Addendum to "Smooth Iteration"

Upon rereading §9 of [NFS] we realized that both the formulation and proof of §9 Lemma 4 suffered from ambiguity. Here we redo that work, culminating in Theorem 17, which plays the role of [NFS] §9 Lemma 4. Our references will be to the more detailed and careful [CM] rather than to [NFS]. W develop the theory of pseudo projecta *ab ovo*, given a knowledge of [CM] Ch 1, Ch 2. We also make use of the notation:

$$\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle,$$

defined in §3.2 of Ch 3.

(*Note* We apologize for earlier having posted a sloppy version of this addendum which introduced new errors.)

§3.6.1 Pseudo projecta

In order to prove Theorem P90, we must redo §2.6, allowing "pseudo projecta" to play the role of the real projecta:

Def Let $M = \langle J_\alpha^A, B \rangle$ be acceptable. Then $\rho = \langle \rho_i \mid i < \omega \rangle$ is a good sequence of pseudo projecta for M iff the following hold:

(a) ρ_i is p.r. closed if $i > 0$.

(b) $\omega \leq \rho_{i+1} \leq \rho_i \leq \rho_M^i$ for $i < \omega$.

(c) $J_{\rho_i}^A$ is cardinally absolute in M (i.e. if $\gamma \in J_{\rho_i}^A$ is a cardinal in $J_{\rho_i}^A$, then it is a cardinal in M).

(Note $\rho_0 < \rho_M^0 = On_M$ is not excluded. Moreover, ρ_i itself need not be a cardinal in M.)

We shall generally write "ρ is good for M" instead of "ρ is a good sequence of pseudo projecta for M".

Def Let ρ be good for $M = J_\alpha^A$.

$H_i = H_i(M, \rho) =: |J_{\rho_i}^A|$ for $i < \omega$.

We adopt the same language with typed variables v^i ($i<\omega$) as before. The formula classes $\Sigma_h^{(n)}$ ($h,n<\omega$) are defined exactly as before. The satisfaction relation:

$$M \models \varphi[x_1,\ldots,x_n] \bmod \rho$$

is defined as before except that the variables v^i now range over $H_i = H_i(M,\rho)$ instead of $H^i = H^i_M$. A relation $R(x_1^{i_1},\ldots,x_n^{i_n})$ is $\Sigma_1^{(n)}(M,\rho)$ (or $\Sigma_1^{(n)}$ (M mod ρ)) iff it is M-definable mod ρ by a $\Sigma_1^{(n)}$ formula. Similarly for $\underline{\Sigma}_1^{(n)}$, Σ^*, $\underline{\Sigma}^*$. We then define:

<u>Def</u> $\sigma : M \xrightarrow[\Sigma_1^{(n)}]{} M'$ mod (ρ,ρ')

iff the following hold:

(a) ρ is good for M and ρ' is good for M'

(b) $\sigma''H_i \subset H_i'$ for $i<\omega$, where

$H_i = H_i(M,\rho)$, $H_i' = H_i(M',\rho')$

(c) Let φ be $\Sigma_1^{(n)}$, $\varphi = \varphi(v_1^{i_1},\ldots,v_p^{i_p})$ where $i_1,\ldots,i_p \le n$. Then:

$$M \models \varphi[\vec{x}] \bmod \rho \leftrightarrow M' \models \varphi[\sigma(\vec{x})] \bmod \rho'$$

for all $x_1,\ldots,x_p \in M$ s.t. $x_l \in H_{i_l}$ ($l=1,\ldots,p$)

We also define:

Def $\sigma : M \xrightarrow[\Sigma^*]{} M' \bmod (\rho, \rho')$ iff

σ is $\Sigma_0^{(n)}$-preserving mod (ρ, ρ') for $n < \omega$

As before, this is equivalent to:

σ is $\Sigma_1^{(n)}$-preserving mod (ρ, ρ') for $n < \omega$.

We also write:

$\sigma : M \xrightarrow[\Sigma_1^{(n)}]{} M' \bmod \rho'$

to mean:

$\sigma : M \xrightarrow[\Sigma_1^{(n)}]{} M' \bmod (\rho, \rho')$,

where $\rho = \langle \rho_M^i \mid i < \omega \rangle$.

(Similarly for $\sigma : M \xrightarrow[\Sigma^*]{} M' \bmod \rho'$.)

Lemma 1 Let $\sigma : M \xrightarrow[\Sigma_1^{(n)}]{} M'$. Let ρ be good for M and define ρ' by:

$\rho_i' = \sigma(\rho_i)$ if $\rho_i < \rho_M^i$; $\rho_i' = \rho_{M'}^i$ if not.

Then $\sigma : M \xrightarrow[\Sigma_1^{(n)}]{} M' \bmod (\rho, \rho')$.

Abstand

(Whence, if σ is fully Σ^*-preserving, it is also Σ^*-preserving modulo (ρ, ρ').)

proof.

Clearly ρ' is good for M'. Now let R be $\Sigma_j^{(n)}(M,\rho)$. Then R is uniformly $\Sigma_j^{(n)}(M)$ in the finite set:

$$u = u(M,\rho) =: \langle \rho_i \mid \rho_i < \rho_M^0 \wedge \rho_i \neq \rho_h \text{ for } h < i \rangle$$

But then, if R' is $\Sigma_1^{(n)}(M',\rho')$ by the same definition, it is $\Sigma_j^{(n)}(M')$ in $\sigma(u)$ by the same definition.

QED (Lemma 1)

<u>Lemma 2</u> Let $\sigma : M \longrightarrow_{\Sigma^*} M'$ and let ρ, ρ' be as in Lemma D100. Let $\kappa = \mathrm{crit}(\sigma)$, where $\rho_{i+1} \leq \kappa < \rho_i$. Define ρ'' by:

$$\rho_j'' =: \rho_j' \text{ for } j \neq i, \ \rho_i'' =: \sup \sigma''\rho_i.$$

Then:

$$\sigma : M \longrightarrow_{\Sigma^*} M' \bmod (\rho, \rho'')$$

proof.

ρ'' is still good for M'. By induction on n it then follows that σ is $\Sigma_1^{(n)}$-preserving modulo (ρ, ρ'').

QED (Lemma 2)

One might expect that most of §2.6 will not go through with pseudo projecta in place of projecta, since $\langle H_i, B\rangle$ is <u>not</u> necessarily amenable when B is $\Sigma_0^{(i)}(M,\rho)$. As it turns out, however, a great many proofs in §2.6 do not use this property (in contrast to the treatment in §2.5). In particular, Lemmas 2.6.3 — 2.6.16 go through without change. Similarly, the definition of a good function can be relativized to a good ρ in place of $\langle \rho_M^n \mid n<\omega\rangle$. We define

$$\mathfrak{G}_m = \mathfrak{G}_m(M,\rho)\ ;\ \mathfrak{G}^* = \mathfrak{G}^*(M,\rho)$$

exactly as before with ρ in place of $\langle \rho_M^i \mid i<\omega\rangle$. Lemmas 2.6.22 — 2.6.25 then go through exactly as before. Leaving the definition of <u>good</u> $\Sigma_1^{(n)}$ <u>definition</u> unchanged, we get the following version of Lemma 2.6.27:

Let F be a good $\Sigma_1^{(n)}$ function mod ρ. There is a good $\Sigma_1^{(n)}$ definition which defines F mod ρ.

