

## § 2 Iterating Subproper and Subcomplete Forcing

We essentially repeat our earlier definition.

Def A complete BA  $\mathbb{B}$  is subproper iff for sufficiently large cardinal  $\theta$ :

Let  $\mathbb{B} \in H_\theta$  and  $N = \langle L_\sigma[A], A, \mu \rangle$  where  $\sigma > \theta$  is

regular and  $H_\theta \subset N$ . Let  $\sigma: \bar{N} \prec N$  be countable and full. Let  $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\mu}, \bar{\lambda}_i) = \theta, \mathbb{B}, \mu, \lambda_i$  ( $1 \leq i \leq m < \omega$ ) where  $\bar{\lambda}_i \in (\omega_1, \theta)$  is regular and  $\bar{\mathbb{B}} \in \bar{\lambda}_i$  for  $i = 1, \dots, m$ .

Set  $\bar{\lambda}_0 = 0 \cap \bar{N}$ . For any  $a \in \bar{\mathbb{B}} \setminus \{0\}$  there is  $b \subset \sigma(a)$  which forces that if  $G \ni b$  is  $\mathbb{B}$ -generic, there is  $\sigma_0 \in V[G]$  with:

(a)  $\sigma_0: \bar{N} \prec N$

(b)  $\sigma_0(\bar{\theta}, \bar{\mathbb{B}}, \bar{\mu}, \bar{\lambda}_i) = \theta, \mathbb{B}, \mu, \lambda_i$  ( $i = 1, \dots, m$ )

(c)  $\sup \sigma_0 \text{ " } \bar{\lambda}_i = \sup \sigma \text{ " } \lambda_i$  ( $i = 0, \dots, m$ )

(d)  $\bar{G} = \sigma_0^{-1} \text{ " } G$  is  $\bar{\mathbb{B}}$ -generic over  $\bar{N}$ .

It is trivial to see that a subproper forcing cannot collapse  $\omega_1$ . (Clearly every proper forcing is subproper, since properness is just the case  $\sigma_0 = \sigma$ .)

Note  $\sigma_0$  extends uniquely in  $V[G]$  to a map

$\sigma_0^*: \bar{N}[G] \prec N[G]$  s.t.  $\sigma_0^*(\bar{G}) = G$ . But

$\bar{N}[G]$  is then full in  $V[G]$ . Thus the stage is set to handle a subproper

$\tilde{\mathbb{B}} \in V[G]$ .

Def We say that  $\mu$  verifies the subproperness of  $\mathbb{B}$  if the above holds for all cardinals  $\theta \geq \mu$ .

Def Let  $N = L_\alpha[A]$ ,  $\sigma: \bar{N} \prec N$  be as above. We then say that  $\sigma$  witnesses the subproperness of  $\mathbb{B}$  wrt.  $N$ . We also say that  $X = \text{rng}(\sigma)$  witnesses the subproperness of  $\mathbb{B}$ .

An apparent weakening of the notion of subproperness is:

Def IB is weakly subproper iff there is  $\aleph_1$ -t. for sufficiently large cardinals  $\theta$ , if  $N = L_\theta[A]$  is as above and  $X < N$  is countable and full  $\aleph_1$ -t.  $\aleph_1, \theta, IB \in N$ , then  $X$  witnesses the subproperness of IB.

However:

Lemma 1 If IB is weakly subproper, then it is subproper.

prf.

Let  $\theta_0$  be least  $\aleph_1$ -t. for some  $z \in H_{\theta_0}$  if  $\theta \geq \theta_0$  is a cardinal and  $N$  is as above and  $X < N$  is countable and full  $\aleph_1$ -t.  $\theta, IB, z \in X$ , then  $X$  witnesses subproperness.

At  $\mu \geq \theta_0$   $\aleph_1$ -t.  $\mu = \overline{V}_\mu$ , let  $\theta_0^\mu$  be defined as above in  $V_\mu$  rather than  $V$ . Clearly  $\theta_0^\mu \leq \theta_0^{\mu'} \leq \theta_0$  for  $\mu \leq \mu'$ . But then there is  $\mu_0$   $\aleph_1$ -t.  $\theta_0^{\mu_0} = \theta_0$ . At  $\mu \geq \mu_0$ ,

let  $A_\mu =$  the set of  $z \in H_{\theta_0}$  s.t.

if  $\theta_0 \leq \theta < \mu$ ,  $\theta$  is a cardinal,  $N$  is as above,  $X < N$  is countable and full s.t.  $\theta, IB, z \in X$ , then  $X$  witnesses subproperness. Then  $\mu_0 \leq \mu \leq \mu' \rightarrow$

$\rightarrow A_\mu \supset A_{\mu'} \subset H_{\theta_0}$ . Hence there is

$\mu_1$  s.t.  $A_\mu = A_{\mu_1}$  for  $\mu_1 \leq \mu$ . But

then  $A_{\mu_1} = A_\infty$ , where  $A_\infty$  is defined as above with  $\infty$  in place of  $\mu$ .

Now let  $\theta > \mu_1$  be a cardinal + let  $N = L_\tau[A]$  be as above. Then

$\langle \theta_0^\mu \mid \mu = \overline{\overline{\mu}} < \theta \rangle$  is  $N$ -

- definable in  $\theta, IB$ . Hence

$\mu_0$  is  $N$ -definable in  $\theta, IB$ . Hence

$\langle A_\mu \mid \mu_0 \leq \mu \wedge \forall \mu \in H_\theta \wedge \overline{\overline{\mu}} = \mu \rangle$  is

$N$ -definable in  $\theta, IB$ . Hence

$\mu_1, A_{\mu_1} = A_\infty$  are  $N$ -definable in

$\theta, IB$ . Now let  $X < N$  be full and countable s.t.  $\theta, IB \in X$ .

Then  $A_\infty \cap X \neq \emptyset$ . Hence  $X$

witnesses subproperness. QED

Lemma 2 Let  $B_0 \subseteq B_1$  where  $B_0$  is subproper and  $H_\theta \check{B}_1 / \check{G}_0$  is subproper. Then  $B_1$  is subproper.

prf.

Choose  $\theta$  large enough that  $B_1 \in H_\theta$ ,  $\theta$  verifies the subproperness of  $B_0$  and  $H_\theta \check{\theta}$  verifies the subproperness of  $\check{B}_1 / \check{G}_0$ .

Let  $N = L_\tau[A]$  where  $\tau > \theta$  is regular and  $H_\theta \subset N$ . Let  $X \subset N$  be full and countable s.t.  $\theta, B_0, B_1 \in X$ . By Lemma 1 it suffices to show that  $X$  witnesses the subproperness of  $B_1$ . Let  $a \in X \cap (B \setminus \{0\})$ . Let  $\lambda_1, \dots, \lambda_m \in X$  s.t.  $\lambda_i \in (a \uparrow \theta)$  is regular and  $\lambda_1 < \lambda_2$ .

Let  $\sigma: \bar{N} \xrightarrow{\sim} X$ ,  $\sigma(\bar{a}, \bar{B}_0, \bar{B}_1, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\theta}) = a, B_0, B_1, \lambda_1, \lambda_2, \theta$ . Let  $\bar{\lambda}_0 = 0 \cap \bar{N}$ . By the subproperness of  $B_0$  there is  $b_0 \in a_0 = \bar{a} \uparrow b_0$  which forces the existence of  $\sigma_0 \in V[G_0]$  s.t.

- (a)  $\sigma_0: \bar{N} \subset N$
- (b)  $\sigma_0(\bar{a}, \bar{\lambda}_i, \bar{B}_0, \bar{B}_1, \bar{\theta}) = a, \lambda_i, B_0, B_1, \theta$  ( $i=1, \dots, m$ )
- (c)  $\sup \sigma_0 \bar{\lambda}_i = \check{\lambda}_i = \sup \sigma \bar{\lambda}_i$  ( $i=0, \dots, m$ )
- (d)  $\bar{G}_0 = \sigma_0^{-1} \check{G}_0$  is  $\bar{B}_0$ -generic over  $\bar{N}$ , where  $G_0$  is  $B_0$ -generic over  $V$ .

But then  $\sigma_0$  extends uniquely to a  $\sigma_0^*: \bar{N}[\bar{G}] \subset N[G]$  s.t.  $\sigma_0^*(\bar{G}) = G$ .

Hence  $X_0 = \text{rang}(\sigma_0^*) \subset N[G_0]$  is full in  $V[G_0]$ .

Note that  $b_0 \in h_0(a) = \left[ \frac{\check{a}}{G_0} \neq 0 \right]_{B_0}$ . Hence

$a' = a/G_0 \neq 0$  in  $B' = B_0/G_0$  and  $a' \in X_0$ .

But then there is  $b' \subset a'$  in  $B'$  which forces the existence of  $\sigma' \in V[G_0][G']$  (where  $G'$  is  $B'$ -generic over  $V[G_0]$ ) s.t.

(a')  $\sigma' : \bar{N}[\bar{G}_0] \subset N[G_0]$

(b')  $\sigma'(\bar{\lambda}, \bar{\lambda}_i, \bar{B}_0, \bar{B}_1, \bar{B}', \bar{G}_0, \bar{\theta}) = \lambda, \lambda_i, B_0, B_1, B', G_0, \theta$

where  $i=1, \dots, m$  and  $\bar{B}' = \sigma'^{-1}(B') = \bar{B}_1/\bar{G}_0$ .

(c')  $\text{sup } \sigma' \text{ " } \bar{\lambda}_i = \tilde{\lambda}_i \text{ (} i=0, \dots, m \text{)}$

(d')  $\bar{G}' = \sigma'^{-1}G'$  is  $\bar{B}'$ -generic over  $\bar{N}[\bar{G}_0]$ .

