## §2 Insertions

We continue to assume that $M$ is uniquely normally iterable.

Let $I, I'$ be normal iterations of $M$ of length $\eta+1, \eta'+1$ respectively. Let $\nu \in M'_{\eta'}$ be s.t. $E_\nu^{M'_{\eta'}} \neq \emptyset$. (Hence $\nu \neq \nu'_i$ for $i < \eta'$, since $\nu'_i$ is a cardinal in $M'_{\eta'}$.) Set: $F = E_\nu^{M'_{\eta'}}$ and:
$\kappa = \mathrm{crit}(F)$, $\lambda = \lambda(F) =: F(\kappa)$, $\tau = \tau(F) =: \kappa^{+J_\nu^{E^{M'_{\eta'}}}}$

Set:
$s =$ the least $s$ s.t. $s = \eta'$ or $s < \eta'$ and $\nu < \nu'_s$
$t =$ " " $t$ " $t = s$ or $t < s$ and $\kappa < \lambda'_t$

We assume:

(*) $I | t+1 = I' | t+1$. Moreover, if $t < \eta$, then $\nu_t \geq \nu'_t$. (Hence $\tau < \lambda'_t < \nu'_t$.)

Under this assumption we define:

$$W(I, I', \nu) = \langle I^*, I'', e \rangle \text{ s.t.}$$

(a) $I''$ properly extends $I' | s+1$

(b) Either $I^* = I$ or $I^* = I | t+1$

(c) $e$ inserts $I^*$ into $I''$

(d) $lh(I'') = s+2+(lh(I^*) - (t+1))$

(e) $e(i) = i$ for $i \leq t$ and $e(t+1+i) = s+2+i$

(f) $\nu''_s = \nu$

(Hence $\tilde{e}(t) = s+1$, $e(t) = t$, $\tilde{e}(i) = e(i)$ for $i \neq t$.)

(Hence $M''^{*}_{s} = M_t \| \mu$ where $\mu$ is maximal s.t. $\tau$ is a cardinal in $M_t \| \mu$.)

We define $e, I^*, I''$ as follows. We first set: $I''|s+1 = I'|s+1$. We then define $I''|s+2$ to be the unique extension of $I''|s+1$ with: $\nu_s'' = \nu$. We define an in insertion $e\restriction s+3$ of $I|t+1$ into $I''|s+2$ by:

$e(i) = i$ for $i \le t$ ; $e(t+1) = s+2$,

$\tilde{e}(t) = s$, $\tilde{\sigma}_t = \tilde{\pi}_t$ , where:

$$\tilde{\pi}_t : M_s''^* \xrightarrow[F]{*} M''_{s+1}$$

and $M_s''^* = M_t'\|\mu = M_t\|\mu$, where $\mu$ is largest s.t. $\tau$ is a cardinal in $M_t$.

We then define $I^*, I'', e$ by cases as follows:

<u>Case 0</u> $\eta = t$.

Then $I^* = I = I'|t+1$. $I'' = I''|s+2$, and $e = e\restriction s+2$ as above.

<u>Case 1</u> $\eta > t$ and $\tau$ is not a cardinal in $M_\eta$.

Then $\tau$ is not a cardinal in $M_t\|\nu_t = M_\eta\|\nu_t$, since $\nu_t$ is a cardinal in $M_\eta$.

We set: $I^* = I|t+1$, $I'' = I''|s+2$, $e = e\restriction s+3$.

All conditions are satisfied.

<u>Case 2</u> The above cases fail.

Then $\eta > t$ and $\tau$ is a cardinal in $M_\eta$.

~~Hence~~ $\tau$ is a cardinal in $M_t\|\nu_t = M_\eta\|\nu_t$.

Hence $M_s''^* = M_t\|\mu$ where $\mu \ge \nu_t$.

Hence $\tilde{\sigma}_t(\nu_t)$ is defined.

Extend $I''|s+2$ to a potential iteration of length $s+3$ by setting: $\nu''_{s+1} = \tilde{\sigma}_t(\nu_t)$. This gives us a potential insertion of length $s+2$. By our extension lemma we can extend this to an insertion of $I$ into $I''$ s.t.

$$e(t+1+i) = s+2+i$$

and $lh(I'') = \eta''+1 =: e(\eta+1)$.