Even some of §2.7 remain valid for pseudo projecta. In §2.7.1 we defined $\Gamma^0(\tau, M)$ (τ being a cardinal in M) as the set of maps $f \in M$ s.t. $\mathrm{dom}(f) \in H = H_\tau^M$. In §2.7.2 we then introduced $\Gamma^n = \Gamma^n(\tau, M)$ for the case that $n > 0$ and $\tau \leq \rho_M^n$, defining Γ^n to be the set of f s.t.:

(a) $\mathrm{dom}(f) \in H = H_\tau^M$

(b) For some $i < n$ there is a good $\Sigma_1^{(i)}(M)$ function G and a parameter $p \in M$ s.t.:

$$f(x) = G(x, p) \text{ for all } x \in \mathrm{dom}(f).$$

Lemma 2.7.10 then told us that, whenever $\pi : M \longrightarrow_{\Sigma_0^{(n)}} M'$, there is a canonical way of assigning to each $f \in \Gamma^n$ a definable partial map $\pi'(f)$ on M'. This continues to hold if $\pi : M \longrightarrow_{\Sigma_0^{(n)}} M'$ mod p. The extended version of 2.7.10 reads:

<u>Lemma 2.7.10</u> Let $\pi : M \longrightarrow_{\Sigma_0^{(n)}} M'$ mod p. There is a unique map π' which assigns to each $f \in \Gamma^n(\tau, M)$ a function $\pi'(f)$ with the following property:

(*) $\pi'(f) : \pi(\mathrm{dom}(f)) \to M'$. Moreover, if $f(x) = G(x,p)$ for all $x \in \mathrm{dom}(f)$, where G is a good $\Sigma_1^{(i)}(M)$ function for an $i < n$ and $p \in M$, then $\pi'(f)(x) = G'(x, \pi(p))$ for $x \in \pi(\mathrm{dom}(f))$, where G' is a good $\Sigma_1^{(i)}(M', \rho)$ function by the same good definition.

The proof is exactly as before. As before we get:

<u>Lemma 2.7.11</u> Let n, τ, π, π' be as above. Then $\pi'(f) = \pi(f)$ for $f \in \Gamma^0(\tau, M)$.

Abstand

Thus, again, we could unambiguously write $\pi(f)$ instead of $\pi'(f)$ for f. <u>However</u>, this is only unambiguous if we have previously specified the good sequence ρ. π' depends not only on π but also on the good sequence ρ. For this reason we shall write: $\pi_\rho(f)$ for $\pi'(f)$. We can omit the subscript ρ if the good sequence is clear from the context.

In §3.2 we then considered the special case that $\tau = \kappa + M$ where κ is a cardinal in M. (This is mainly of interest when there is an extender F on M at κ.) We then set:

$$\Gamma^n_*(\kappa, M) =: \{ f \in \Gamma^n(\kappa, M) \mid \mathrm{dom}(f) = \kappa \},$$

We also set:

$\Gamma^*(\kappa, M) =: \Gamma^n_*(\kappa, M)$ where $n \leq \omega$ is maximal s.t. $\kappa < \rho^n_M$.

Let us call p a <u>defining parameter</u> for $f \in \Gamma^*(\kappa, M)$ iff either $p = f$ or else:

$f(\xi) = G(\xi, p)$ for all $\xi < \kappa$

where G is a good $\Sigma^{(i)}_1(M)$ function for an $i < n$. By Lemma 2.6.25 we can then conclude:

<u>Fact 1</u> Let $R(\vec{x}, y_1, \ldots, y_r)$ be a $\Sigma^{(n)}_0(M)$ relation. Let $f_i \in \Gamma^n_*(\kappa, M)$ have a defining parameter p_i for $i = 1, \ldots, r$. Then the relation:

Displ. $Q(\vec{x}, \vec{\xi}) \longleftrightarrow: R(\vec{x}, f_1(\xi_1), \ldots, f_r(\xi_r))$

is $\Sigma^{(n)}_0(M)$ in the parameters $\kappa, p_1, \ldots, p_r$.

Moreover, if:

Displ. $\sigma : M \xrightarrow[\Sigma^{(n)}_0]{} M'$ mod ρ

and R' has the same $\Sigma^{(n)}_0(M, \rho)$ definition, then the relation:

$Q'(\vec{x}, \vec{\xi}) \longleftrightarrow : R'(\vec{x}, \sigma_p(f_1)(\xi_1), \ldots, \sigma_p(f_r)(\xi_r))$ in $\Sigma_1^{(n)}(M', p)$ in $\kappa, \sigma(p_1, \ldots, \sigma(p_r)$ by the same definition as Q.

Now let $a_1, \ldots, a_m \in M$ and set;

define $X = \{\langle\vec{\xi}\rangle \mid R(\vec{a}, \vec{f}(\xi))\}$.

Then $X \in H_M^n$, since $\langle H_M^n, Q\rangle$ is amenable.

<u>Fact 2</u> Let $R, R', Q, Q', f_1, \ldots, f_r, \sigma, M, M'$ be as in Fact 1. Let $\vec{a}, X$ be as above.

$$\sigma(X) = \{\langle\vec{\xi}\rangle \in \sigma(\kappa) \mid R'(\sigma(\vec{a}), \sigma_p(\vec{f})(\vec{\xi}))\}.$$

proof (sketch)

We know:

$$\bigwedge \vec{\xi} < \kappa (\langle\vec{\xi}\rangle \in X \longleftrightarrow Q(\vec{a}, \vec{\xi}))$$

which is $\Pi^{(n)}(M)$ in the parameters $H_\kappa^M, \vec{a}, \vec{p}$ (We use here the fact that κ, and the Gödel n-tuple function on κ are H_κ^M-definable.) But then the corresponding $\Pi_0^{(n)}(M', p)$ statement holds of $H_n(M', p), \sigma(\vec{a}), \sigma(\vec{a}), \sigma(\vec{p})$.

QED (Fact 2)

Note σ is Σ_1 preserving mod p, if $n > 0$. But then $\kappa' = \sigma(\kappa)$ is a cardinal in M', since it is a cardinal in $H_0 = H(M'; p)$ and p_0 is cardinally absolute in M'.

We now recall the Q-quantifier:

$Qz^i \varphi(z^i) =: \bigwedge u^i \bigvee v^i (v^i \supset u^i \wedge \varphi(v^i))$.

By a $Q^{(i)}$ formula we mean any formula of the form $Qz^i \varphi(z^i)$, where $\varphi(v^i)$ is $\Sigma_1^{(i)}$.

We write:

$$\sigma : M \xrightarrow[Q^*]{} N \mod (p, p')$$

to mean that σ is elementary mod (p, p') with respect to $Q^{(n)}$ formulae for all $n < \omega$.

Clearly, if σ is Q^* preserving mod (p, p'), then it is Σ^*-preserving mod (p, p').