We may assume  $b' = b \cdot G_0$ , where  $b \in B_1/G_0$  and  $b_0$  forces that  $b$  forces the above to hold. But then there is  $b_1 \in B_1$  s.t.  $b \cdot G_0 = b_1/G_0$ . Since  $b_0 \Vdash b = \check{b}_1/G_0 \neq 0$ , we have  $b_0 \subset h_0(b_1)$ . Hence  $b \neq 0$ , where  $b = b_0 \cap b_1$ . Since  $b_0 \Vdash \check{b}_1/G_0 \subset \check{a}/G_0$ , we have  $b = b_0 \cap b_1 \subset a$ . Now let  $G_1 \ni b$  be  $B_1$ -generic. Set:

$$G_0 = G_1 \cap B_0; G' = \{ b/G_0 \mid b \in G_1 \}$$

Then  $G_0$  is  $B_0$ -generic and  $G'$  is  $B' = B_1/G_0$ -generic over  $V[G_0]$ . Since  $b_0 \in G_0$ , there is  $\sigma_0 \in V[G_0]$  satisfying (a)-(d) above. This gives  $\sigma_0^* : \bar{N}[\bar{G}_0] \subset N[G_0]$  s.t.  $\sigma_0^*(\bar{G}_0) = G_0$ , and  $b' = b \cdot G_0 = b_1/G_0$ . Since  $b' \in G'$ , there is  $\sigma' \in V[G_0][G'] = V[G_1]$

satisfying (a') - (d') above. Set:

$$\sigma_1 = \sigma' \upharpoonright \bar{N}. \text{ Then}$$

$$(a) \sigma_1 : \bar{N} \prec N$$

$$(b) \sigma_1(\bar{\alpha}, \bar{\lambda}_i, \bar{B}_1, \bar{\theta}) = \alpha, \lambda_i, B_1, \theta \quad (i=1, \dots, m)$$

$$(c) \sup \sigma_1 \text{ " } \bar{\lambda}_i = \tilde{\lambda}_i \quad (i=0, \dots, m)$$

It remains only to show:

Claim  $\bar{G}_1 = \sigma_1^{-1} \text{ " } G_1$  is  $\bar{B}_1$ -generic over  $\bar{N}$ ,

pf.

Clearly  $G_1 = G_0 * G' = \{b \mid b/G_0 \in G'\}$ .

Set  $\bar{G}_1 = \bar{G}_0 * \bar{G}'$ . Then  $\bar{G}_1$  is  $\bar{B}_1$ -generic over  $\bar{N}$ , since  $\bar{G}_0$  is  $\bar{B}_0$ -generic over  $\bar{N}$  and  $\bar{G}'$  is  $\bar{B}'$ -generic over  $\bar{N}[\bar{G}_0]$ . But

$$b \in \bar{G}_1 \iff b/\bar{G}_0 \in \bar{G}' \iff \sigma'(b/\bar{G}_0) = \sigma_1(b)/\bar{G}_0 \in G' \\ \iff \sigma_1(b) \in G_1.$$

QED (Lemma 2)

We recall the definition of subcompleteness:

Def  $\bar{B}$  is subcomplete iff for suit. large cardinal  $\theta$ :  $\forall \bar{B} \in H_\theta$  and  $N, \bar{\alpha}, \sigma, \lambda_i, \bar{N}$ :  $\bar{B}, \bar{\theta}, \bar{\alpha}, \bar{\lambda}_i$  are as in the def of "subproper", and  $\bar{G}$  is  $\bar{B}$ -generic over  $\bar{N}$ , then there is  $b \in \bar{B}$  which forces the existence of  $\sigma_0$  satisfying (a) - (d).

[The difference is that  $\bar{G}$  was chosen in advance.]

We leave it to the reader to see that subcomplete forcing adds no reals. It is also straightforward to define the notion "weakly subcomplete" and prove the analogue of Lemma 1.

By a slight modification of the proof just given we obtain:

Lemma 3 Lemma 2 holds with "subcomplete" in place of "subproper".

proof (sketch)

$\bar{G}_1$  is given at the outset and we set:

$$\bar{G}_0 = \bar{G}_1 \cap \bar{B}_0 ; \bar{G}' = \{ b/\bar{G}_0 \mid b \in \bar{G}_1 \}.$$

Then  $\bar{G}_1 = \bar{G}_0 * \bar{G}'$ ,  $\bar{G}_0$  is  $\bar{B}_0$ -generic over  $\bar{N}$  and  $\bar{G}'$  is  $\bar{B}'$ -generic over  $\bar{N}[\bar{G}_0]$ , where

$$\bar{B}' = \bar{B}_1 / \bar{G}_0. \text{ As before we are assuming}$$

$\sigma : \bar{N} \prec N$  to be full and

$$\sigma(\bar{\alpha}, \bar{\lambda}, \bar{B}_0, \bar{B}_1, \bar{\theta}) = \alpha, \lambda, B_0, B_1, \theta.$$

Let  $b_0 \in B_0 \setminus \{0\}$  force the existence of  $\sigma_0 \in V[G_0]$  satisfying (a)-(d). Then  $\sigma_0$  extends uniquely to  $\sigma_0^* : \bar{N}[\bar{G}_0] \prec N[G_0]$  as before, where  $\bar{N}[\bar{G}_0]$  is full in  $V[G_0]$ .

Thus  $\sigma_0^*(\bar{B}') = B' = B_1 / G_0$ . Since  $\bar{G}'$  is  $\bar{B}'$ -generic over  $\bar{N}[\bar{G}_0]$  and  $B'$  is subcomplete, there is  $b' \in B' \setminus \{0\}$

forcing the existence of  $\sigma'$  satisfying (a')-(d'). Let  $b, b_1$  be as before.



Just as before,  $b = b_0 \cap b_1 \neq 0$ . We then let  $G_1 \ni b$  be  $\mathbb{B}_1$ -generic and finish the proof exactly as before. QED (Lemma 3)

Without proof we mention:

Lemma 4 Lemma 2 holds with "semisubproper" in place of "subproper."

The proof is again essentially the same, but the modification is somewhat more complex.

(It facilitates proofs of this sort if we replace "full" by "weakly full" (in the sense of the next section § 3), in the definition of "semisubproper". This is an inessential weakening.)

Theorem 5 Let  $IB = \langle IB_i, 1 \leq i \leq d \rangle$  be an RCS iteration s.t.  $\text{It}_i (IB_{i+1} / G_i \text{ is subproper})$  for  $i < d$ .

Suppose moreover that  $\bar{3} \leq \bar{1B}_3$  and  $IB_{\bar{3}+1}$  collapses  $\bar{1B}_3$  to  $\omega_1$  for  $\bar{3} < d$ . Then each  $IB_{\bar{3}}$  is subproper, proof

By induction on  $i$  we show:

(\*)  $\text{It}_h (IB_i / G_h \text{ is subproper})$  for  $h \leq i$ ,

$h=0$  then gives the desired result. We note that if  $G_h$  is  $IB_h$ -generic and  $\bar{1B} = IB / G_h$ ,

then  $\bar{1B}_{i-h} = IB_i / G_h$  for  $h \leq i \leq d$ . Moreover

$\text{It}_{\bar{1B}_{i-h}} (IB_{(i-h)+1} / G_{i-h} \text{ is subproper})$  for  $h \leq i < d$

holds in  $V[G_h]$ . (To see this, let  $\bar{G}$

be  $\bar{1B}_{i-h}$ -generic over  $V[G_h]$ . Then

$G_h \times \bar{G} = G_i$  is  $IB_i$ -generic over  $V$ .

Moreover  $\bar{1B}_{i-h+1} / \bar{G} = (IB_{i+1} / G_h) / \bar{G} =$

$= IB_{i+1} / G_i$  is subproper in  $V[G_i] =$

$= V[G_h][\bar{G}]$ .)

This means in practice that once we have shown  $IB_i$  to be subproper, we can

simply repeat the proof in  $V[G_h]$  to

show that  $IB_i / G_h = \bar{1B}_{i-h}$  is subproper, where  $G_h$  is  $IB_h$ -generic.

Case 1  $i=0$ . Trivial since  $\mathbb{B}_0 = \{0, 1\}$  is subproper,

Case 2  $i = j+1$ .

$h=i$  is trivial.

$h=0$ :  $\mathbb{B}_j$  is subproper by the ind. hyp and  $\mathbb{B}_{j+1}/G_j$  is subproper, hence  $\mathbb{B}_{j+1}$  is subproper by Lemma 2. QED

To get the result for other  $h \leq j$ , simply repeat the proof in  $V[G_h]$ .

QED (Case 2)

Case 3  $i = \lambda$ ,  $\text{Lim}(\lambda)$ .

$h = \lambda$  is again trivial. For the usual reasons it will suffice to prove it for  $h=0$ .

Case 3.1  $\text{cf}(\lambda) \leq \max(\omega_1, \overline{\mathbb{B}_i})$  for an  $i < \lambda$ .

Then there is  $\gamma < \lambda$  s.t.  $\mathbb{B}_\gamma \leq \omega_1$ .

Since we know that  $\mathbb{B}_\gamma$  is subproper, it suffices by Lemma 2 to show:

Claim  $\mathbb{B}_\lambda / G_\gamma$  is subproper

But this says that, if  $G_\gamma$  is  $\mathbb{B}_\gamma$ -generic, then  $\tilde{\mathbb{B}}_{\lambda-\gamma}$  is subproper, where

$\tilde{\mathbb{B}} = \mathbb{B}/G_\gamma$ . Hence we may assume

w.l.o.g. that  $\text{cf}(\lambda) \leq \omega_1$  in  $V$

(taking  $V$  as  $V[G_\gamma]$ ,  $\mathbb{B}$  as  $\tilde{\mathbb{B}}$  and

$\lambda$  as  $\lambda-\gamma$ .)