We then have $\tilde{e}(t) = s$, $e(t) = t$, and $\tilde{e}(i) = e(i)$ for $i \neq t$. We of course set: $I^* =: I$.

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This defines $W(I, I', \nu) = \langle I^*, I'', e \rangle$.

Lemma 1 If $l \in (t, s]$ and $k > s$, then $l \not<_{T''} k$.

proof.

It suffices to show that if $i \geq s$, then $T''(i+1) \notin (t, s]$. For $i = s$ we have $T''(s+1) = t \notin (t, s]$. Now let $i > s$.

Then $i+1 = e(l+1) = \tilde{e}(l)+1$ for some $l$.

Let $h = T(l+1)$. Then $T''(i+1) = \tilde{e}(h)$.

If $h < t$, then $\tilde{e}(h) = h \notin (t, s]$.

If $t = h$ and $\nu_l < \nu$, then

$\nu''_i = \tilde{\sigma}_i(\nu_l) = \tilde{\sigma}_t(\nu_l) = \tilde{\pi}_t(\nu_l) = \nu_l$

and hence $T''(i+1) = t \notin (t, s]$

If $\nu_l \geq \nu$, then $\tilde{\sigma}_t(\nu_l) = \tilde{\pi}_t(\nu_l) \geq$

$\geq \tilde{\pi}_t(\nu) = \lambda''_s$. Hence $T''(i+1) = s+1 \notin (t, s]$

QED

Lemma 2 Let $\overline{M} = M_\eta \| \mu$ where $\mu$ is maximal s.t. $\tau$ is a cardinal in $M_\eta \| \mu$. Let $n \leq \omega$ be maximal s.t. $u < \rho^n_{\overline{M}}$, $\overline{\eta}+1 = lh(I^*)$. Then

(a) $\overline{M}$ is $\Sigma_0^{(n)}$-extendible by $F$

(b) Let $\overline{\pi} : \overline{M} \xrightarrow[F]{(n)} \overline{M}'$. There is a unique $\sigma : M' \xrightarrow[\Sigma_0^{(n)}]{} M''_{\tilde{e}_{\overline{\eta}}}$ s.t.

$$\sigma(\overline{\pi}(f)(\alpha)) = \tilde{\sigma}_{\overline{\eta}}(f)(\alpha)$$

for $f \in \Gamma^*(u, \overline{M})$, $\alpha < \lambda$.

Proof

Case 1 Then $\overline{\eta} = t$, $\overline{M} = M''^*_{\lambda}$, $\overline{\pi} = \pi''_{t, t+1} = \tilde{\pi}_t = \tilde{\sigma}_t$

Hence $M''_{\tilde{e}_\eta} = M''_{\tilde{e}_t} = M''_{t+1} = \overline{M}'$ and we take $\sigma = \mathrm{id}$.

Case 2 Then $\overline{M} = M''^*_t$ where $t = \lambda$, $\overline{\pi} = \pi''_{t, t+1} = \tilde{\pi}_t = \tilde{\sigma}_t$ and $\overline{\eta} = t$. Take $\sigma = \mathrm{id}$.

Case 3

$\tilde{\sigma}_\eta \upharpoonright \tau = \sigma_\eta \upharpoonright \tau = \tilde{\sigma}_t \upharpoonright \tau = \tilde{\pi}_t = \pi'_{t, t+1}$, $e_\eta = \tilde{e}_\eta$ ;

Hence $\tilde{\sigma}_\eta(X) = F(X)$ for $X \in \mathbb{P}(u) \cap M_\eta = \mathbb{P}(u) \cap M_h$

Thus;

$\langle \tilde{\sigma}_\eta \upharpoonright M_\eta, \mathrm{id} \upharpoonright \lambda \rangle : \langle M_\eta, F \rangle \longrightarrow \langle M''_{e_{\tilde{\eta}}}, F \rangle$,

But $\tilde{\sigma}_\eta : M_\eta \xrightarrow[\Sigma^*]{} M''_{e_{\tilde{\eta}}}$ ; hence $\tilde{\sigma}_\eta$ is strictly $\Sigma_0^{(n)}$- preserving.