At $p = \langle p_M^i \mid i < \omega \rangle$, we write:

$$\sigma : M \xrightarrow[Q^*]{} N \mod p.$$

In the following assume:

(1) $\sigma : M \xrightarrow[\Sigma^*]{} N \bmod \rho'$.

We define a <u>minimal</u> good sequence:

$$\rho = \min \rho' = \min(\sigma, N, \rho')$$

with the following properties:

(a) $\sigma : M \xrightarrow[Q^*]{} N \bmod \rho$

(b) $\sup \sigma '' \rho^i_M \leq \rho^i \leq \rho'_i$ for $i < \omega$

(c) Let φ be $\Sigma^{(i)}_0$. Let $x \in M$, $z_1, \ldots, z_p \in H_i(N, \rho)$

Then:

$N \models \varphi[\vec{z}, \sigma(x)] \bmod \rho \longleftrightarrow N \models \varphi[\vec{z}, \sigma(x)] \bmod \rho'$.

(d) $\rho = \min \rho$.

Abstand,

We define ρ as follows:

<u>Def</u> Let $\sigma : M \xrightarrow[\Sigma^*]{} N \bmod \rho'$.

$\rho_i(0) =: \sup \sigma '' \rho^i_M$

$\rho_i(n+1) =:$ the supremum of all $F(\eta)$

s.t. $\eta < \rho_{i+1}(n)$ and F is a $\Sigma^{(i)}_1(N, \rho')$

map to ρ'_i in parameters from $\operatorname{rng}(\sigma)$

$\rho_i =: \sup_{n<\omega} \rho_i(n)$

$\rho = \langle \rho_i \mid i < \omega \rangle$.

<u>Note</u> We shall henceforth refer to this important definition as MIN.

Lemma 3 $\rho_i(n) \leq \rho_i(n+1)$

proof

We show by induction on n that it holds for all $i < \omega$.

Case 1 $n = 0$

At $\xi < \rho^i_M$, then $\sigma(\xi) = F(0)$, where $F =$ $F =$ the constant function ξ. But then F is $\Sigma_1^{(i)}(N, \rho')$ in $\sigma(\xi)$. Hence $\sigma(\xi) < \rho_i(1)$.

Case 2 $n > 0$

Then $\rho_{i+1}(n) \geq \rho_{i+1}(n-1)$. Hence:

$$F''\rho_{i+1}(n) \supset F''\rho_{i+1}(n-1)$$

for all F which is a $\Sigma_1^{(i)}(N, \rho')$ map to ρ_i'. The conclusion is immediate.

QED (Lemma 4)

Lemma 4 : $\rho_i(n)$ is p.r. closed for $i > 0$.

proof.

We show by induction on n that it holds for all $i > 0$.

Case 1 $n = 0$.

$\sigma \upharpoonright J^A_{\rho^i_M} : J^A_{\rho^i_M} \xrightarrow[\Sigma_0]{} J^A_{\rho_i}$ cofinally, where ρ^i_M is p.r. closed

Case 2 $n > 0$. Let $n = m+1$.

Then $\rho_i(m)$ is p.r. closed. Let

f be a monotone p.r. function on On.
It suffices to show:

<u>Claim</u> $f''\rho_i(n) \subset \rho_i(n)$.

Let $\nu < \rho_i(n)$. Then $\nu < F(\eta)$ where $\eta < \rho_{i+1}(n)$ and F is $\Sigma_1^{(i)}(N,\rho')$ to ρ_i' on $\sigma(x)$.
But then $f\circ F$ is $\Sigma_1^{(i)}(N,\rho')$ to ρ_0', since ρ_i' is p.r. closed. Hence
$f(\nu) < f\circ F(\eta) < \rho_i(n)$. QED (Lemma 4).

<u>Corollary 5</u>. ρ_i is p.r. closed for $i > 0$.

<u>Def</u> $H_i(n) = H_i(N,\sigma,\rho_i(n)) =: |J^{AN}_{\rho_i(n)}|$

$H_i = H_i(N,\rho) =: |J^{AN}_{\rho_i}|$

<u>Lemma 6</u>

(a) $H_i(0) = \bigcup \sigma'' H^i_M$

(b) $H_i(n+1)$ = the union of all $F(x)$ s.t. $x \in H_{i+1}(n)$ and F is $\Sigma_1^{(i)}(N,\rho')$ to ρ_i' in parameters from rng(σ).

(c) $H_i = \bigcup_n H_i(n)$

proof

(c) is immediate. (a) is immediate since:

$$\sigma \upharpoonright H^i_M : H^i_M \xrightarrow[\Sigma_0]{} H_i(0) \text{ cofinally.}$$

We prove (b). Let $y = F(x)$, where F, x are as in (b).

Claim $y \in H_i''(n+1)$

proof.

We recall the function $\langle S^A_\nu \mid \nu < \infty \rangle$ s.t. for all limit α:

$J^A_\alpha = \bigcup_{\nu<\alpha} S^A_\nu$ and $\langle S^A_\nu \mid \nu < \alpha \rangle$ is uniformly $\Sigma_1(J^A_\alpha)$

Since $\rho_{i+1}(n)$ is p.r. closed, there is a $\Sigma_1(H_{i+1}(n))$ map f of $\rho_{i+1}(n)$ onto $H_{i+1}(n)$. Set:

$g(x) =:$ the least ν s.t. $x \in S_\nu$.

Then $\tilde{F}(\xi) \simeq g F f(\xi)$ is a $\Sigma^{(i)}_1(N, \rho')$ map to ρ'_i in parameters from rng(σ). Hence, where $f(\eta) = x$, we have $y \in S^A_{\tilde{F}(\eta)} \subset H_i(n+1)$

QED (Lemma 6)

By the Definition MIN and Lemma P102;

<u>Lemma 7</u> Let $\rho = \min \rho'$. Then:

- $\sigma'' \rho^{\delta}_{M} \subset \rho_{\delta} \leq \rho'_{\delta} \leq \rho^{\delta}_{N}$
- $\rho_{\delta} = \sup X$, where X is the set of all $F(\nu)$ s.t. $\nu < \rho_{\delta+1}$ and F is a $\Sigma_1^{(i)}(N, \rho')$ map to ρ'_{δ} in some $\sigma(x)$.

Similarly by Lemma 6:

Clubtand

<u>Lemma 8</u> Let $\rho = \min \rho'$. Then:

- $\sigma'' H^{\delta}_{M} \subset H_{\delta} \subset H'_{\delta} \subset H^{\delta}_{N}$
- $H_{\delta} = \bigcup X$ where X is the set of all $F(z)$ s.t. $z \in H_{\delta+1}$ and F is a $\Sigma_1^{(i)}(N, \rho')$ map to H'_{δ} in some $\sigma(x)$.

We can now show:

Lemma 9 ρ is good for N.

proof.

By Lemma 10 6 we have:

$\omega \leq \rho_{i+1} \leq \rho_i \leq \rho_i' \leq \rho_N^i$.

Moreover ρ_i is pr. closed for $i > 0$ by Lemma P 103.

It remains only to show:

Claim H_i is cardinally absolute w.r.t. N.

proof.