Let  $\theta$  be big enough that  $\mathbb{B} \in H_\theta$  and  $\theta$  verifies the subproperness of  $\mathbb{B}_i$  for  $i < \lambda$ .  
 Let  $\tau > \theta$  be regular and set:  $N = \langle L_\tau[A], A, \dots \rangle$   
 where  $H_\theta \subset N$ . Let  $\sigma: \bar{N} \prec N$ ,  $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}) = \theta, \mathbb{B}, \lambda$ , where  $\bar{N}$  is countable and full.

Claim  $\sigma$  witnesses the subproperness of  $\mathbb{B}_\lambda$  wrt.  $N$ .  
 In other words, if  $\sigma(\bar{\lambda}) = \lambda$  and  $\sigma(\bar{\lambda}_i) = \lambda_i$

for  $i = 1, m, m$ , where  $\lambda_i$  is regular with  $\bar{\mathbb{B}}_\lambda < \lambda_i$  ( $i = 1, m, m$ ), and given  $a \in \mathbb{B}_\lambda \setminus \{0\}$ , there is  $b < a$ ,  $b \in \mathbb{B}_\lambda \setminus \{0\}$ , s.t. whenever  $G \ni b$  is  $\mathbb{B}_\lambda$ -generic, there is  $\tilde{\sigma} \in V[G]$  s.t.

- (a)  $\tilde{\sigma}: \bar{N} \prec N$
- (b)  $\tilde{\sigma}(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{\lambda}_i, \bar{\lambda}_i) = \theta, \mathbb{B}, \lambda, \lambda, \lambda_i$  ( $i = 1, m, m$ )
- (c)  $\sup \tilde{\sigma}'' \bar{\lambda}_i = \tilde{\lambda}_i = \sup \sigma'' \lambda_i$  for  $i = 0, m, m$ , where  $\bar{\lambda}_0 = 0 \cap \bar{N}$ .
- (d)  $g = \tilde{\sigma}^{-1}'' G$  is  $\bar{\mathbb{B}}_\lambda$ -generic over  $\bar{N}$ .

proof.  
 Let  $\langle \bar{x}_i \mid i < \omega \rangle$  enumerate  $\bar{N}$  and let  $\langle \bar{\Delta}_i \mid i < \omega \rangle$  enumerate the  $\Delta \in \bar{N}$  s.t.  $\Delta$  is  $\bar{\mathbb{B}}_\lambda$ -strongly  $\bar{\mathbb{B}}_\lambda$ -dense in  $\bar{\mathbb{B}}_\lambda \setminus \{0\}$ . Since  $\text{cf}(\lambda) \leq \omega_1$  there is  $\bar{f} \in \bar{N}$  s.t.  $f = \sigma(\bar{f})$  maps  $\omega_1$  to  $\lambda$  s.t.  $\sup f'' \omega_1 = \lambda$ . Pick  $\bar{z}_i \in \text{rng}(f)$  s.t.  $\langle \bar{z}_i \mid i < \omega \rangle$  is monotone and cofinal in  $\bar{\lambda}$  with  $\bar{z}_0 = 0$ . Set  $\bar{z}_i = \sigma(\bar{z}_i)$ .

We first define a sequence  $\langle b_n \mid n < \omega \rangle$  in  $\mathbb{R}(b_n, \bar{B} \upharpoonright \bar{X})$  in  $\bar{N}$ ,  $b_{n+1}^* \subset b_n^*$ , and  $b_{n+1} \upharpoonright \bar{Z}_{n+1} = b_n \upharpoonright \bar{Z}_{n+1}$  as follows:

Pick  $b_0$  s.t.  $b_0^* \subset \bar{a}$  and  $b_0^* \in \Delta_0$ . Given  $b_n$  we construct  $b_{n+1}$  s.t.

$\{a \in \bar{B}_{\bar{Z}_{n+1}} \mid a \cap b_{n+1}^* \in \Delta_{n+1}\}$  is dense

below  $\tilde{a} = h_{\bar{Z}_{n+1}}(b_n^*) = (b_n \upharpoonright \bar{Z}_{n+1})^*$ .

We accomplish this as follows:

Set  $\Delta = \{b \mid \mathbb{R}(b, \overline{B} \cap \overline{\lambda}) \wedge b^* \subset b_n^* \wedge b^* \in \Delta_{n+1}\}$ .

Then  $\{b^* \mid b \in \Delta\}$  is dense below  $b_n^*$  in  $\overline{B}_X \setminus \{0\}$ . Thus  $\Delta' = \{h_{\overline{\Sigma}_{n+1}}(b^*) \mid b^* \in \Delta\}$  is dense below  $\tilde{a} =_{\text{df}} h_{\overline{\Sigma}_{n+1}}(b_n^*)$  in  $\overline{B}_X \setminus \{0\}$ .

Let  $A$  be a maximal antichain in  $\Delta'$ . Then  $\cup A = \tilde{a}$ . For each  $a \in A$  choose a  $b_a \in \Delta$  s.t.  $a = h_{\overline{\Sigma}_{n+1}}(b_a^*) = (b_a \upharpoonright \overline{\Sigma}_{n+1})^*$ . Set:

$$b_{n+1}(i) = \begin{cases} b_n(i) & \text{if } i < \overline{\Sigma}_{n+1} \\ \bigcup_{a \in A} (a \upharpoonright b_a(i)) \cup \tilde{a} & \text{if } i \geq \overline{\Sigma}_{n+1} \end{cases}$$

We claim that  $b_{n+1}$  has the desired properties. We first show that it is a good sequence for  $\overline{B} \cap \overline{\lambda}$ .

(1)  $h_i(b_{n+1}(i)) = 1$

Trivial for  $i < \overline{\Sigma}_{n+1}$ . Now let  $i \geq \overline{\Sigma}_{n+1}$ . Then

$$h_i(b_{n+1}(i)) = \bigcup_{a \in A} a \cup \tilde{a} = \tilde{a} \cup \tilde{a} = 1,$$

and since  $h_i(b_a(i)) = 1$ . QED(1)

(2)  $h_i(b_{n+1}^*) = (b_{n+1} \upharpoonright i)^*$ .

We recall the disjoint distributive law which holds in every complete BA:

(DDL) Let  $b = \bigcup_{i \in I} b_i$ , where  $b_i \cap b_j = 0$  for  $i \neq j$ .

Let  $a_i^j \subset b_i$  for  $i \in I, j \in J$ . Then

$$\bigcap_i \bigcup_j a_i^j = \bigcup_i \bigcap_j a_i^j,$$

wh.

$$\bigcap_i \bigcup_j a_i^j = b \cap \bigcap_i \bigcup_j a_i^j = \bigcup_i (b_i \cap \bigcap_j a_i^j) =$$

$$= \bigcup_i \bigcap_j (b_i \cap a_i^j) = \bigcup_i \bigcap_j a_i^j. \text{ QED(DDL)}$$

As a step toward proving (2) we first note:

$$(3) \text{ If } j \geq \bar{j}_{m+1}, \text{ then } (b_{m+1} \uparrow j)^* = \bigcup_{a \in A} (b_a \uparrow j)^*.$$

w.t.

$$\begin{aligned} (b_{m+1} \uparrow j)^* &= \tilde{a} \cap \bigcap_{i \in [\bar{j}_{m+1}, j)} b_{m+1}(i) \\ &= \tilde{a} \cap \bigcap_{i \in [\bar{j}_{m+1}, j)} \bigcup_{a \in A} (a \cap b_a(i)) \\ &= \tilde{a} \cap \bigcup_{a \in A} \bigcap_{i \in [\bar{j}_{m+1}, j)} (a \cap b_a(i)) \\ &= \tilde{a} \cap \bigcup_{a \in A} (b_a \uparrow j)^* = \bigcup_{a \in A} (b_a \uparrow j)^* \end{aligned}$$

since  $a = (b_a \uparrow \bar{j}_{m+1})^*$  for  $a \in A$ , QED(3)

We now prove (2). For  $j \in [\bar{j}_{m+1}, \lambda)$  have:

$$\begin{aligned} h_j(b_{m+1}^*) &= h_j\left(\bigcup_{a \in A} b_a^*\right) = \bigcup_{a \in A} h_j(b_a^*) = \\ &= \bigcup_{a \in A} (b_a \uparrow j)^* = (b_{m+1} \uparrow j)^*. \end{aligned}$$

For  $j < \bar{j}_{m+1}$  we have  $h_{\bar{j}_{m+1}}(b_{m+1}^*) = \bigcup A = \tilde{a}$

by (3) and hence:

$$\begin{aligned} h_j(b_{m+1}^*) &= h_j h_{\bar{j}_{m+1}}(b_{m+1}^*) = h_j(\tilde{a}) = \\ &= h_j((b_{m+1} \uparrow \bar{j}_{m+1})^*) = (b_{m+1} \uparrow j)^* = (b_{m+1} \uparrow j)^*. \end{aligned}$$

QED(2)

By (1) + (2)  $b_{m+1}$  is a good sequence:

$$(4) \text{ GS}(b_{m+1}, \bar{B} \uparrow \bar{\lambda}).$$

But then:

(5)  $P(b_{m+1}, \overline{B} \cap \overline{A})$

pf. Let  $c \in U_{\overline{X}}(b_{m+1})$ . Then  $c \in \overline{B}_i$  and  $c \subset (b_{m+1} \cap i)^*$  for some  $i$ . We may assume  $i \geq \overline{\Sigma}_m$ . Hence

$$c \subset \bigcup_{a \in A} (b_a \cap i)^*. \text{ Let } c' = c \cap (b_a \cap i)^* \neq \emptyset.$$

Then  $c' \in U_{\overline{X}}(b_a)$  and there is  $d \subset c'$  s.t.  $d \in S_{\overline{X}}(b_a)$ . It follows easily that  $d \in S_{\overline{X}}(b_{m+1})$ . QED(5)

Finally we note:

(6)  $\{a \in \overline{B}_{\overline{\Sigma}_{n+1}} \setminus \{0\} \mid a \cap b_{m+1}^* \in \Delta_{n+1}\}$  is

dense below  $\tilde{a} = h_{\overline{\Sigma}_{n+1}}(b_m)$  in  $\overline{B}_{\overline{\Sigma}_{n+1}} \setminus \{0\}$ .

pf. It suffices to show that it is pre-dense. But  $\cup A = \tilde{a}$  and

$$a \cap b_{m+1}^* = a \cap \bigcup_{a' \in A} b_{a'}^* = b_a^* \in \Delta_{n+1}$$

for  $a \in A$ . QED(6)

This completes the construction of  $\langle b_m \mid m < \omega \rangle$ .