QED by (Lemma P51)

## Insertations

**Def** $\mathbb{S} = \langle\langle I^i \mid i < \eta\rangle, \langle \nu_i \mid i+1 < \eta\rangle, \langle e^{ij} \mid i \leq_T j\rangle, T\rangle$ is a normal insertion iteration of $I_0$ (insertation for short) iff the following hold:

(a) $\eta \geq 1$ and each $I^i = \langle\langle M^i_h\rangle, \langle \nu^i_h\rangle, \langle \pi^i_{hj}\rangle, T^i\rangle$ is a normal iteration of $M$ of length $\eta_i + 1$.

(b) $T$ is a tree on $\eta$ s.t. $iTj \to i < j$

(c) $F_i =: E^{M^i_{\eta_i}}_{\nu_i} \neq \emptyset$. Moreover, $\nu_i < \nu_j$ for $i < j$.

Set: $\kappa_i =: \mathrm{crit}(F_i)$, $\lambda_i = \lambda(F_i) =: F_i(\kappa_i)$,

$\tau_i = \tau(F_i) =: \kappa^{+J^E_{\nu_i}}$ where $E = E^{M^i_{\eta_i}}$.

(d) $e^{ij}$ inserts an $I^i|\mu$ into $I^j$. Moreover $\tilde{e}^{hj} = \tilde{e}^{ij} \circ \tilde{e}^{hi}$ and $e^{ii}$ is the identical insertion of $I_i$ to itself. (Hence $\mathrm{dom}(\tilde{e}^{ij}) = \mu$ and $\tilde{e}^{hj} = \tilde{e}^{ij} \circ \tilde{e}^{hi}$)

(e) Set: $s = s_i =:$ the least $s$ s.t. $s = \eta_i$ or $s < \eta_i$ and $\nu_i < \nu^i_s$. Then $I^i|s+1 = I^j|s+1$. and $\nu^j_s = \nu_i$ for $i < j \leq \eta$.

(f) Let $i+1 < \eta$. Let $h$ be least s.t. $h = i$ or $h < i$ and $\kappa_i < \lambda_h$. Then $h$ is the immediate predecessor of $i+1$ in $T$. (In symbols: $h = T(i+1)$.)

Set:
$t = t_i =$ the least $t$ s.t. $t = s_i$ or $t < s_i$ and $\kappa_i < \lambda^i_t$.
Then the condition $(*)$ for forming $W(I^h, I^i, \nu_i)$ is satisfied! At $h = i$, this is clear. Now let $h < i$. Then

$\kappa < \lambda_n = \lambda^i_{s_h}$ by (e). Hence $t \leq s_h$. (*) then follows easily by (e).

(g.) Let $h = T(i+1)$; Then $e = e^{h, i+1}$ inserts $I^i_*$ into $I^{i+1}$, where $\langle I^i_*, I, e\rangle = W(I^h, I^i, \nu_i)$.

(i?)

Def $i+1$ is a truncation point iff $\tau_i$ is not a cardinal in $M^h_{\eta_h}$, where $h = T(i+1)$. (This is the only case in which $I^i_* \neq I^h$ can occur.)

(h) If $\lambda < \eta$ is a limit ordinal, then $T``\{\lambda\}$ is cofinal in $\lambda$. Moreover $T``\{\lambda\}$ contains at most finitely many truncation points.

(i) If $\lambda$ is as above and $(h, \lambda)_T =: \{i \mid h \underset{T}{<} i \underset{T}{<} \lambda\}$ has no truncation point, then $e^{i,\lambda}$ inserts $I^h$ into $I^\lambda$ and:

$I^\lambda, \langle \tilde{e}^{i,\lambda} \mid h \underset{T}{\leq} i \underset{T}{<} \lambda\rangle$

is a good limit of:

$\langle I^i \mid h \underset{T}{\leq} i \underset{T}{<} \lambda\rangle, \langle \tilde{e}^{ij} \mid h \underset{T}{\leq} i \underset{T}{\leq} j \underset{T}{<} \lambda\rangle.$

Note We also call $I^\lambda, \langle \tilde{e}^{i\lambda} \mid i \underset{T}{<} \lambda\rangle$ the limit of $\langle I^i \mid i \underset{T}{<} \lambda\rangle, \langle \tilde{e}^{ij} \mid i \underset{T}{\leq} j \underset{T}{<} \lambda\rangle$, since we know: $\tilde{e}^{hj} = \tilde{e}^{ij} \cdot \tilde{e}^{hi}$ for $h \underset{T}{\leq} i \underset{T}{\leq} j$.

