappa^+}$

and $\sigma''(\tau) = \text{sup } \sigma'' \ll \tau$ whenever $\tau \geq \bar{\kappa} + \bar{N}$ is regular in \bar{N} .

Let $\sigma' = \sigma_{N'}$, $\sigma'' = \sigma_{N''}$. Let L' be the infinitary language on $L_{\sigma'}(W')$ comprising the base theory, a new constant σ and the further axioms:

- $\sigma : \bar{N} \prec N''$ cofinally
- $\sigma(\bar{\alpha}, \bar{\beta}, \bar{\lambda}_i, \bar{\kappa}) = \sigma'(\bar{\alpha}), \sigma'(\bar{\beta}), \sigma'(\bar{\lambda}_i), \sigma'(\bar{\kappa})$
- $\text{sup } \sigma'' \tau = \sigma'(\tau)$ whenever $\tau \geq \bar{\kappa}$ is regular in \bar{N}
- $\sigma \circ \bar{\sigma}$ is Prinsy generic over N'

Then L' is consistent, since $\langle H_{\kappa^+}, \sigma' \rangle$ models L' .

But $\langle N', \sigma' \rangle$ is the liftup of $\langle \bar{N}, \sigma'' \upharpoonright H_{\bar{\kappa}}^{\bar{N}} \rangle$ since $\sigma'' \upharpoonright H_{\bar{\kappa}}^{\bar{N}} = \sigma \upharpoonright H_{\bar{\kappa}}^{\bar{N}}$.

Hence there is unique $\pi : N' \prec N''$ cofinally, s.t. $\pi \upharpoonright H_{\sigma'(\bar{\kappa})}^{N'} = \text{id}$ and $\pi \sigma' = \sigma''$. But then L''

is consistent, where \mathcal{L}'' has the same definition over $L_{\delta''}(N'')$ in the parameters $\bar{\pi}, \bar{\rho}, \bar{\lambda}, \bar{g}, \bar{u}, \sigma''(\bar{\pi}), \sigma''(\bar{\rho}), \sigma''(\bar{\lambda}), \sigma''(\bar{u})$. (Note that $\sigma'(\bar{u}) = U \cap H_{\sigma'(\bar{u} + \bar{N})}^{N'}$, and $\sigma''(\bar{u}) = U$, since $\bar{u} = \{x \mid \forall z \langle z, x \rangle \in \bar{\rho}\}$.)

Now generically collapse δ'' to ω .

In the resulting model $V[\bar{\sigma}]$

let σ_1 be a solid model of \mathcal{L}'' .

Set $\sigma_1 = \bar{\sigma} \circ \sigma$, $g = \sigma_1 \circ \bar{g}$. Then

(2) g is Prkry generic over V

since $\sigma''(\bar{u}) = U$,

Since g is Prkry generic over N'' and N'' is regular in $L_{\delta''}(N'')$, g is also Prkry generic over $L_{\delta''}(N'')$; hence

(3) $L_{\delta''}(N''[g])$ is admissible.

Let \mathcal{L}^* be the language on $L_{\delta''}(N''[g])$ with the base

a constant $\bar{\sigma}$, the axioms of \mathcal{L}'' ,

and the axiom: $\underline{g} = \bar{\sigma}'' \underline{\bar{g}}$.

Then \mathcal{L}^* is consistent, since

$\langle H_{\kappa^{++}}^V[G], \sigma'' \rangle$ is a model. But

$\mathcal{L}^* \in V[g]$. We now virtually repeat the proof of (6) in §3.4 to get:

(4) In $V[g]$ there is σ^* s.t.,

- $\sigma^* : \bar{N} \prec N''$ cofinally
- $\sigma^*(\bar{\alpha}, \bar{\beta}, \bar{\lambda}_i) = \sigma''(\bar{\alpha}), \sigma''(\bar{\beta}), \sigma''(\bar{\lambda}_i)$ ($i=1, \dots, m$)
- $\sigma^* \circ \bar{g} = g$
- $\sup \sigma^* \upharpoonright \tau = \sigma^*(\tau)$ whenever $\tau \geq \bar{\kappa} + \bar{N}$ & regular in \bar{N} .

proof (sketch)

We work in $V[g]$. As before let

$Y \prec H_{\kappa^{++}}$ be countable s.t. $\bar{N}, N'', \sigma'' \in Y$,

As before $Y \cap \sigma''(\tau)$ is cofinal in $\sigma''(\tau)$ whenever $\tau = \bar{\kappa}$ or $\tau \geq \bar{\kappa} + \bar{N}$ is regular in \bar{N} . Let $k : \bar{H} \xrightarrow{\sim} Y$, $k(\bar{N}'') = N''$,

$k(\bar{\sigma}'') = \sigma''$, $k(\bar{\delta}'') = \delta''$, $k(\bar{\mathcal{L}}^*) = \mathcal{L}^*$,

Then $k \upharpoonright \bar{N}_\alpha : \bar{N}_\alpha \prec N_\alpha$ cofinally. Moreover

$k \upharpoonright \bar{\sigma}''(\tau)$ is cofinal in $\sigma''(\tau)$ whenever

$\tau = \bar{\kappa}$ or $\tau \geq \bar{\kappa} + \bar{N}$ is regular in \bar{N} ,

$\bar{\mathcal{L}}^*$ is consistent & hence has a solid

model \mathcal{M} . Let $\sigma^* = \dot{\sigma} \upharpoonright \mathcal{M}$. Then