By induction on  $n < \omega$  we construct:

•  $\langle c_n \mid n < \omega \rangle$  s.t.  $R(c_n, \mathbb{B} \upharpoonright \bar{\Sigma}_n)$

(Hence  $c_0 = \emptyset, c_0^* = 1, \text{ and } \omega \bar{\Sigma}_0 = 0$ )

•  $\langle \dot{\sigma}_n \mid n < \omega \rangle, \langle \dot{u}_n \mid n < \omega \rangle, \langle \dot{g}_n \mid n < \omega \rangle$

s.t.  $\dot{\sigma}_n, \dot{u}_n, \dot{g}_n \in \mathcal{V} \mathbb{B}_{\bar{\Sigma}_n}$

We inductively verify:

(a)  $c_{n+1} \upharpoonright \bar{\Sigma}_n = c_n$

(b)  $\dot{\sigma}_0 = \check{\sigma}, \dot{g}_0 = \{\check{1}\}, \dot{u}_0 = \check{u}_0$ , where

$\check{u}_0 = \langle \check{x}, \check{x}_1, \dots, \check{x}_m, \check{f}, \check{\mathbb{B}}, \check{\theta}, \check{\lambda} \rangle$

(c)  $c_n^*$  forces the following to hold in  $\mathcal{V} \mathbb{B}_{\bar{\Sigma}_n}$ :

•  $\dot{g}_n$  is  $\check{\mathbb{B}}_{\bar{\Sigma}_n}$ -generic over  $\check{N}$

•  $(b_n \upharpoonright \bar{\Sigma}_n)^* \in \dot{g}_n$

•  $\dot{\sigma}_n : \check{N}[\dot{g}_n] \prec N[G_n] \wedge \dot{\sigma}_n(\dot{g}_n) = \dot{G}_n$

(where  $\dot{G}_n$  is the canonical generic name in  $\mathcal{V} \mathbb{B}_{\bar{\Sigma}_n}$ )

(d) At  $n > 0$ , then  $c_n^*$  forces:

•  $\dot{\sigma}_n(\dot{u}_{n-1}) = \dot{\sigma}_{n-1}(\dot{u}_{n-1})$

•  $\dot{u}_n = \langle \check{x}_n, \check{z}_n^0, \dots, \check{z}_n^m, \dot{u}_{n-1}, \dot{g}_n, \check{b}_n \rangle$ ,

where:

•  $\check{z}_n^l =$  the least  $z < \check{\lambda}_l$  s.t.  $\dot{\sigma}_n(z) \geq \check{\Sigma}_n^l$

(where  $\langle x_i \mid i < \omega \rangle$  enumerates  $\check{N}$ ,  $\langle \check{\Sigma}_n^l \mid n < \omega \rangle$

is a monotone, cofinal sequence in

$\check{\lambda}_i = \sup \sigma \check{\lambda}_i$  ( $i = 0, m, m$ ), and

$\check{\lambda}_0 = 0_m \cap \check{N}$

Since  $c_n = c_{n+1} \upharpoonright \bar{\Sigma}_n$  and  $\tilde{\lambda} = \sup_n \bar{\Sigma}_n$  is  $\omega$ -  
 - cofinal in  $V$ , we conclude that

$$\mathbb{P}(c, \mathbb{B} \upharpoonright \tilde{\lambda}), \text{ where } c = \bigcup_m c_m.$$

But then  $c^* \in \mathbb{B}_{\tilde{\lambda}} \setminus \{0\}$  and  $h_{\bar{\Sigma}_n}(c^*) = c_n^*$ .

Let  $G \ni c^*$  be  $\mathbb{B}_{\tilde{\lambda}}$ -generic. Set:

$$G_n = G \cap \mathbb{B}_{\bar{\Sigma}_n} = \dot{G}_{\bar{\Sigma}_n}^G; \quad g_n = \dot{g}_{\bar{\Sigma}_n}^{G_n}, \quad \sigma_n = \dot{\sigma}_n^{G_n}$$

Note that  $\sigma_n(\bar{\Sigma}_h) = \bar{\Sigma}_h$  for all  $m, h < \omega$ ,

since  $\sigma_n(\bar{f}) = \bar{f}$  for all  $n$ . Then

$$\sigma_n : \bar{N}[g_n] \prec \tilde{N}[G_n] \wedge \sigma_n(g_n) = G_n \quad (n < \omega).$$

Since  $\sigma_m(x_h) = \sigma_h(x_h)$  for  $h < m$ , we can

define a new map  $\tilde{\sigma} : \bar{N} \prec \tilde{N}$  by:

$$\tilde{\sigma}(x) =_{\mathbb{N}} \sigma_n(x) \text{ for suit. large } n.$$

$$\text{Since } \sigma_m(g_{m-1}) = \sigma_{m-1}(g_{m-1}) = G_{m-1} = G \cap \mathbb{B}_{\bar{\Sigma}_{m-1}}$$

for  $m > 0$ , it follows easily that:

$$\sigma_m(g_h) = G_h = G \cap \mathbb{B}_{\bar{\Sigma}_h} \text{ for } h \leq m < \omega,$$

$$\text{Thus } g_m = \sigma_{m+1}^{-1} \upharpoonright G_m = \tilde{\sigma}^{-1} \upharpoonright G_m \quad (m < \omega).$$

$$\text{Set } g = \tilde{\sigma}^{-1} \upharpoonright G. \text{ Then } g \cap \bar{\mathbb{B}}_{\bar{\Sigma}_n} = g_n.$$

Claim 1  $g$  is  $\bar{\mathbb{B}}_{\tilde{\lambda}}$ -generic over  $\bar{N}$ ,

$$\text{p.f. } (b_m \upharpoonright \bar{\Sigma}_{m+1})^* \in g_{m+1} \subset g \text{ for } m < \omega,$$

$$\text{where } (b_m \upharpoonright \bar{\Sigma}_{m+i+1})^* = (b_{m+1} \upharpoonright \bar{\Sigma}_{m+i+1})^*.$$

Hence  $(b_m \upharpoonright \bar{\Sigma}_{n+i})^* \in \mathfrak{g}$  for  $n+i < \omega$ . Hence

$$b_m^* = \bigcap_{i < \omega} (b_m \upharpoonright \bar{\Sigma}_{n+i})^* \in \mathfrak{g}. \text{ But}$$

$\tilde{\Delta} = \{a \in (b_m \upharpoonright \bar{\Sigma}_n)^* \mid a \upharpoonright b_m^* \in \Delta_n\}$  is dense in  $\bar{B}_{\bar{\Sigma}_n} \setminus \{0\}$ . Hence  $\mathfrak{g} \cap \tilde{\Delta} \neq \emptyset$ , since

$\mathfrak{g}$  is  $\bar{B}_{\bar{\Sigma}_n}$ -generic over  $\bar{N}$ . Let  $a \in \mathfrak{g} \cap \tilde{\Delta}$ .

Then  $a \upharpoonright b_m^* \in \mathfrak{g}$ , where  $a \upharpoonright b_m^* \in \Delta_n$ .

Hence  $\mathfrak{g} \cap \Delta_n \neq \emptyset$  for all  $n$  QED (Claim 1)

Claim 2  $\sup \tilde{\sigma} \text{ " } \bar{\lambda}_i = \tilde{\lambda}_i = \sup \sigma \text{ " } \lambda_i$   
for  $i = 0, \dots, n$

( $\geq$ )  $\tilde{\sigma}(z_n^i) = \sigma_n(z_n^i) \geq \bar{\Sigma}_n^i$  where  $\sup \bar{\Sigma}_n^i = \lambda_n^i$

( $\leq$ )  $\tilde{\sigma}(x) = \sigma_n(x) = \tilde{\lambda}_i$  for some  $n$  if  $x < \bar{\lambda}_i$ .

QED (Claim 2)

(Note that by Claim 2, we have:

$\tilde{\sigma} : \bar{N} \prec \tilde{N}$  cofinally, since  $\lambda_0 = 0 \cap \bar{N}$ .)

Trivially

Claim 3  $\tilde{\sigma}(\bar{\theta}, \bar{B}, \bar{\lambda}, \bar{\lambda}_i) = \theta, B, \lambda, \lambda_i$

This proves that  $\sigma$  witnesses the  
subproperness of  $B_\lambda$ .

It remains only to carry out the  
construction of  $C_n$  ( $n < \omega$ ).