Def If $\mathfrak{I} = \langle\langle I^i\rangle, \langle \nu^i\rangle, \langle e^{ij}\rangle, T\rangle$ is an iteration of length $\eta$ and $0 < i \leq \eta$, set:

$\mathfrak{I} | i = \langle\langle I^j \mid j < i\rangle, \langle \nu^j \mid j+1 < i\rangle, \langle e^{hj} \mid h \underset{T}{\leq} j \underset{T}{<} i\rangle, T \cap i^2\rangle$

Obviously:

Lemma 3.0 If $\mathcal{I}$ is an iteration and $0 < i \leq lh(\mathcal{I})$, then $\mathcal{I}/i$ is an iteration.

Lemma 3.2 If $I$ is a normal iteration of $M$ of length $\eta+1$, then $\langle I \rangle =: \langle\langle I \rangle, \emptyset, \langle id \restriction \eta+2 \rangle, \emptyset \rangle$ is an iteration of length 1.

Lemma 3.3 Let $\mathcal{I} = \langle\langle I^i \rangle, \langle \nu^i \rangle, \langle e^{ij} \rangle, T \rangle$ be an iteration of length $\gamma+1$ and $\eta_i + 1 = lh(I^i)$ for $i \leq \gamma$. Let $E_\nu^{M^\gamma_{\eta_\gamma}} \neq \emptyset$, where $\nu > \nu^i$ for $i < \gamma$. Then there is a unique extension of $\mathcal{I}$ to an iteration $\mathcal{I}'$ of length $\gamma+2$ s.t. $\mathcal{I}'/\gamma+1 = \mathcal{I}$ and $\nu'_\gamma = \nu$.

A more difficult theorem is:

Thm 4 Let $\mathcal{I}$ be an iteration of limit length $\gamma$. There is a unique extension $\mathcal{I}'$ s.t. $\mathcal{I}'/\gamma = \mathcal{I}$ and $lh(\mathcal{I}') = \gamma+1$.

The proof will require many sublemmas.

From now on let

$$\mathcal{I} = \langle\langle I^\zeta \rangle, \langle \nu^\zeta \rangle, \langle e^{\zeta\mu} \rangle, T \rangle$$

be the iteration of length $\gamma$.

Let $I^\zeta = \langle\langle M_i^\zeta \rangle, \langle \nu_i^\zeta \rangle, \langle \pi_{ij}^\zeta \rangle, T^\zeta \rangle$ be of length $\eta_\zeta + 1$.

**Lemma 4.0** Let $\xi < \mu < \eta$. Then

(a) $s_\xi < s_\mu$

(b) $\nu_\xi = \nu^\mu_{s_\xi}$ (hence $\xi <_i \eta_\mu$)

proof.

We first note that (b) → (a), since $\nu_\mu > \nu_\xi = \nu^\mu_{s_\xi}$ and $s_\mu$ is least s.t. $s = \eta_\mu \vee (s < \eta_\mu \wedge \nu_\mu < \nu^\mu_s)$.

But (b) holds by ( ) in the def of iteration.

QED (4.0)

**Lemma 4.1** Let $\xi + 1 \leq_T \mu$. Then $e^{\xi+1,\mu} \restriction s_\xi + 1 = \mathrm{id}$.

proof. (by induction on $\mu$)

For $\mu = \xi + 1$ it is trivial. Now let $\xi + 1 \leq_T \mu + 1$ and let it hold at $\gamma = T(\mu+1)$. Then $\xi < \gamma$ and $s_\mu \geq \lambda_\xi = \lambda^\mu_{s_\xi}$. Hence $t_\mu \geq s_\xi + 1$ and $e^{\gamma,\mu+1} \restriction t_\mu + 1 = \mathrm{id}$. Hence:

$$e^{\xi+1,\mu+1} = e^{\gamma,\mu+1} \cdot e^{\xi+1,\gamma} = \mathrm{id}$$

Now let $\mu$ be a limit ordinal and let it hold at $\gamma$ for $\xi + 1 \leq_T \gamma \leq_T \mu$. It holds at $\mu$ since:

$$I_\mu, \langle e^{i\mu} \mid \xi + 1 \leq_T i \leq_T \mu \rangle$$

is the direct limit of:

$$\langle I_i \mid i \leq_T \mu \rangle, \langle e^{ii'} \mid \xi + 1 \leq_T i \leq_T i' \leq_T \mu \rangle.$$

QED (Lemma 4.1)

Def $\hat{s}_\gamma = \text{lub}\{s_\xi \mid \xi < \gamma\}$.