We know: $H_i = \bigcup X$, where X = the set of $F(z)$ s.t. $z \in H_{i+1}$ and F is a $\Sigma_1^{(i)}(N, \rho')$ map to $H_i' = H_i(N, \rho')$. Moreover H_i' is cardinally absolute in N.

(1) Let $\alpha \in X$. Then $\bar{\bar{\alpha}}^N \in X$ and there is $f \in X$ s.t. $f : \bar{\bar{\alpha}}^N \xrightarrow{\text{onto}} \alpha$.

proof Suppose not.

Define a $\Sigma_1(H_i)$ map by:

$F(\beta) \simeq$ the $<_{JA}$-least pair $\langle \gamma, f \rangle$ s.t. $\gamma < \beta$ and $f : \gamma \xrightarrow{\text{onto}} \beta$.

Then $F''X \subset X$. Set:

$\alpha_0 = \alpha$; $\alpha_{i+1} = (F(\alpha_i))_0$.

By induction on i it follows that α_i exists and $\alpha_i \in X$. But then $\alpha_{i+1} < \alpha_i$ for $i < \omega$. Contradiction!

QED(1)

Now let α be a cardinal in H_i but not in N. Then $\alpha \notin X$ by (1). But $\alpha < \beta$ for a $\beta \in X$. Hence $\overline{\overline{\beta}}^N > \alpha$. (Otherwise, letting $\gamma = \overline{\overline{\beta}}^N \leq \alpha$, we have $\gamma \in X \subset H_i$ and there is $f \in X \subset H_i$ s.t. $f : \gamma \xrightarrow{\text{onto}} \beta$. Hence there is $g \in H_i$ s.t. $g : \gamma \xrightarrow{\text{onto}} \alpha$, since $0 < \alpha < \beta$. Hence α is not a cardinal in H_i.) But then, letting $\gamma = \overline{\overline{\beta}}^N$, α is a cardinal in J_γ^A and γ is a cardinal in N. Hence α is a cardinal in N by acceptability.

QED (Lemma 9)

Abstand.

We now verify property (c) for $\rho = \min \rho'$.

Abstand

<u>Lemma 10</u>: Let $\overline{B}(\vec{w}^i)$ be $\Sigma_0^{(i)}(M)$ in the parameter $x \in M$. Let $B'(\vec{w}^i)$ be $\Sigma_0^{(i)}(N, \rho')$ in $\sigma(x)$ and $B(\vec{w}^i)$ be $\Sigma_0^{(i)}(N, \rho)$ in $\sigma(x)$ by the same definitions. Then:

$$\bigwedge \vec{z} \in H_i \, (B(\vec{z}) \leftrightarrow B'(\vec{z}))$$

proof. By induction on i.

The case $i = 0$ is trivial. Now let it hold for h where $i = h+1$. It suffices to prove (a), (b) for $\overline{B}$ which is

$\Sigma_1^{(h)}(M)$ in x. We then have:

$$\bar{B}(\vec{z}) \longleftrightarrow \bigvee u^h \, \bar{D}(u^h, \vec{z})$$

where $\bar{D}$ is $\Sigma_0^{(h)}(M)$ in x;

$$B'(\vec{z}) \longleftrightarrow \bigvee u^h \, D'(u^h, \vec{z})$$

where D' is $\Sigma_0^{(h)}(N, \rho')$ in $\sigma(x)$ by the same definition, and:

$$B(\vec{z}) \longleftrightarrow \bigvee u^h \, D(u^h, \vec{z})$$

where D is $\Sigma_0^{(h)}(N, \rho)$ in $\sigma(x)$ by the same definition.

Define a map F to ρ'_h which is $\Sigma_1^{(h)}(N, \rho')$ in $\sigma(x)$ by:

$$\zeta = F(\vec{z}) \leftrightarrow (\bigvee u \in S_\zeta \, D'(u, \vec{z}) \wedge$$
$$\wedge \bigwedge \zeta' < \zeta \, \bigwedge u \in S_{\zeta'} \, \neg D'(u, \vec{z}))$$

Since $i = h+1$, it follows that:

$$F(\vec{z}) < \rho_h \quad \text{if defined for } \vec{z} \in H_i.$$

Hence for $\vec{z} \in H_i$:

$$B'(\vec{z}) \leftrightarrow \bigvee u \in H'_h \, D'(u, \vec{z})$$
$$\leftrightarrow \bigvee u \in S_{F(\vec{z})} \, D'(u, \vec{z})$$
$$\leftrightarrow \bigvee u \in H_h \, D'(u, \vec{z})$$
$$\leftrightarrow \bigvee u \in H_h \, D(u, \vec{z}) \leftrightarrow B(\vec{z})$$

(by the induction hypothesis).

QED (Lemma 10).

Since $\sigma : M \longrightarrow_{\Sigma^{(i)}} N \bmod \rho'$, we conclude that $\sigma : M \longrightarrow_{\Sigma^{(i)}} N \bmod \rho$.

Since this holds for all $i<\omega$, we conclude:

<u>Corollary 11</u> $\sigma : M \xrightarrow[\Sigma^*]{} N \bmod p$

Another immediate corollary is:

<u>Corollary 12</u> $p = \min(N, \sigma, p)$.

It remain only to prove:

<u>Lemma 13</u> $\sigma : M \xrightarrow[Q^*]{} N \bmod p$.

proof

<u>Assume</u>: $M \models Qu^i \varphi(u^i, x)$ where φ is $\Sigma_1^{(i)}$.

<u>Claim</u> $N \models Qu^i \varphi(u^i, \sigma(x)) \bmod p$

Let $v \in H_i$. Then $v \subset w = G(\bar{w})$, where $\bar{w} \in H_{i+1}$ and G is a $\Sigma_1^{(i)}(N, p)$ map to H_i in parameters von $\operatorname{rng}\sigma$. Let:

$$\varphi = \bigvee z^i \psi(z^i, u^i, x) \text{ where } \psi \text{ is } \Sigma_0^{(i)}.$$

Define a $\Sigma_1^{(i)}(N, p)$ map to H_i in $\sigma(x)$ by:

$F(w) \simeq$ the N-least $\langle z, u\rangle \in H^i$ s.t. $w \subset u \wedge \psi(z, u, \sigma(x))$.

The $\Pi_1^{(i+1)}$ statement:

$\bigwedge a^{i+1}(a^{i+1} \in \operatorname{dom}(G) \to a^{i+1} \in \operatorname{dom}(F \circ G))$

holds in N, since the corresponding statement holds in M by our assumption.

Let $\langle z, u\rangle = FG(\bar{w}) = F(w)$. Then $v \subset w \subset u$ and $\psi(z, u, \sigma(x))$. Hence:

$$N \models Qu\,\varphi(u, \sigma(x)) \bmod p$$

QED (Lemma 13)

Thus $\rho = \min \rho'$ possesses all the properties that we ascribed to it.