The construction of  $c_0 = \phi, \dot{\sigma}_0 = \hat{\sigma}, \dot{u}_0 = \check{u}, g_0 = \{1\}^{\vee}$

has already been given. Now let  $c_n, \dot{\sigma}_n, \dot{u}_n, g_n$  be given s.t. (a) - (d) hold.

Let  $G_n \ni C_n^*$  be  $\mathbb{B}_{\Sigma_n}$ -generic. Set:

$$\sigma_n = \dot{\sigma}_n^{G_n}, u_n = \dot{u}_n^{G_n}, g_n = \dot{g}_n^{G_n}. \text{ Then}$$

$$\sigma_n : \bar{N}[g_n] \hookrightarrow N[G_n] \text{ and } \sigma_n(g_n) = G_n.$$

Hence  $X_n = \text{rng}(\sigma_n) \hookrightarrow N[G_n]$  is full

in  $V[G_n]$ . But  $\tilde{\mathbb{B}} = \mathbb{B}_{\Sigma_{n+1}} / G_n$  is subproper

in  $V[G_n]$ . Clearly  $\sigma_n(\tilde{\mathbb{B}}_{\Sigma_{n+1}} / g_n) = \tilde{\mathbb{B}}$ .

Moreover  $\sup \sigma_n \ulcorner \check{\lambda}_l = \check{\lambda}_l \urcorner = \sup \sigma \ulcorner \lambda_l$

for  $l = 0, \dots, m$ . Since:

$$\begin{aligned} \llbracket (b_m \uparrow \check{\Sigma}_{n+1})^* / g_n \neq 0 \rrbracket_{\mathbb{B}_{\Sigma_n}} &= h_{\check{\Sigma}_n} (b_m \uparrow \check{\Sigma}_{n+1})^* \\ &= h_{\check{\Sigma}_n} h_{\check{\Sigma}_{n+1}} (b_m^*) = (b_m \uparrow \check{\Sigma}_n)^* \in g_n \end{aligned}$$

we have:  $(b_m \uparrow \check{\Sigma}_{n+1})^* / g_n \neq 0$ .

Set:  $\bar{b} = (b_m \uparrow \check{\Sigma}_{n+1})^* / g_n$  and

$$b = \sigma_n(\bar{b}) = (\sigma_n(b_m) \uparrow \check{\Sigma}_{n+1})^* / G_n.$$

Then  $b \in \tilde{\mathbb{B}} \cap X$  and  $b \neq 0$ . Hence

there is a condition  $\tilde{c} \in \tilde{\mathbb{B}} \setminus \{0\}$  s.t.

$\tilde{c} \subset b$  and  $\tilde{c}$  forces the following

to hold:

(\*) Let  $\tilde{G} \ni \tilde{c}$  be  $\mathbb{B}$ -generic over  $V[G_m]$ . Then

there is  $\tilde{\sigma} \in V[G_m][\tilde{G}]$  s.t.

(a)  $\tilde{\sigma} : \bar{N}[q_m] \hookrightarrow N[G_m]$  s.t.  $\tilde{\sigma}(q_m) = G_m$

(b)  $\tilde{\sigma}(u_n) = \sigma_n(u_n)$

(c)  $\sup \tilde{\sigma} \upharpoonright \bar{\lambda}_i = \tilde{\lambda}_i \quad (i = 0, \dots, m)$

(d)  $\tilde{q} = \tilde{\sigma}^{-1} \upharpoonright \tilde{G}$  is  $\mathbb{B}_{\bar{\Sigma}_{m+1}} / q_m$ -generic over  $\bar{N}[q_m]$ .

Note that  $\tilde{\sigma}(b_n) = \sigma_n(b_n)$  by (b). Hence  $\tilde{\sigma}(\bar{b}) = \sigma_n(\bar{b}) = b$ . Since  $\tilde{c} \in b$ , we then have:

$(b_m \upharpoonright \bar{\Sigma}_{m+1}) / q_m \in \tilde{q}$ .

W.l.o.g. we may assume  $\tilde{c} = c^*$ , where  $R(c, (\mathbb{B} \upharpoonright \bar{\Sigma}_{m+1}) / G_m)$ . We may also assume  $c = \dot{c} \dot{G}_m$ , where  $\Vdash_{\bar{\Sigma}_m} R(\dot{c}, (\mathbb{B} \upharpoonright \bar{\Sigma}_{m+1}) / \dot{G}_m)$  and  $c_m$  forces the statements <sup>about  $\dot{c}$</sup>  which ensure (\*) for  $c_m \in G_m$ .

$\Vdash_{\bar{\Sigma}_m} \bar{\Sigma}_m \leq i < \bar{\Sigma}_{m+1}$  set:

$c'(i) = 1$  if  $b \in \mathbb{B}_{i+1}$  s.t.  $\Vdash_{\bar{\Sigma}_m} b / \dot{G}_m = \dot{c}$

Set:  $c'(i) = 1$  for  $i < \bar{\Sigma}_m$ . Then

$R(c', \mathbb{B} \upharpoonright \bar{\Sigma}_{m+1})$ , since  $\Vdash_{\bar{\Sigma}_m} R(\dot{c}' / \dot{G}_m, \mathbb{B} \upharpoonright \bar{\Sigma}_{m+1} / \dot{G}_m)$ .

Moreover  $h_{\bar{\Sigma}_m}(c'^*) = \llbracket \dot{c}'^* / \dot{G}_m \neq 0 \rrbracket_{\mathbb{B}_{\bar{\Sigma}_m}} =$

$= \llbracket \dot{c}'^* \neq 0 \rrbracket_{\mathbb{B}_{\bar{\Sigma}_m}} = 1$ .  $\forall$  we then set:

$$c_{m+1}(i) = \begin{cases} c_m(i) & \text{if } i < \bar{\Sigma}_m \\ c'(i) & \text{if } \bar{\Sigma}_m \leq i < \bar{\Sigma}_{m+1} \end{cases}$$

we have:

$$c_{n+1}^* = c_n^* \cap c'^* \neq 0, \text{ since } h_{\sum_n} (c_n^* \cap c'^*) = c_n^* \cap h_{\sum_n} (c'^*) = c_n^* \neq 0. \text{ Since}$$

$$R(c_{n+1} \uparrow \sum_n, B \uparrow \sum_n) \text{ and}$$

$$(c_{n+1} \uparrow \sum_n)^* \uparrow \sum_n \models R(c_{n+1}/G_n, B \uparrow \sum_{n+1}/G_n),$$

we conclude:  $R(c_{n+1}, B \uparrow \sum_{n+1})$ .

Hence  $c_{n+1}^* \neq 0$  in  $B_{\sum_{n+1}}$ , since  $B$  is an RCS iteration. Now let  $G_{n+1} \ni c_{n+1}^*$  be  $B_{\sum_{n+1}}$ -generic. Then  $G_n = G_{n+1} \uparrow B_{\sum_n}$  is

$B_{\sum_n}$ -generic and  $c_n^* \in G_n$ . Set:

$$\tilde{B} = B_{\sum_{n+1}} / G_n, \quad c = c_{n+1} / G_n,$$

$$\tilde{G} = \{ b / G_n \mid b \in G_{n+1} \},$$

Then  $\tilde{G}$  is  $\tilde{B}$ -generic over  $N[G_n]$  and

$\tilde{c} = c \in \tilde{G}$ . Thus (\*) holds. Set:

$\tilde{g} = \tilde{\sigma}^{-1} \upharpoonright \tilde{G}$ , where  $\tilde{g}$  is given by (\*),  $\tilde{\sigma}$  extends uniquely to a

$$\sigma_{n+1} : N[g_n][\tilde{g}] \prec N[G_n][\tilde{G}]$$

s.t.  $\sigma_{n+1}$

$$g_{n+1} = g_n * \tilde{g} = \{ b \in \tilde{B}_{\sum_{n+1}} \mid b/g_n \in \tilde{g} \},$$

We know:  $G_n * \tilde{G} = G_{n+1}$ . Since

$$\sigma_{n+1}(u_n) = \sigma_n(u_n), \text{ we have } \sigma_{n+1}(g_n) = G_n$$

Hence  $\sigma_{n+1} : \bar{N}[g_{n+1}] \hookrightarrow N[G_{n+1}]$  and

$\sigma_{n+1}(g_{n+1}) = G_{n+1}$ . We also note

that  $(b_n \uparrow \bar{J}_{n+1}) / g_n \in \tilde{g}$ , so;

- $b_{n+1} \uparrow \bar{J}_{n+1} = b_n \uparrow \bar{J}_{n+1} \in g_{n+1}$ .

By our construction:

- $g_{n+1}$  is  $\bar{B}_{\bar{J}_{n+1}}$  - generic over  $\bar{N}$

- $\sigma_{n+1} : \bar{N}[g_{n+1}] \hookrightarrow N[G_{n+1}]$  s.t.  $\sigma_{n+1}(g_{n+1}) = G_{n+1}$

- $\sigma_{n+1}(u_n) = \sigma_n(u_n)$

Now set:

$$u_{n+1} = \langle x_{n+1}, z^0, \dots, z^m, u_n \uparrow g_n, b_n \rangle$$

where  $z^l =$  the least  $z < \bar{\lambda}_l$  s.t.  $\sigma_{n+1}(z) \geq \bar{J}_{n+1}^l$   
 $(l = 0, \dots, m)$ .

All of this is forced by  $c_{n+1}^*$ , so there are terms  $\dot{\sigma}_{n+1}, \dot{g}_{n+1}, \dot{u}_{n+1}$  s.t.

$$\sigma_{n+1} = \dot{\sigma}_{n+1} \uparrow G_{n+1}, g_{n+1} = \dot{g}_{n+1} \uparrow G_{n+1}, u_{n+1} = \dot{u}_{n+1} \uparrow G_{n+1}$$

and the above statements are forced by  $c_{n+1}^*$ .