$\bar{\sigma}^* \in \text{wfc}(\bar{\sigma})$ and:

- $\bar{\sigma}^* : \bar{N} \prec \bar{N}''$ cofinally
- $\bar{\sigma}^*(\bar{\alpha}, \bar{P}, \bar{\lambda}_i) = \bar{\sigma}''(\bar{\alpha}), \bar{\sigma}''(\bar{P}), \bar{\sigma}''(\bar{\lambda}_i)$
- $\bar{g}^* = \bar{\sigma}^* \circ \bar{g}$, where $k(\bar{g}^*) = g$
- $\sup \bar{\sigma}^* \restriction \bar{\alpha} = \bar{\sigma}^*(\bar{\alpha})$ if $\bar{\alpha} = \bar{a}$ or $\bar{\alpha} \geq \bar{a} + \bar{N}$ is regular in \bar{N} ,

But then $\sigma^* = k\bar{\sigma}^*$ has the desired properties.

QED(4)

Since $\langle N'', \sigma'' \rangle = \text{the lift up of } \langle \bar{N}, \bar{\sigma} \restriction H_{\bar{a}++}^{\bar{N}} \rangle$,

there is $\pi_0 : N'' \prec N$ s.t. $\pi_0 \restriction H_{\bar{a}++}^{\bar{N}} = \text{id}$

and $\pi_0 \sigma'' = \sigma$. Set $\sigma_0 = \pi_0 \sigma^*$.

It follows easily that:

- $\sigma_0 : \bar{N} \prec N$ cofinally
- $\sigma_0(\bar{\alpha}, \bar{P}, \bar{\lambda}_i) = \alpha, P, \lambda_i$
- $\sup \sigma_0 \restriction \bar{\alpha}_i = \tilde{\alpha}_i$
- $g = \sigma_0 \circ \bar{g} = \sigma^* \circ \bar{g}$

But g, G are interdefinable in $V[g] = V[\sigma]$,

where G is a P -generic set. Similarly

for \bar{g}, \bar{G} in $\bar{N}[\bar{G}]$. Hence

• $\bar{G} = \sigma_0^{-1} \restriction \bar{G}$.

Since G is P -generic, there must be a $p \in G$ which forces all of this.

QED(Lemma 2)

This proof can easily be modified to show that \mathbb{P} is subproper above μ for each $\mu < \kappa$, in the sense of the definition at the end of § 2;

Letting $\sigma \restriction H_{\kappa}^{\bar{N}} : H_{\kappa}^{\bar{N}} \rightarrow \tilde{H}$ cofinally, we

have $\tilde{H} = H_{\bar{\kappa}}$, where $\bar{\kappa} = \sup \sigma''\kappa = \sigma'(\kappa)$, where $\langle N', \sigma' \rangle$ is defined

as above. But then $\sigma \restriction H_{\bar{\mu}}^{\bar{N}} \in \tilde{H}$ for

$\bar{\mu} < \bar{\kappa}$. We can thus add to \mathcal{L}' the

axiom: $\sigma \restriction H_{\bar{\mu}}^{\bar{N}} = \underline{\sigma \restriction H_{\bar{\mu}}^{\bar{N}}}$. Carrying this

axiom with us through the rest of

the proof, we arrive at $\sigma_0 \restriction H_{\bar{\mu}}^{\bar{N}} =$

$= \sigma \restriction H_{\bar{\mu}}^{\bar{N}}$, QED

Our further examples will all be reversible \mathcal{L} -forcings in the sense of [J]§3. This will be true even of Namba forcing, since we have shown in [J]§6 that Namba forcing is equivalent to such an \mathcal{L} -forcing.

From now on we assume a knowledge of:

[J]§3,

§ 3.6 Namba Forcing

Lemma 3 The forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of [J]§5.

Example 1 (p. 7) is subcomplete.

In this forcing we start with a regular $\beta \geq \omega_2$ s.t. $2^\omega = \omega_1$ and $2^\beta = \beta$. \mathbb{P} then collapses each regular $\delta \in (\omega_1, \beta]$ to ω_1 , making it ω -cofinal without collapsing reals. In [J]§6 we show that if $\beta = \omega_2$, then $\text{BA}(\mathbb{P}) = \text{BA}(\aleph_1)$, where \aleph_1 is Namba forcing. Hence:

Corollary 3.1 If $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$, then Namba forcing is subcomplete.

We assume that the reader has a good understanding of [J]§3. We shall also make use of Corollary 2.8 in [J]§4, which says that if G is \mathbb{P} -generic over V , then G is definable from $\langle M^G, \pi^G, B^G \rangle$ by:

$$\begin{aligned} P \in G &\leftrightarrow (M^P = M^G \upharpoonright (|P|+1) \wedge \pi^P = \pi^G \upharpoonright (|P|+1)^2 \wedge \\ &\wedge B^P = (\pi^G \upharpoonright_{|P|, \omega_1})^{-1} \cap B^G \wedge \\ &\wedge \langle \bar{a}, a \rangle \in \text{FP}_{|P|, \omega_1}^{\pi^G} : \langle M^P \upharpoonright_{|P|}, \bar{a} \rangle \in \langle M, a \rangle. \end{aligned}$$

We set: $M = L_\beta^A = \langle L_\beta[A], A \rangle$, where

$L_\beta[A] = H_\beta$. Set: $N = \langle H_{\beta^+}, M, \langle \dots \rangle \rangle$,