As a corollary of Lemma 13 we get:

abstand

<u>Corollary 14</u> Let $B \subset H_i$ be $\Sigma_1^{(i)}(N,\rho)$ in parameters from $\operatorname{rng}\sigma$. Then $\langle H_i, B\rangle$ is amenable.

proof

Let $\bar{B}$ be $\Sigma_1^{(i)}(M)$ in x and B be $\Sigma_1^{(i)}(N,\rho)$ in $\sigma(x)$ the same definition. Since $\langle H_M^i, \bar{B}\rangle$ is amenable, we have:

$$Qu^i\, Vy^i\; y^i = u^i \cap \bar{B} \quad \text{in } M.$$

But then:

$$Qu^i\, Vy^i\; y^i = u^i \cap B \quad \text{in } N \bmod \rho.$$

Let $u \in H_i$. There is then $v \supset u$, $v \in H_i$ s.t. $u \cap B \in H_i$. Hence $u \cap B = u \cap v \in H_i$.

QED (Corollary 14)

abstand

<u>Def</u> $\sigma : M \xrightarrow[\Sigma^*]{} N \min \rho$ iff

$\sigma : M \xrightarrow[\Sigma^*]{} N \bmod \rho \wedge \rho = \min(N, \sigma, \rho)$.

(Similarly for $\Sigma_i^{(n)}$, $Q_i^{(n)}$, Q^* etc.)

Lemma 15: Let $\pi : M \longrightarrow_{\Sigma^*} M'$. Let $\kappa = \mathrm{crit}(\pi)$, $\lambda \leq \pi(\kappa)$, and suppose an extender F at κ, λ on M to be defined by:

1. $F(X) = \lambda \cap \pi(X)$ for $X \in \mathbb{P}(\kappa) \cap M$.

Let $\sigma : \bar{M} \longrightarrow_{\Sigma^*} M$ min p, where $\sigma(\bar{\kappa}) = \kappa$. Let $\bar{F}$ be a weakly amenable extender at $\bar{\kappa}, \bar{\lambda}$ on $\bar{M}$. Assume:

$\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$, where $g : \bar{\lambda} \to \lambda$.

Let $n \leq \omega$ be maximal s.t. $\bar{\kappa} < \rho^n_{\bar{M}}$.

Define a good sequence ρ^* for M' by:

$$\rho^*_i = \begin{cases} \sup \pi '' \rho_n & \text{if } i = n \\ \pi(\rho_i) & \text{if } i \neq n \text{ and } \rho_i < \rho^i_M \\ \rho^i_{M'} & \text{if } i \neq n \text{ and } \rho_i = \rho^i_M. \end{cases}$$

(Hence $\pi : M \longrightarrow_{\Sigma^*} M'$ mod (ρ, ρ^*) by Lemmas P100 and P101.) Then:

(a) $\bar{M}$ is n-extendible by $\bar{F}$.

(b) Let $\bar{\pi} : \bar{M} \longrightarrow^{(n)}_{\bar{F}} \bar{M}'$. There is a map σ' s.t. $\sigma' : \bar{M}' \longrightarrow_{\Sigma_0^{(n)}} M'$ mod ρ^* and $\sigma' \bar{\pi} = \pi \sigma$, $\sigma' \upharpoonright \bar{\lambda} = g$.

Moreover, σ' is defined by:

$$\sigma'(\bar{\pi}(f)(\alpha)) = ((\pi\sigma)_{\rho^*}(f))(g(\alpha))$$

for $f \in \Gamma^*(\bar{\kappa}, \bar{M})$.

proof.

We obviously have:

$$\pi\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M' \bmod \rho^*$$

It is also clear that n is maximal s.t. $\bar{\kappa} < \rho_n$ and also maximal s.t. $\kappa' = \pi(\bar{\kappa}) < \rho_n^*$.

We now prove (a). We must show that the $\in$-relation $\in^*$ of $\mathbb{D}^*(\bar{F}, \bar{m})$ is well founded. Let $\langle f, \alpha \rangle, \langle f', \alpha' \rangle \in \mathbb{D}^*$. Set:

$$e = \{\langle \xi, \zeta \rangle < \bar{\kappa} \mid f(\xi) \in f'(\zeta)\}.$$

Then:

$$\begin{aligned}
\langle f, \alpha \rangle \in^* \langle f', \alpha' \rangle &\longleftrightarrow \langle \alpha, \alpha' \rangle \in \bar{F}(e) \\
&\longleftrightarrow \langle g(\alpha), g(\alpha') \rangle \in F(\sigma(e)) \\
&\longleftrightarrow \langle g(\alpha), g(\alpha') \rangle \in \pi\sigma(e) \\
&\longleftrightarrow (\pi\sigma)_{\rho^*}(f)(g(\alpha)) \in (\pi\sigma)_{\rho^*}(f')(g(\alpha')),
\end{aligned}$$

(The second line uses the assumption: $\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \to \langle M, F \rangle$. The third uses: $F(X) = \lambda \cap \pi(X)$. The ~~fourth~~ uses Fact 2, which we established earlier in this section.) QED (a)

We now prove (b). Let $\bar{R}'$ be a $\Sigma_0^{(n)}(\bar{M})$ relation and let R' be $\Sigma_0^{(n)}(M')$ by the same definition. We claim that:

$\sigma' : \overline{M}' \xrightarrow[\Sigma_0^{(n)}]{} M'$, where σ' is defined by:

$$\sigma'(\overline{\pi}(f)(\alpha)) = (\pi\sigma)_{\rho^*}(f)(g(\alpha))$$

for $f \in \Gamma^*(\overline{u}, \overline{M})$, $\alpha < \overline{\lambda}$.

Let $\overline{R}'$ be a $\Sigma_0^{(n)}(\overline{M}')$ relation and let R' be $\Sigma_0^{(n)}(M', \rho^*)$ by the same definition.

Let $\alpha_1, \ldots, \alpha_m < \overline{\lambda}$ and $f_1, \ldots, f_m \in \Gamma^*(\overline{u}, \overline{M})$.

Writing e.g. $\vec{f}(\vec{\alpha})$ for $f_1(\alpha_1), \ldots, f_m(\alpha_m)$, it suffices to show:

<u>Claim</u> $\overline{R}'(\overline{\pi}(\vec{f})(\vec{\alpha})) \longleftrightarrow R'(\pi\sigma(\vec{f}), g(\vec{\alpha}))$.

proof.
Let $\overline{R}$ be $\Sigma_0^{(n)}(\overline{M})$ and R be $\Sigma_0^{(n)}(M, \rho)$ by the same definition. Set:

$$e = \{\langle \vec{\xi} \rangle \mid \overline{R}(\vec{f}(\vec{\xi}))\}$$

Then:

$$\begin{aligned}
\overline{R}'(\overline{\pi}(\vec{f})(\vec{\alpha})) &\longleftrightarrow \langle \vec{\alpha} \rangle \in \overline{F}(e) \\
&\longleftrightarrow \langle g(\vec{\alpha}) \rangle \in F(\sigma(e)) \\
&\longleftrightarrow \langle g(\vec{\alpha}) \rangle \in \pi\sigma(e) \\
&\longleftrightarrow R'((\pi\sigma)_{\rho^*}(\vec{f})(g(\vec{\alpha})))
\end{aligned}$$

QED (Lemma 15)