This completes the construction.

QED (Case 3.1)

Case 3.2 Case 3.1 fails

Then  $\lambda > \omega_1$  is regular and  $\bar{B}_i < \lambda$  for  $i < \lambda$ .  
 Hence  $\lambda$  is regular in  $V^{B_i}$  for  $i < \lambda$ . Thus,  
 by the definition of RCS-iteration, we have;  
 $B_\lambda$  is the minimal completion of  $\bigcup_{i < \lambda} B_i$   
 (i.e.  $\bigcup_{i < \lambda} B_i \setminus \{0\}$  is dense in  $B_\lambda \setminus \{0\}$ ).

We again let  $\theta$  s.t.  $B \in H_\theta$  and  $\theta$  verifies  
 the subproperness of  $B_i$  for  $i < \lambda$ . We  
 claim that  $\theta$  verifies the subproperness  
 of  $B_\lambda$ . Let  $\kappa > \theta$  be regular and let  
 $N = \langle \bar{L}[A], A, \dots \rangle$  s.t.  $H_\theta \subset N$ . Again let  
 $\sigma: \bar{N} \prec N$ ,  $\sigma(\bar{\theta}, \bar{B}, \bar{\lambda}) = \theta, B, \lambda$ , where  $\bar{N}$  is countable  
 and full.

Claim  $\sigma$  witnesses the subproperness of  $B_\lambda$  w.t.  $N$ .

We again let  $\sigma(\bar{\lambda}) = \lambda$  and suppose that  
 $\sigma(\bar{\lambda}_i) = \lambda_i$  for  $i = 1, \dots, m$ , where  $\lambda_i$  is regular  
 and  $\bar{B}_\lambda < \lambda_i$  for  $i = 1, \dots, m-1$ . We set  
 $\lambda_m = \lambda$ . We are given  $a \in B_\lambda \setminus \{0\}$  and  
 claim that there is  $b \in a$ ,  $b \in B_\lambda \setminus \{0\}$  s.t.  
 whenever  $G \ni b$  is  $B_\lambda$ -generic, then there  
 is  $\tilde{G} \in V[G]$  s.t. (a)-(d) hold as before.

Note We are of course not constrained  
 to prove  $\sup \tilde{G} \bar{\lambda}_m = \sup \sigma \lambda_m$ , since  
 we do not have  $\bar{B}_\lambda < \bar{\lambda}_m$ . However, this  
 will come out of the proof, and



including  $\lambda = \lambda_m$  in our list of regular cardinals facilitates our proof. We shall, of course, exploit the fact that  $\overline{B}_i < \lambda$  for  $i < \lambda$ .

To prove this we again pick a cofinal monotone sequence  $\langle \bar{\xi}_i \mid i < \omega \rangle$  in  $\bar{\lambda}$  with  $\bar{\xi}_0 = 0$ . However, we are no longer able to enforce that  $\sigma_h(\bar{\xi}_m) = \sigma(\bar{\xi}_m)$ , where  $\langle \sigma_h \mid h < \omega \rangle$  is the sequence of maps we intend to add, converging to  $\tilde{\sigma}$ . This will make our construction more complex. We will be able to enforce  $\sup \sigma_h \upharpoonright \bar{\lambda} = \sup \sigma \upharpoonright \bar{\lambda}$ .

We again let  $\langle x_i \mid i < \omega \rangle$  enumerate  $\bar{N}$  and  $\langle \Delta_i \mid i < \omega \rangle$  enumerate the strongly dense subsets of  $\overline{B}_\lambda = \sigma^{-1}(B_\lambda)$  in  $\bar{N}$ . We define the sequence  $\langle b_i \mid i < \omega \rangle$  exactly as before. Set  $\tilde{\lambda} = \tilde{\lambda}_m = \sup \sigma \upharpoonright \bar{\lambda}$ . Then  $\tilde{\lambda} < \lambda$ .  $B_{\tilde{\lambda}}$  will now in large part play the role that  $B_\lambda$  played in Case 3.1.

We again choose a monotone cofinal sequence  $\langle \bar{\xi}_i^l \mid i < \omega \rangle$  in  $\tilde{\lambda}_l = \sup \sigma \upharpoonright \bar{\lambda}_l$  for  $l = 0, \dots, m$ . We take  $\bar{\xi}_i^m = \bar{\xi}_i$ , where  $\langle \bar{\xi}_i \mid i < \omega \rangle$  is defined as above.

We wish to construct  $c_n$  ( $n < \omega$ ) which will play the same role as in Case 3.1. A particular  $c_n^*$  should force the existence of  $\sigma_n, g_n$  s.t.:

(1)  $g_n$  is  $\overline{B_{\overline{\lambda}_n}}$ -generic over  $\overline{N}$

$$\bullet b_n \upharpoonright \overline{\lambda}_n \in g_n$$

$$\bullet \sigma_n : N[g_n] \prec N[G_n] \text{ s.t. } \sigma(g_n) \neq G_n.$$

However,  $c_n^*$  cannot itself fix the value of  $\sigma_n(\overline{\lambda}_n)$ , so it makes no sense to require  $R(c_n, B_{\sigma_n(\overline{\lambda}_n)})$ . The best we can require is that  $R(c_n, B_{\overline{\lambda}_n})$  and that if  $G_{\overline{\lambda}_n} \ni c_n^*$  is  $B_{\overline{\lambda}_n}$ -generic, then (1) holds with  $G_n = G_{\overline{\lambda}_n} \cap B_{\sigma_n(\overline{\lambda}_n)}$ .

If  $\alpha = \langle \nu_0, \dots, \nu_{n-1} \rangle$  is any monotone sequence in  $\overline{\lambda}$  we simultaneously construct

$$\text{an } e_\alpha \in B_{\nu_{n-1}} \quad (e_\alpha = 1 \text{ if } n=0)$$

s.t.  $e_\alpha \cap c_n^*$  fixes the value of

$\sigma_n(\overline{\lambda}_h)$  as  $\nu_h$  for  $h < n$ . We will

then have  $e_\alpha \cap c_n^* \in B_{\nu_{n-1}}$ .

Moreover, if  $n > 0$ , then  $e_{\lambda} \subset C_{n-1}^*$ .

(We can, of course, have  $e_{\lambda} = 0$ , but not for all  $\lambda$ .)

Def For  $m < \omega$  let  $S_m =$  the set of monotone  $\lambda: m \rightarrow \tilde{\lambda}$ . Set  $S = \bigcup_n S_n$ . For  $\lambda \in S$  let  $|\lambda| = \text{dom}(\lambda) = m$  if  $\lambda \in S_m$ . We also let  $\max(\lambda) = \sup \text{rng}(\lambda)$  (hence  $\max(\emptyset) = 0$ ).

By induction on  $n$  we construct:

- $c_n$  s.t.  $\mathbb{P}(c_n, \mathbb{B}_{\tilde{\lambda}})$

- $\langle e_{\lambda} \mid \lambda \in S_m \rangle$  s.t.  $e_{\lambda} \in \mathbb{B}_{\max(\lambda)}$

- $\langle \dot{\sigma}_{\lambda} \mid e_{\lambda} \neq 0 \rangle, \langle \dot{u}_{\lambda} \mid e_{\lambda} \neq 0 \rangle, \langle \dot{g}_{\lambda} \mid e_{\lambda} \neq 0 \rangle$

s.t.  $\dot{\sigma}_{\lambda}, \dot{g}_{\lambda}, \dot{u}_{\lambda} \in \sqrt{\mathbb{B}_{\max(\lambda)}}$ .

We inductively verify:

(a).  $e_{\emptyset} = 1 \in \mathbb{B}_0$ ,  $e_{\lambda \nu} \in \mathbb{B}_{\max(\lambda)}$

(where  $\lambda \nu = \lambda \restriction \langle \nu \rangle$ )

- $e_{\lambda \nu} \cap e_{\lambda \nu'} = 0$  if  $\nu \neq \nu'$

- $\bigcup_{\nu} e_{\lambda \nu} = e_{\lambda} \cap C_{|\lambda|}^*$  (hence  $e_{\lambda \nu} \subset C_{|\lambda|}^*$ )

- $e_{\lambda} \subset C_{|\lambda|}(i)$  for  $i \geq \max(\lambda)$

(hence  $e_{\lambda} \cap C_{|\lambda|}^* = e_{\lambda} \cap (C_{\lambda \restriction \max(\lambda)}^*) \in \mathbb{B}_{\max(\lambda)}$ )

- $C_m(i) \subset C_m(i)$  for  $m < n$

(hence  $C_n^* \subset C_m^*$ )

- $e_{\lambda} \cap C_{|\lambda|}(i) = e_{\lambda} \cap C_m(i)$  for  $n \geq |\lambda|, i < \max(\lambda)$

(hence  $e_{\alpha} \cap (c_{|\alpha|} \uparrow \max(\alpha))^* = e_{\alpha} \cap (c_m \uparrow \max(\alpha))^*$   
for  $|\alpha| \leq n$ .)

(b)  $\sigma_{\emptyset} = \sigma^*$ ,  $g'_{\emptyset} = \{1\}$ ,  $u_{\emptyset} = \check{u}_{\emptyset}$ , where

$u_{\emptyset} = \langle x_0, \bar{\alpha}, \bar{\alpha}_1, \dots, \bar{\alpha}_m, \bar{B}, b_0 \rangle$ , where  
 $\langle x_i, i < \omega \rangle$  enumerates  $\bar{N}$ .