<u>Lemma 4.2</u> Let $\gamma = T(\xi+1)$. Then $\hat{s}_\gamma \leq t_\xi \leq s_\gamma$.

proof

$t_\xi$ = the least $t$ s.t. $u_\xi < \lambda^\xi_t$. Hence

(1) $\hat{s}_\gamma \leq t_\xi$,

since if $i < \gamma$, then $\lambda_i = \lambda^\gamma_i \leq u_\xi$. Hence $s_i < t_\xi$.

(2) $t_\xi \leq s_\gamma$

proof.

This is trivial for $\gamma = \xi$. Let $\gamma < \xi$. Then

$u_\xi < \lambda_\gamma = \lambda^\xi_{s_\gamma}$. Hence $t_\xi \leq s_\gamma$.

QED (4.2)

Def $X$ is <u>in limbo</u> <u>at</u> $\mu$ iff $X \subset \hat{s}_\mu$ and there is no pair $i, j$ s.t. $i \in X$, $j \geq \hat{s}_\mu$ and $i <_T j$.

By Lemma 1 we have: $(t_\xi, s_\xi]$ is in limbo at $\xi+1$. We can improve this to:

<u>Lemma 4.3</u> If $\xi+1 \leq_T \mu$, then $(t_\xi, s_\xi]$ is in limbo at $\mu$.

proof.

By induction on $\mu$

<u>Case 1</u> $\mu = \xi+1$. By Lemma 1.

<u>Case 2</u> $\mu = \delta+1 \geq_T \xi+1$.

Let $\gamma = T(\delta+1)$. Then it holds at $\gamma$, Moreover $\hat{s}_\gamma \le t_\delta \le s_\gamma$. Let $i \in (t_\xi, s_\xi]$ and $i <_{T^\mu} j$ where $j \ge \hat{s}_\mu = s_\delta + 1$. If $j > s_\mu$, then $j = s_\delta + 2 + l$ and hence $e^{\gamma,\mu}(k) = j$ where $k = t_\delta + 1 + l$. Since $e^{\gamma,\mu}(i) = i$, we conclude: $i <_{T^\gamma} k > t_\delta \ge \hat{s}_\gamma$. Contradiction!

If $j = \hat{s}_\mu$, we note that $t_\delta \le_{T^\mu} j$, hence $i \le_{T^\mu} t_\delta$. Since $e^{\gamma,\mu}$ takes $i, t_\delta$ to themselves, we conclude: $i \le_{T^\gamma} t_\delta \ge \hat{s}_\gamma$. Contradiction! QED (Case 2)

Case 3 $\mu$ is a limit ordinal.

Suppose $i \in (t_\xi, s_\xi]$ with $i <_{T^\mu} h$, $h \ge \hat{s}_\mu$. Then $h = e^{\gamma+1,\mu}(\bar{h})$ for a $\gamma$ s.t.

$\xi + 1 \le_{T^\mu} \gamma + 1 <_{T^\mu} \mu$. But $e^{\gamma+1,\mu} \upharpoonright s_\gamma + 1 = \mathrm{id}$ by Lemma 4.1. Hence $\bar{h} > s_\gamma$. Hence $\bar{h} \ge \hat{s}_\gamma = s_\gamma + 1$, we know that $i \not<_{T^{\gamma+1}} \bar{h}$, Hence $i \not<_{T^\mu} h$. Contradiction! QED (Lemma 4.3)

Def $\hat{I} = \bigcup_{\xi < \eta} I^\xi | s_\xi + 1$

Then $\hat{I}$ is a normal iteration of length $\hat{\eta} =: \mathrm{lub}\{s_\xi \mid \xi < \eta\}$. For $i < \hat{\eta}$ set:

Def $\gamma(i) =$ the least $\gamma$ s.t. $i \le s_\gamma$.

(Hence $\hat{s}_\gamma \le i \le s_\gamma$.)