We would like to prove something much stronger, namely that $\bar{M}$ is $*$-extendible by $\bar{F}$ and that:

$$\sigma' : \bar{M}' \xrightarrow[\Sigma^*]{} M' \text{ mod } \rho^*.$$

For this we must strengthen the condition: $\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$. In §3.2 we helped ourselves in a similar situation by strengthening the relation $\longrightarrow$ to $\longrightarrow^*$. However, $\longrightarrow^*$ is too strong for our purposes and we adopt the following weakening:

<u>Def</u> $\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow^{**} \langle M, F \rangle$ iff the following hold:

(a) $\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$

(b) Let $\bar{\alpha} < lh(\bar{F})$, $\alpha = g(\bar{\alpha})$. There are $\bar{G}, G, \bar{H}, H$ s.t., letting:

$\bar{\kappa} = \text{crit}(\bar{F})$, $\kappa = \text{crit}(F)$

we have:

(i) $\bar{G}, \bar{H}$ are $\Sigma_1(\bar{M})$ in a $\bar{q} \in \bar{M}$ and G, H are $\Sigma_1(M)$ in $q = \sigma(\bar{q})$ by the same definition.

(ii) $\bar{G} = \bar{F}_{\bar{\alpha}}$, $\bar{H} = \bar{M} \cap {}^{\bar{\kappa}}\mathbb{P}(\bar{\kappa})$

(iii) $G \subset F_\alpha$

(iv) $H \subset \{X \in {}^{\kappa}\mathbb{P}(\kappa) \mid \bigwedge \xi < \kappa \,(X_\xi \text{ or } \kappa \setminus X_\xi \in G)\}$.

Note $\longrightarrow^*$ implies $\longrightarrow^{**}$, since if $G = F_\alpha$, then we can take: $H = M \cap {}^\kappa\mathbb{P}(\kappa)$.

Note Let $\bar{X} \in \bar{M} \cap {}^{\bar{\kappa}}\mathbb{P}(\bar{\kappa})$. Let $X = \sigma(\bar{X})$, then $X \in H$. Hence $\bigwedge \xi < \kappa\, (X_\xi$ or $\kappa \setminus X_\xi \in G)$

Note $\longrightarrow^{**}$ follows from the condition:

$\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow^{***} \langle M, F \rangle$,

which says that the following hold:

(a) $\langle \sigma, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$

(b) Let $\bar{\alpha} < lh(\bar{F})$, $\alpha = g(\bar{\alpha})$. There are $\bar{P}$, P s.t., letting $\bar{\kappa} = crit(\bar{F})$, $\kappa = crit(F)$, we have:

(i) $\bar{P}$ is $\Sigma_0(\bar{M})$ in a $\bar{q} \in \bar{M}$ and P is $\Sigma_0(M)$ in $q = g(\bar{q})$ by the same definition

(ii) $\bigvee y\, \bar{P}(y, X) \longleftrightarrow X \in \bar{F}_{\bar{\alpha}}$

(iii) $\bigvee y\, P(y, X) \longrightarrow X \in F_\alpha$

(iv) Let $X \in ({}^{\bar{\kappa}}\mathbb{P}(\bar{\kappa})) \cap \bar{M}$. Then:

$$\bigvee y \bigwedge i < \bar{\kappa}\, (\bar{P}(y, X(i)) \vee \bar{P}(y, \bar{\kappa} \setminus X(i)))$$

We can then set:

$G = \{X \mid \bigvee y\, P(y, X)\}$,

$H = \{X \in {}^\kappa\mathbb{P}(\kappa) \cap M \mid \bigvee y \bigwedge i < \kappa\, (P(y, X_i) \vee P(y, (\kappa \setminus X_i))\}$

We don't know whether the reverse implication holds.

Lemma 16: Let $\pi, \sigma, \overline{M}, M, \overline{M}', M', \rho, \rho^*, \overline{F}, F, \overline{\pi}, \sigma'$ be as in Lemma 14. Assume:

$\langle \sigma, q \rangle : \langle \overline{M}, \overline{F} \rangle \longrightarrow^{**} \langle M, F \rangle$.

Then $\overline{M}$ is $*$-extendible by $\overline{F}$ and:

$\sigma' : \overline{M}' \longrightarrow_{\Sigma^*} M' \bmod \rho^*$

Proof

$\overline{F}$ is then close to $\overline{M}$. Hence $\overline{M}$ is $*$-extendible by $\overline{F}$.

By induction on i we now show:

Claim $\sigma' : \overline{M}' \longrightarrow_{\Sigma_1^{(i)}} M' \bmod \rho^*$

For $i > n$ this is given. Now let $i = n$. We prove a somewhat stronger claim:

Subclaim 1 Let $\overline{A} \subset \overline{\kappa}$ be $\Sigma_1^{(n)}(\overline{M}')$ in $\overline{a} \in \overline{M}'$ and $A \subset \kappa$ be $\Sigma_1^{(n)}(M')$ in $a = \sigma'(\overline{a})$ by the same definition. There is $\overline{s} \in \overline{M}$ s.t. $\overline{A}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{s}$ and A is $\Sigma_1^{(n)}(M)$ in $s = \sigma(\overline{s})$ by the same definition.

(We leave it to the reader to see that this proves the Claim for the case $i = n$.)

We now prove the Subclaim. Let:

$\overline{A}(i) \longleftrightarrow \bigvee y\, \overline{P}'(y, i, \overline{a})$,

$A(i) \longleftrightarrow \bigvee y\, P'(y, i, a)$

where $\overline{P}'$ is $\Sigma_0(\overline{M}')$ and P' is $\Sigma_0(M'; \rho^*)$ by the same definition.

Let $\bar{P}$ be $\Sigma_0^{(m)}(\bar{M})$ and P be $\Sigma_0^{(m)}(M)$ by the same definition. Let $\bar{a} = \bar{\pi}(f)(\bar{\alpha})$ and $a = \pi\sigma(f')(\alpha)$, where $\alpha = g(\bar{\alpha})$. Let $\bar{p}$ be a "defining parameter" for f (i.e. either $\bar{p} = f$ or else $f(\xi) = B(\xi, \bar{p})$ where B is a good $\Sigma_1^{(i)}(\bar{M})$ function for an $i < n$). Then $p = \sigma(\bar{p})$ is in the same sense a defining parameter for $\sigma(f)$ and $p' = \pi\sigma(\bar{p})$ is a defining parameter for $\pi\sigma(f)$, (the good definition of B remaining unchanged). Finally, let $\bar{G}, G, \bar{H}, H$, ...

by the principle:

$$\langle \sigma, q\rangle : \langle \bar{M}, \bar{F}\rangle \longrightarrow^{**} \langle M, F\rangle \text{ for } \bar{\alpha}, \alpha = g(\bar{\alpha}).$$

Since $\langle \bar{M}', \bar{\pi}\rangle$ is the extension of $\langle \bar{M}, \bar{F}\rangle$, we know that: $\bar{\pi}\,``H^n_{\bar{M}}$ is cofinal in H^n_M.