(c)  $e_{\alpha} \cap c_{|\alpha|}^*$  forces the following in  $V^{(B_{\max(\alpha)})}$

- $g'_{\alpha}$  is  $\check{B}_{\check{\alpha}_{|\alpha|}}$ -generic over  $\check{N}$

- $(b_{|\alpha|} \uparrow \check{\alpha}_{|\alpha|})^* \in g'_{\alpha}$

- $\sigma_{\alpha} : \check{N}[g'_{\alpha}] \prec \check{N}[G]$  and  $\sigma_{\alpha}(g'_{\alpha}) = G$ ,  
where  $G$  is the canonical generic name.

(d) If  $\alpha = \bar{\alpha} \nu$ , then  $e_{\alpha} \cap c_{|\alpha|}^*$  forces:

- $\sigma_{\alpha}(u_{\bar{\alpha}}) = \check{u}_{\bar{\alpha}}$

- $u_{\alpha} = \langle \check{x}_{|\alpha|}, \check{z}_{\alpha}^0, \dots, \check{z}_{\alpha}^m, g'_{\alpha}, b_{|\alpha|}^{\nu}, \check{\alpha}_{|\alpha|}^{\nu} \rangle$ , where

- $\check{z}_{\alpha}^l =$  the least  $z < \check{\alpha}_l$  s.t.  $\sigma_{\alpha}(z) \geq \check{\alpha}_{|\alpha|}^l$   
( $l = 0, \dots, m$ )

(e)  $e_{\alpha \nu} \upharpoonright_{\max(\alpha)} \sigma_{\alpha}(\check{\alpha}_{|\alpha|}^{\nu}) = \check{\nu}$

Note By (a) we easily have:  $\alpha(i) \neq \alpha'(i) \rightarrow e_{\alpha} \cap e_{\alpha'} = \emptyset$   
for  $\alpha, \alpha' \in S$ .

We shall delay the construction of

$c_m, e_{\alpha}, \sigma_{\alpha}, u_{\alpha}, g'_{\alpha}$  and the verification  
of (a)-(e) until later

We note:

(1) Let  $n > 0$ . Then  $\bigcup_{|A|=n} e_A = c_{n-1}^*$

prf. Ind. on  $n$ .

$$n=1: \bigcup_{\lambda} e_{\lambda} = \bigcup_{\nu} e_{\emptyset \nu} = e_{\emptyset} \cap c_{\emptyset}^* = 1 = c_0^*$$

$$n=m+1: \bigcup_{\lambda} e_{\lambda} = \bigcup_{|A|=m} \bigcup_{\nu} e_{A\nu} =$$

$$= \bigcup_{|A|=m} (e_A \cap c_m^*) = c_{m-1}^* \cap c_m^* = c_m^* \quad \square \text{ED}$$

Our intention is to fuse the  $c_m$  ( $m < \omega$ ) into a c.r.t.  $\mathcal{R}(c, \mathcal{B} \upharpoonright \tilde{\lambda})$  just as in Case 3.1, but it will be somewhat trickier in the present case. We set:

$$\bar{c}(i) = \bigcap_{m < \omega} c_m(i) \quad , \quad c(i) = \bar{c}(i) \cup \text{Th}_c(\bar{c}(i)).$$

Claim (a)  $\text{tp}(c \upharpoonright i) = (\bar{c} \upharpoonright i)^*$  for  $i \leq \tilde{\lambda}$

(Hence  $(\bar{c} \upharpoonright i)^* = (c \upharpoonright i)^*$ )

(b)  $\mathcal{R}(c \upharpoonright i, \mathcal{B} \upharpoonright \tilde{\lambda})$  for  $i \leq \tilde{\lambda}$ .

The proof will be by induction on  $i$ .

In the following suppose that  $j < \tilde{\lambda}$  and that (a), (b) hold for  $i < j$ . Suppose moreover that  $a \in \bigcup_i (\bar{c} \upharpoonright i)$  - i.e. there is  $h < i$  s.t.

$$a \in (\bar{c} \upharpoonright h)^* \text{ and } a \in \mathcal{B}_h.$$

(2)  $a \cap c_m^* \neq \emptyset$  for all  $m < \omega$ .

prf.

Clearly  $a \in \bigcup_i (c_m \upharpoonright i)$ ; let  $a \in (\bar{c} \upharpoonright h)^*$ ,  $a \in \mathcal{B}_h$ ,  $h < i$ .

Then  $a \cap c_m^* \neq \emptyset$  and  $h_i(a \cap c_m^*) = a \cap (c_m \upharpoonright i)^* = a \neq \emptyset$ .  $\square \text{ED}$

(3) For  $m < \omega$  there is  $\lambda \in S_m$  s.t.  $a \cap e_{\lambda} \cap c_m^* \neq \emptyset$ .

prf.  $\bigcup_{|A|=m+1} e_{A'} = c_m^*$ . Hence  $a \cap e_{\lambda'} \neq \emptyset$  for an

$\lambda' = \lambda \nu$ ,  $|\lambda| = m$ . Hence  $a \cap e_{\lambda'} \cap c_m^* \neq \emptyset$ .  $\square \text{ED}$ .

(4) Let  $a \in e_n \cap c_m^* \neq 0$  s.t.  $\max(|z|) < i$ ,

Then  $a \in U_i(\bar{c} \cap i)$  (Moreover,

$a \in e_n \cap c_m^* \in B_h$ ,  $a \in e_n \cap c_m^* \subset (c \cap h)^*$  for  
 $a \cap h < i$  s.t.  $h \geq \max(|z|)$ .

proof

Let  $\mu = \max(|z|)$ . Then  $e_n \cap c_m^* = e_n \cap (c_m \cap \mu)^* =$   
 $= e_n \cap (c_m \cap \mu)^*$  for all  $m \geq n$  by (a). Hence  
 $e_n \cap c_m^* = e_n \cap (\bar{c} \cap \mu)^* = e_n \cap (c \cap \mu)^*$  by the inch hyp.

Let  $a \in B_h$ ,  $a \subset (c \cap h)^*$ ,  $h < i$ . Then  
 $a \in e_n \cap c_m^* \subset (c \cap i)^* \cap (c \cap \mu)^* = (c \cap j)^*$  where  
 $j = \max(\mu, h)$ . Obviously  $a \in e_n \cap c_m^* \in B_j$ .

QED (4)

(5) There is  $\lambda$  s.t.  $\max(|z|) \geq i$  and  
 $a \in e_n \cap c_{|\lambda|}^* \neq 0$ .

proof.

Let  $n < \omega$  be big enough that  $\sum_m^1 = \sigma(\bar{\xi}_m) \geq \gamma$   
 (hence  $m > 0$ ). Set  $a' = a \in e_n \cap c_m^* \neq 0$  where  
 $|z| = m$ . If  $\max(|z|) \geq i$ , we are done.

If not, let  $\mu = \max(|z|) < i$ . By (4) we have!  
 $a' \in U_i(\bar{c} \cap i)$ ,  $a' \in B_j$ ,  $a' \subset (c \cap j)^*$  where  
 $\mu \leq j < i$ . Note that

$$e_n \cap c_m^* \upharpoonright_{\mu} \sigma_{\lambda}^1(\bar{z}^1) \geq \sum_m^v \geq \gamma.$$

For each  $\xi < \bar{\lambda}$  set  $d_{\xi} = e_n \cap c_m^* \cap [\bar{z}^1 = \xi] \upharpoonright_{B_{\mu}}$

Then  $d_{\xi} \in B_{\mu}$  and  $e_n \cap c_m^* = \bigcup_{\xi} d_{\xi}$ .

Hence  $\tilde{a} = a' \cap d_{\xi} \neq 0$  for some  $\xi < \bar{\lambda}$ .

Hence  $\tilde{a} \in B_j$ ,  $\tilde{a} \subset (c \cap j)^*$ . Hence  
 $\tilde{a} \in U_i(\bar{c} \cap i)$ .

Now let  $\xi < \bar{\xi}_p$ , where  $m < p$ . Let  $\tilde{a} \in e_{\lambda} \cap c_{p+1}^* \neq 0$ , where  $|\lambda'| = p+1$ .

Claim  $\max(\lambda') \geq \gamma$

proof.

Let  $\tilde{a} \in e_{\lambda} \cap c_{p+1}^* \in G$ , where  $G$  is  $\mathbb{B}_p$ -generic.

Then  $e_{\lambda} \cap c_m^* \in G$ . Hence  $z_m^1 = \xi$ , where

$z_m^1 = \tilde{z}_m^1 G$ , and  $\sigma_m(\xi) \geq \bar{\xi}_m^1 \geq \gamma$ , where

$\sigma_m = \tilde{\sigma}_m G$ . But  $e_{\lambda} \cap e_{\lambda'} \neq 0$ ; hence  $\lambda = \lambda' \upharpoonright m$ .

Hence  $\sigma_p(z_m^1) = \sigma_m(z_m^1)$  and

$\gamma \leq \sigma_p(\xi) \leq \sigma_p(\bar{\xi}_p) = \max(\lambda')$ . QED (5)

We now prove the Claim by induction on  $i \leq \tilde{\lambda}$ .

Case 1  $i=0$ . Trivial since  $\bar{e} \cap i = \emptyset$

Case 2  $i=j+1$  (hence  $i < \tilde{\lambda}$ ).

Then  $(e \cap i)^* = (\bar{e} \cap i)^*$  and  $R(e \cap i, \mathbb{B} \upharpoonright \tilde{\lambda})$ .

We must show:

Claim  $h_i(\bar{e} \cap i) \supseteq (e \cap i)^*$ .