Lemma 4.4 Let $i \leq_{\tilde{T}} j$. Then $\gamma(i) \leq_T \gamma(j)$.

proof.

Suppose not. Let $i, j$ be a counterexample.

Then $\gamma(i) \not\leq_T \gamma(j)$, $i < j$, and $\gamma(i) < \gamma(j)$.

Set: $\gamma = \gamma(j)$. There is $\mu + 1 \leq_T \gamma$ s.t.

$T(\mu+1) < \gamma(i) < \mu + 1$. Set $\tau = T(\mu+1)$.

Then $s_\tau < i$, since $\tau < \gamma(i)$. Hence

$t_\mu \leq s_\tau < i$ by Lemma 4.2. But

$i \leq \cancel{s_{\gamma(i)}} \leq s_\mu$, since $\gamma(i) \leq \mu$.

Hence $i \in (t_\mu, s_\mu]$. But $j \geq \hat{s}_\gamma$;

hence $i \not\leq_{T^\gamma} j$ by Lemma 4.3.

Hence $i \not\leq_{\tilde{T}} j$, since $I^\gamma | s_\gamma + 1 = \tilde{I} | s_\gamma + 1$.

Contradiction! QED (Lemma 4.4)

Lemma 4.5 Let $\tau = T(\xi+1) \leq \mu$. Then

$\mathrm{crit}(\tilde{e}^{\tau,\mu}) = t_\xi$ and $\tilde{e}^{\tau,\mu}(t_\xi) \leq \hat{s}_\mu$.

proof (Induction on $\mu$)

Case 1 $\mu = \xi + 1$

$\tilde{e}^{\tau,\xi+1}(t_\xi) = s_\xi + 1 = \hat{s}_{\xi+1} > t_\xi$,

$\tilde{e}^{\tau,\xi+1}(i) = \tilde{e}^{\tau,\xi+1}(i) = i$ for $i < t_\xi$.

Case 2 $\mu = \delta + 1$. Let $\gamma = T(\delta+1)$

$\tilde{e}^{\tau,\mu}(t_\xi) = \tilde{e}^{\gamma\mu} \cdot \tilde{e}^{\tau\gamma}(t_\xi) \leq$

$\leq \tilde{e}^{\gamma\mu}(\hat{s}_\gamma) \leq \tilde{e}^{\gamma\mu}(t_\delta) = s_\delta + 1 = \hat{s}_\mu$.

By the ind. hypothesis we have:

$e^{\tau,\mu}(t_\zeta) = \tilde{e}^{\gamma\mu}\tilde{e}^{\tau\gamma}(t_\zeta) \ge e^{\tau\gamma}(t_\zeta) > t_\zeta$,

and, for $i < t_\zeta$:

$e^{\tau\mu}(i) = \tilde{e}^{\gamma,\mu}\,\tilde{e}^{\tau\gamma}(i) = \tilde{e}^{\gamma,\mu}(i)$ where $i \le t_\zeta \le t_\delta$.

Hence $e^{\tau\mu}(i) = \tilde{e}^{\gamma,\mu}(i) = i$. QED (Case 2)

Case 3 $\mu$ is a limit ordinal.

Then $\tilde{e}^{\tau,\mu} \restriction t_\zeta = id$, since

$\eta_\mu, \langle \tilde{e}^{\delta,\mu} \mid \delta \le_T \mu \rangle$ is the direct limit of $\langle \eta_\delta \mid \delta < \mu \rangle, \langle \tilde{e}^{\delta,\gamma} \mid \delta \le_T \gamma \le_T \mu \rangle$, and

$\tilde{e}^{\tau,\gamma} \restriction t_\zeta = id$ for $\gamma < \mu$.

(But $\tilde{e}^{\tau,\mu}(t_\zeta) = \tilde{e}^{\delta,\mu} e^{\tau,\delta}(t_\zeta) > t_\zeta$)

Claim $\tilde{e}^{\tau,\mu}(t_\zeta) \le \hat{\lambda}_\mu$.

proof. Let $h < \tilde{e}^{\tau,\mu}(t_\zeta)$, where $\tau \le_T \gamma \le_T \mu$. Then $h = \tilde{e}^{\gamma,\tau}(\bar{h})$, where $\gamma \ge \zeta$. Thus $\bar{h} < \tilde{e}^{\tau,\gamma}(t_\zeta) \le \hat{\lambda}_\gamma \le t_\delta$ where $\gamma = T(\delta+1)$, $\delta + 1 \le_T \mu$. Hence

Hence $h = e^{\gamma,\mu}(\bar{h}) = \bar{h} < \hat{\lambda}_\gamma \le \hat{\lambda}_\mu$, since $\tilde{e}^{\gamma\mu} \restriction t_\delta = id$. by Case 2.