Thus:

$$(1) \quad \bar{A}(i) \leftrightarrow \bigvee u \in H^n_{\bar{M}} \bigvee y \in \bar{\pi}(u)\, \bar{P}'(y, i, \bar{\pi}(f)(\bar{\alpha}))$$

$$\leftrightarrow \bigvee u \in H^n_{\bar{M}}\ \bar{\alpha} \in \bar{\pi}(\bar{X}(i, u))$$

$$\leftrightarrow \bigvee u \in H^n_{\bar{M}}\ \bar{X}(i, u) \in \bar{G},$$

~~abtand~~

where $\bar{X}(i, u) = \{\xi < \bar{u} \mid \bar{P}(y, i, f(\xi))\}$.

Thus $\bar{A}$ is $\Sigma_1^{(m)}(\bar{M})$ in $\bar{p}, \bar{q}, \bar{\pi}$. We now show that A is $\Sigma_1^{(m)}(M)$ in p, q, π by the same definition. Set:

$$H_n = H_n(M, \rho), \quad H_n' = H_n(M', \rho^*).$$

It is easily seen that the relation:

$Q(u,i,\xi) \longleftrightarrow : (u \in H_n \wedge \bigvee y \in u\, P(y, i, \sigma_\rho(f)(\xi)))$
is $\Sigma_0^{(n)}(M, \rho)$ in p; and the relation:
$Q'(u,i,\xi) \longleftrightarrow : (u \in H'_n \wedge \bigvee y \in u\, P'(y, i, (\pi\sigma)_{\rho^*}(\xi))$
is $\Sigma_0^{(n)}(M', \rho^*)$ in p' by the same definition.
Set: $X(u,i) = \{\xi < \alpha \mid Q(u,i,\xi)\}$. Then
$X(u,i) \in H_n$, since $\langle H_n, Q \rangle$ is amenable
by Lemma 10 and hence is rud closed.
Since $\rho^*_n = \sup \sigma''\rho_n$, we know that $\pi''H_n$
is cofinal in H'_n. Thus

$$\begin{aligned}(2)\ A(i) &\longleftrightarrow \bigvee u \in H_n \bigvee y \in \pi(u)\, P'(y, i, ((\pi\sigma)_{\rho^*}(f)(\alpha))\\ &\longleftrightarrow \bigvee u \in H_n\, Q(\pi(u), i, \alpha)\\ &\longleftrightarrow \bigvee u \in H_n\, \alpha \in \pi(X(u,i)) \wedge X\\ &\longleftrightarrow \bigvee u \in H_n\, \alpha \in \pi(X(u,i))\\ &\longleftrightarrow X(u,i) \in F_\alpha .\end{aligned}$$

If $F_\alpha = G$, we would be finished, but G might be a proper subset of F_α. (Moreover, we don't even know that F_α is M-definable in parameters. However, we can prove:

$$(3)\ A(i) \longleftrightarrow \bigvee u \in H_n\, X(u,i) \in G,$$

which establishes Subclaim 1. The direction $(\leftarrow)$ is trivial by (2), since $G \subset F_\alpha$. We prove $(\rightarrow)$. Assume $A(i_0)$, where $i_0 < u$.

We must show that $u \in H_n$ can be chosen large enough that $X(u, i_0) \in G$. We know that it can be chosen large enough that $X(u, i_0) \in F_\alpha$. Since $\rho = \min\{M, \sigma, \beta\}$, we also know that the set of $S(\xi)$ s.t. S is a partial $\Sigma_1^{(n)}(M; \rho)$ map to H_n in a parameter $s = \sigma(I)$ and $\xi < \rho_{n+1}$ is cofinal in H_n. (This uses Lemma P107.) Hence we can assume w.l.o.g. that $u = S(\xi_0)$ for a $\xi_0 < \rho_{n+1}$. Now set:

$$Y(v) =: \{X(v, i) \mid i < u\} \text{ for } v \in H_n.$$

Then $Y(v) \in H_n$ by the rud closure of $\langle H_n, Q\rangle$. Moreover, the function Y is $\Sigma_1(\langle H_n, G\rangle)$ and hence is a $\Sigma_1^{(n)}(M, \rho)$ function. Hence $Y \cdot S$ is $\Sigma_1^{(n)}(M, \rho)$ in s. Let $\bar{S}$ be $\Sigma_1^{(n)}(\bar{M})$ in $\bar{I}$ and $\bar{Y}$ be $\Sigma_1^{(n)}(\bar{M})$ by the same definitions. The $\Pi^{(n+1)}(M, \rho)$ statement:

$$\bigwedge \xi < \rho_{n+1} \ (\xi \in \operatorname{dom}(Y \cdot S) \to Y \cdot S(\xi) \in H)$$

is true, since the corresponding statement:

$$\bigwedge \xi < \rho_{\bar{M}}^{n+1} \ (\xi \in \operatorname{dom}(\bar{Y} \cdot \bar{S}) \to \bar{Y} \cdot \bar{S}(\xi) \in \bar{H})$$

is true in $\bar{M}$. Since $u = S(\xi_0)$, it follows that $Y(u) \in H$ and:

$$X(u, i_0) \in G \vee (u \setminus X(u, i_0)) \in G.$$

But $G \subset F_\alpha$. $(u \setminus X(u, i_0)) \in G$ is therefore impossible, since we would then have:

$$X(u, i_0) \cap (u \setminus X(u, i_0)) = \emptyset \in F_\alpha.$$

Hence, $X(u, i_0) \in G$. QED (Subclaim 1)

Abstand

<u>Subclaim 2</u> $\sigma' : \bar{M}' \xrightarrow[\Sigma_1^{(n)}]{} M' \bmod p^*$.

proof.

Let Q be $\Sigma_1^{(n)}(M', p^*)$ and $\bar{Q}$ be $\Sigma_1^{(n)}(\bar{M}')$ by the same definition. Set

$$P(i, x) \longleftrightarrow (i = 0 \wedge Q(x)),$$
$$\bar{P}(i, x) \longleftrightarrow (i = 0 \wedge \bar{Q}(x))$$

Set:

$$A(x) = \{i \mid P(i, x)\}, \quad \bar{A}(x) = \{i \mid \bar{P}(i, x)\}$$

Then A is the characteristic function of Q and $\bar{A}$ is the characteristic function of $\bar{Q}$. But $A(\sigma'(x)) = \bar{A}(x)$ for $x \in \bar{M}'$ by Sublemma 1. QED (Subclaim 2)

QED

A slight reformulation of Subclaim 1 yields:

<u>Subclaim 3</u> Let A be $\Sigma_1^{(n)}(M';p^*)$ in $p = \sigma(\bar{p})$. Let $\bar{A}$ be $\Sigma_1^{(n)}(\bar{M}')$ in $\bar{p}$ by the same definition

Set: $\bar{H} = H_{\bar{\kappa}}^{\bar{M}}$, $H = H_{\kappa}^{M}$. Then $A \cap H$ is $\Sigma_1^{(n)}(M, p)$ in a $q = \sigma(\bar{q})$ and $\bar{A} \cap \bar{H}$ is $\Sigma_1^{(n)}(\bar{M})$ in $\bar{q}$ by the same definition.

proof.