(Hence  $(e \cap i)^* = (\bar{e} \cap i)^*$ ,  $h_j(e \cap i) = 1$  and hence  $R(e \cap i, \mathbb{B} \upharpoonright \tilde{\lambda})$ )

Suppose not. Let  $a = (\bar{e} \cap i)^* \setminus h_i(\bar{e} \cap i)$ .

Then  $a \in U_i(\bar{e} \cap i)$ , since  $a \in \mathbb{B}_i$ ,  $a \in (\bar{e} \cap i)^*$ .

By (5) we can find  $m < i$ ,  $\alpha \in S_m$

s.t.  $\max(\lambda) \geq i$  and  $e_{\lambda} \cap c_m^* \cap a \neq 0$ .

Choose  $m$  minimal for this property.

Then  $r = \bar{r} \nu$ , where  $\max(\bar{r}) \leq j'$ . Hence  $e_r \in B_{j'}$ . By (a) we have

$$e_r \cap c_m(h) = e_r \cap c_m(h) \text{ for } m \geq n, h \in I'.$$

Hence  $e_r \cap a \cap \bar{c}(j') \neq 0$ . But

$$h_j(e_r \cap a \cap \bar{c}(j')) = (e_r \cap a) \cap h_j(\bar{c}(j')) = 0$$

Contradiction!  $\square$  (Case 2)

Case 3  $i' = \gamma < \bar{\lambda}$ ,  $\text{Lim}(\gamma)$ .

By the induct. hypothesis (a) holds below  $\gamma$ .

Hence  $GS(c \upharpoonright \gamma, B \upharpoonright \gamma)$ .

Claim  $R(c \upharpoonright \gamma, B \upharpoonright \gamma)$

(Hence  $R(c \upharpoonright \gamma, B)$  since  $B$  is an RCS iteration.)

Let  $a \in U_\eta(c \upharpoonright \gamma)$ ,  $a \in B_{i'} \setminus \{0\}$ ,  $a \in (c \upharpoonright i')^*$  where  $i' < \gamma$ . We must find  $a' \in a$  s.t.  $a' \in S_\eta(c \upharpoonright \gamma)$ .

Case 3.1 There is  $a' \in a$  s.t.  $a' \in U_\eta(c \upharpoonright \gamma)$  and  $\forall j' < \gamma$   $a' \upharpoonright_{j'}(x') = \omega$ .

Then trivially  $a' \in S_\eta(c \upharpoonright \gamma)$ .

Case 3.2 Case 3.1 fails.

Let  $s \in S_m$  s.t.  $\max(s) \geq \gamma$  and  $e_s \cap c_m \cap a \neq 0$ , with  $m$  chosen minimally. Then  $s = \bar{s} \nu$ ,  $\nu \geq \gamma$  and  $\max(\bar{s}) < \gamma$ . Set:

$$a' = a \cap e_{\bar{s}} \cap c_{m-1}^*$$

with  $a' \in (c \upharpoonright j')^*$ ,  $a' \in B_{j'}$  for  $j' \geq \max(\bar{s})$  by (6).



But  $a \cap e_1 \subseteq a'$  and  $a \cap e_1 \in B_1$ , since  $e_1 \in B_{\max(\bar{\lambda})}$ . Hence  $a \cap e_1 \in U_\gamma(c \cap \gamma)$ .

But then  $a \cap e_1 \in U_\gamma(c_m \cap \gamma)$ . Hence there is  $\tilde{a} \subseteq a$  s.t.  $\tilde{a} \in S_\gamma(c_m \cap \gamma)$ .

Hence  $\tilde{a} \subseteq (c_m \cap i)^*$  for all  $i < \gamma$ , since Case 3.1 fails. Moreover  $\tilde{a} \in B_i$  for any  $i < \gamma$ .

By (a),  $e_1 \cap (c_m \cap i)^* = e_1 \cap (c_m \cap i)^*$  for all  $m \geq 1$ ,  $i < \gamma$ , since  $\max(\bar{\lambda}) \geq \gamma$ .

Hence  $\tilde{a} \subseteq (c \cap i)^*$  for  $i < \gamma$ , since  $a \subseteq e_1$ .

Hence  $\tilde{a} \in S_\gamma(c \cap \gamma)$ . QED (Case 3, 2)

Case 4  $i = \tilde{\lambda}$ .

Then  $R(c \cap j, B \cap \tilde{\lambda})$  for  $j < \tilde{\lambda}$  and  $cf(\tilde{\lambda}) = \omega$  in  $V$ . The conclusion is trivial.

QED (Claim)

Since  $R(c, B \cap \tilde{\lambda})$  and  $B$  is an RCS iteration, we conclude:  $R(c, B)$ . (In particular,  $R(c, B \cap \tilde{\lambda})$ )

We now show that  $c$  has the desired properties. Let  $G$  be  $\mathbb{B}_\lambda$ -generic with  $c^* \in G$ . We claim that there is  $\sigma_0 \in V[G]$  s.t.

(a)  $\sigma_0 : \bar{N} \prec N$

(b)  $\sigma_0(\bar{\theta}_i, \bar{B}_i, \bar{\alpha}_i, \bar{\lambda}_i) = \theta_i, B_i, \alpha_i, \lambda_i \quad (i=1, \dots, m)$

(c)  $\sup \sigma_0 \text{ " } \bar{\lambda}_i = \tilde{\lambda}_i \quad (i=0, \dots, m)$

(d)  $\bar{a} \in \mathfrak{g} = \sigma_0^{-1} \text{ " } G$  and  $\mathfrak{g}$  is  $\bar{B}$ -generic over  $\bar{N}$ .

We know that  $c_n^* \in G$  for  $n < \omega$ . Since  $C_n^* = \bigcup_{\alpha \in \omega} E_\alpha$ , we know: For each  $n$  there is exactly one  $\alpha_n$  s.t.  $|\alpha_n| = n$  and  $E_{\alpha_n} \in G$ .

Set  $\alpha = \bigcup_n \alpha_n = \langle \tilde{\alpha}_i \mid i < \omega \rangle$ . Since

$C_n^* \cap E_{\alpha_n} \Vdash \dot{\sigma}_{\alpha_n} : \check{N}[\dot{g}_{\alpha_n}] \prec \check{N}[G_n]$ , we

have:  $\sigma_n : \bar{N}[g_n] \prec N[G_n]$ , where

$g_n = \dot{g}_{\alpha_n} \text{ " } G = \sigma_n^{-1} \text{ " } G_n$ ,  $G_n = \dot{G}_n \text{ " } G = G \cap \mathbb{B}_{\tilde{\alpha}_n}$ ,

and  $\sigma_n(g_n) = G_n$ . Clearly  $\sigma_n(\tilde{\alpha}_i) = \tilde{\alpha}_i$

for  $i \leq n$ , and  $\sigma_n(x_i) = \sigma_i(x_i)$  for  $i \leq n$ .

Hence we can define  $\tilde{\sigma} : \bar{N} \prec N$  exactly as before and get:

$\sup \tilde{\sigma} \text{ " } \lambda_l = \tilde{\lambda}_l \quad (l=0, \dots, m)$

exactly as before. Finally, note that

$E_{\alpha_n} \cap C_n^* \Vdash (b_n \text{ " } \tilde{\alpha}_n) \in \dot{g}_{\alpha_n}$ ; hence  $b_n \in \mathfrak{g}$ ,

whence follows - exactly as before -

that  $\mathfrak{g}$  is  $\bar{B}$ -generic over  $\bar{N}$ . But

$b_n \subset \bar{a}$ , Hence  $\bar{a} \in \mathfrak{g}$ . QED

We have thus shown that  $B_\lambda$  is subproper. All that remains is to define  $c_n, e_n, \sigma_n$  etc. and verify (a)-(e). By recursion on  $n$  we define

$$\Gamma_n = \langle c_n, \langle e_\lambda \mid |\lambda|=n \rangle, \langle \sigma_\lambda \mid |\lambda|=n \wedge e_\lambda \neq 0 \rangle, \langle \dot{u}_\lambda \mid |\lambda|=n \wedge e_\lambda \neq 0 \rangle, \langle \dot{q}_\lambda \mid |\lambda|=n \wedge e_\lambda \neq 0 \rangle \rangle$$

and verify (a)-(e) (e.g. (e) will then be verified for  $|\lambda| \leq n$ ).  $\Gamma_n$  is defined by (a), (b) for  $n=0$ . The verifications are trivial. Now let  $\Gamma_n$  be given s.t. (a)-(e) hold. Before proceeding further, we note that by the disjoint distributive law used in the construction of  $\langle b_n \mid n < \omega \rangle$ , the following holds in all complete BA's:

(1) Let  $b = \bigcup_{i \in I} b_i$ ,  $b_i \cap b_j = 0$  for  $i \neq j$ , and  $a_i \leq b_i$ . Then  $\bigwedge_j (a_j \cup (b \setminus b_j)) = \bigcup_i a_i$ .

Proof. Set  $a_i^j = \begin{cases} a_i & \text{if } i=j \\ b_i & \text{if } i \neq j \end{cases}$ . Then

$$\bigwedge_j (a_j \cup (b \setminus b_j)) = \bigwedge_i \bigcup_i a_i^j = \bigcup_i \bigwedge_j a_i^j = \bigcup_i a_i. \quad \text{QED (1)}$$

We now define  $e_\lambda$  for  $|\lambda|=n+1$ :

$$(2) e_{\lambda \nu} = c_n^* \cap e_\lambda \cap \left[ \dot{v} = \dot{\sigma}_\lambda \left( \bigcup_{|\lambda|=n} \right) \right] B_{\max(2)}$$

for  $|\lambda|=n$ .