QED (Lemma 4.5)

(Hence $\tilde{e}^{\tau\mu} \restriction t_\zeta = e^{\tau,\mu} \restriction t_\zeta = id$, since $\tilde{e}(h) \ge e(h) \ge h$.

Now let $b$ be a cofinal branch in $\mathfrak{T}$.

Set: $\delta_\xi = \bigcap_{\xi < i \in b} \mathrm{dom}(e^{\xi, i})$ for $\xi \in b$

Set $\hat{e}^{\xi,i} =: \tilde{e}^{\xi,i} \restriction \delta_\xi$.

Let $\Delta, \langle \tilde{e}^\xi \mid \xi \in b\rangle$ be the direct limit of

$\langle \delta_\xi \mid \xi \in b\rangle, \langle \hat{e}^{\xi,i} \mid \xi \leq i \text{ in } b\rangle$

Assume w.l.o.g. that $\mathrm{wfc}(\Delta)$ is transitive (and $<_\Delta \cap \mathrm{wfc}(\Delta)^2 = \in \cap \mathrm{wfc}(\Delta)^2$).

(To simplify notation we also call $\Delta, \langle \hat{e}^\xi \mid \xi \in b\rangle$ the direct limit of $\langle \eta_\xi \mid \xi \in b\rangle$, $\langle \hat{e}^{\xi,i} \mid \xi \leq i \text{ in } b\rangle$.)

__Lemma 4.6__ $\Delta$ is well founded (hence transitive).

prf.

Using Lemma 4.5 it is easily seen that, letting $\tilde{\kappa}_\tau = t_\xi$ for $\tau = T(\xi+1)$, $\xi+1 \in b$, we have:

(a) $\tilde{e}^{\tau\mu} \restriction \tilde{\kappa}_\tau = \mathrm{id}$ ($\tau \leq \mu$ in $b$)

(b) $\tilde{\kappa}_\xi < \tilde{e}^{\tau,\xi+1}(\tilde{\kappa}_\tau) \leq \tilde{\kappa}_{\xi+1}$ ($\tau = T(\xi+1)$, $\xi+1 \in b$)

(c) $\tilde{e}^{\tau,\xi+1}(\tilde{\kappa}_\tau + j) = \tilde{e}(\tilde{\kappa}_\tau) + j$

(d) $\delta_\lambda = \bigcup_{i \in \lambda \cap b} \tilde{e}^{i\lambda} {}'' \delta_i$ for limit $\lambda \in b$

(Note that $\tilde{e}^{\tau,\xi+1}(\tilde{\kappa}_\tau) = s_\xi + 1$ for $\tau = T(\xi+1)$, $\xi+1 \in b$.)

The conclusion is immediate by the lemma on direct limits in the next pages.

Facts about direct limits: Let $\delta$ be a limit ordinal.

Let $\Delta$, $\langle e_i \mid i<\eta\rangle$ be the direct limit of:

$\langle \delta_i \mid i<\eta\rangle$, $\langle e_{ij} \mid i\leq j<\eta\rangle$

where $e_{ij}:\delta_i \to \delta_j$ is order preserving and the $e_{ij}$ commute. Then there is an induced order $<_\Delta$ on $\Delta$. Let $\tilde{\Delta} = \mathrm{wfc}(\langle \Delta, <_\Delta\rangle)$. We assume w.l.o.g. that $\tilde{\Delta}$ is transitive and $<_\Delta \cap \tilde{\Delta}^2 = \in \cap \tilde{\Delta}^2$. We further assume:

$\kappa_i < \delta_i$ s.t. $e_{ij} \restriction \kappa_i = \mathrm{id}$ for all $j\geq i$

(a) $e_{ij}\restriction \kappa_i = \mathrm{id}$ for $j \geq i$

(b) $\kappa_i < e_{i,i+1}(\kappa_i) \leq \kappa_{i+1}$

(c) $e_{i,i+1}(\kappa_i + j) = e_{i,i+1}(\kappa_i) + j$,

(d) $\delta_\lambda = \bigcup_{i<\lambda} e_{i\lambda}{}''\delta_i$ for limit $\lambda$.