$H = J_{\kappa}^{E}$, where $E = E^M$ and $\bar{H} = J_{\bar{\kappa}}^{\bar{E}}$ where $\bar{E} = E^{\bar{M}}$. But $\kappa, \bar{\kappa}$ are pr closed. Let $f : \kappa \xrightarrow{\text{onto}} H$ be pr in E and let $\bar{f} : \bar{\kappa} \xrightarrow{\text{onto}} \bar{H}$ be pr in $\bar{E}$ by the same definition. Apply Subclaim 1 to $B = f^{-1}{}''A$, $\bar{B} = \bar{f}^{-1}{}''\bar{A}$.

Then $B \subset \kappa$ is $\Sigma_1^{(n)}(M, p)$ in a $q = \sigma(\bar{q})$ and $\bar{B} \subset \bar{\kappa}$ is $\Sigma_1^{(n)}(\bar{M})$ in $\bar{q}$. But then the same holds for $A = f''B$, $\bar{A} = \bar{f}''\bar{B}$.

QED (Subclaim 3)

For $i \geq n$, we know: $\rho_{\bar{M}}^{i} = \rho_{\bar{M}'}^{i}$, so we can write $\bar{\rho}^{i} =: \rho_{\bar{M}}^{i}$. By the definition of p^*, we know: $p_i = p_i^*$ for $i \geq n$.

We can also set:

$\bar{H}^i = H_{\bar{M}}^{i} = H_{\bar{M}'}^{i}$; $H_i^{*} = H_i(M, p) = H_i(M', p^*)$.

We now prove:

<u>Subclaim 4</u> Let $i > n$. Let $\bar{A}$ be $\Sigma_1^{(i)}(\bar{M}')$ in $\bar{a} \in \bar{M}'$ and let A be $\Sigma_1^{(i)}(M;\rho^*)$ in $a = \sigma'(\bar{a})$ by the same definition. Then there are $\bar{B}, B$, $\bar{q}, q$ s.t.

(a) $\bar{B}$ is $\Sigma_0^{(i)}(\bar{M})$ in $\bar{q} \in M$

(b) B is $\Sigma_0^{(i)}(M,\rho)$ in $q = \sigma(\bar{q})$ by the same definition,

(c) $\bar{A} \cap H_i^i = \bar{B} \wedge H_i^i$

(d) $A \cap H_i = B \cap H^i$

proof.
By induction on i. Let it hold below i.
Then w.l.o.g. we can assume:

(1) $\bar{A}(x) \longleftrightarrow \langle \bar{H}^i, \bar{P} \cap \bar{H}^i \rangle \models \varphi[x]$ for $x \in \bar{H}^i$
where φ is Σ_1 and $\bar{P}$ is $\Sigma_0^{i-1}(\bar{M}')$ in $\bar{a}$.

(2) $A(x) \longleftrightarrow \langle H^i, P \cap H^i \rangle \models \varphi[x]$ for $x \in H^i$
where φ is the same Σ_1 formula and P is $\Sigma_0^{i-1}(M')$ in a by the same definition.

But then there are $\bar{Q}, Q, \bar{q}, q$ s.t.

(3) $\bar{P} \cap H^i = \bar{Q} \cap \bar{H}^i$, where $\bar{Q}$ is $\Sigma_1^{i-1}(\bar{M})$ in $\bar{q} \in \bar{M}$

(4) $P \wedge H^i = Q \cap H^i$, where $\bar{Q}$ is $\Sigma_1^{i-1}(M,\rho)$ in $q = \sigma(q)$ by the same definition.

This is by Sublemma 3 if $i = n+1$, and otherwise by the induction hypothesis.

QED (Sublemma 4)

The Claim then follows easily, since σ is Σ^*-preserving mod ρ^*.

QED (Lemma 16)

We can then go one further and set:

$$\rho' = \min(M', \sigma', \rho^*).$$

It then follows that:

$$\pi``\rho_i \subset \rho_i' \Sigma \rho_i^* \text{ for } i < \omega.$$

To see that $\pi``\rho_i \subset \rho_i'$, we recall that $\rho_i' = \sup_{n<\omega} \rho_i'(n)$, where the sequence $\langle \rho_i'(n) \mid i < \omega \rangle$ is defined from ρ^*, M', σ' by a canonical recursion on n by the ~~definition~~ MIN.

But since $\rho = \min(M, \sigma, \rho)$, we have: $\rho_i = \sup_{n<\omega} \rho_i(n)$, where $\langle \rho_i(n) \mid i < \omega \rangle$ is defined from ρ, M, σ by the same induction on n. Since $\pi\sigma' = \pi\sigma$, it follows easily by induction on n that:

$$\pi``\rho_i(n) \subset \rho_i'(n) \text{ for } i < \omega.$$

The details are left to the reader.

Putting all of this together:

<u>Theorem 17</u> Let $\pi : M \xrightarrow[\Sigma^*]{} M'$ with critical point κ. Let $\lambda \leq \pi(\kappa)$ and let the extender F at κ, λ on M be defined by:

displ

$$F(X) = \pi(X) \cap \lambda.$$

Let $\sigma : \overline{M} \xrightarrow[\Sigma^*]{} M$ minp with $\sigma(\overline{\kappa}) = \kappa$. Assume:

displ

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \longrightarrow^{**} \langle M, F \rangle$$

where $\overline{F}$ is a weakly extender at $\overline{\kappa}, \overline{\lambda}$ on $\overline{M}$. Then:

Cbstand

(a) $\overline{M}$ is $*$-extendable by $\overline{F}$, giving $\overline{\pi} : \overline{M} \xrightarrow[\overline{F}]{*} \overline{M}'$.

Clstand

(b) There are σ', ρ' s.t.

(i) $\sigma' : \overline{M}' \xrightarrow[\Sigma^*]{} M'$ minp'

(ii) σ' is defined by:

displ

$$\sigma'(\overline{\pi}(f)(\alpha)) = (\pi\sigma)_\rho(f)(g(\alpha))$$

for $\alpha < \overline{\lambda}$, $f \in \Gamma^*(\overline{\kappa}, \overline{M})$. (Hence $\sigma'\overline{\pi} = \pi\sigma$ and $\sigma' \upharpoonright \overline{\lambda} = g$.)

(iii) $\pi``\rho_i \subset \rho_i' \leq \pi(\rho_i)$ for $i < \omega$

(taking $\pi(\rho_i) = On_{M'}$ if $\rho_i = On_M$).

Abstand

(c) The above, in fact, holds for:

displ

$$\rho' =: \min(\rho^*) = \min(M', \sigma', \rho^*).$$

where ρ^* is defined by:

displ

$$\rho_i^* = \begin{cases} \sup{}``\rho_i & \text{if } \rho_{i+1} \leq \kappa_i \\ \pi(\rho_i) & \text{if } \kappa_i < \rho_{i+1} \text{ and } \rho_i < \rho_M^i \\ \rho_{M'}^i & \text{if } \kappa_i < \rho_{i+1} \text{ and } \rho_i = \rho_M^i \end{cases}$$

Ulstand

This is the most important result on pseudo projecta.