Then: (1) $\kappa_j \geq \kappa_i$ for $j>i$

proof

Otherwise $e_{ij+1}(\kappa_j) > \kappa_j$ where $\kappa_j < \kappa_i$, contradicting (a)

(2) $\kappa_j > \kappa_i$ for $j>i$ (Hence $\kappa_i = \mathrm{crit}(e_i)$)

prf $\kappa_j \geq \kappa_{i+1} > \kappa_i$ by (b)

(3) Let $e_i(h) \in \tilde{\Delta}$ and $e_{ij}(h+l) = e_{ij}(h)+l$ for $j \geq i$ and $h+l < \mu \leq \delta_i$. Then;

and $e_i(h+l) = e_i(h)+l$ for $l<\mu$; hence $e_i(h)+\mu \subset \tilde{\Delta}$

proof. Suppose not.

Let $l$ be the least counterexample. Then

$e_i(h+l) > k > h+m$ for $m<l$

Let $j\geq i$ s.t. $e_j(\bar{k}) = k$. Then

$e_{ij}(h+l) = e_{ij}(h)+l > \bar{k} \geq e_{ij}(h)+l$,

Contr! QED (3)

By (3), taking $h=0$, we have:

(4) $\kappa_i \subset \tilde{\Delta}$ and $e_i \upharpoonright \kappa_i = \mathrm{id}$.

(5) Let $e_{ij}(h) \geq \kappa_j$. Then $e_{ij}(h+l) = e_{ij}(h)+l$ for $l<\delta_i$.

proof. Ind. on $j \geq i$.

Trivial for $j=i$. Now let it hold at $k \geq i$ and let $j=k+1$. Then $e_{ik}(h) \geq \kappa_k$, since otherwise

$e_{ij}(h) = e_{k,k+1}\, e_{ik}(h) = e_{ik}(h) < \kappa_k < \kappa_j$.

Hence $e_{ik}(h+l) = e_{ik}(h)+l$ and

$e_{ij}(h+l) = e_{kj}\, e_{ik}(h+l) = e_{kj}(e_{ik}(h)+l) =$

$e_{kj}(h)+l$, since if $e_{ik}(h) = \kappa_k + a$, then

$e_{k,k+1}(h+l) = e_{k,k+1}(\kappa_k+a+l) = e_{k,k+1}(\kappa_k)+a+l$

$= e_{k,k+1}(\underbrace{\kappa_k+a}_{h}) + l$.

Now let $j$ be a limit ordinal. Then

$\langle \delta_j, \langle e_{ij} \mid i<j \rangle \rangle$ is the limit of

$\langle \delta_i \mid i<j \rangle, \langle e_{ih} \mid i \leq h < j \rangle$, and

we apply (3). QED(5)

<u>Thm</u> $\Delta$ is well founded (hence transitive).

proof.

<u>Case 1</u> For all $i < \eta$, $h < \delta_i$ there is $j > i$ s.t. $e_{ij}(h) < \kappa_j$.

Then $e_i(h) = e_j e_{ij}(h) \in \kappa_j$, since $e_j \upharpoonright \kappa_j = \mathrm{id}$.

Thus $\Delta = \bigcup_i \mathrm{rng}(e_i) \subset \bigcup_i \kappa_i$.

QED (Case 1)

<u>Case 2</u> Case 1 fails.

Then there is $i_0$ s.t. for some $h < \delta_{i_0}$ we have $e_{i_0 j}(h) \geq \kappa_j$ for all $j \geq i_0$.

Since $e_{jk} e_{i_0 j}(h) = e_{i_0 k}(h) \geq \kappa_k$

for $k \geq j \geq i_0$, there is for each $j \geq i_0$ a least $h_j$ s.t. $e_{jl}(h_j) \geq \kappa_l$ for all $l \geq j$.

<u>Claim</u> $e_{ij}(h_i) = h_j$ for $i_0 \leq i \leq j$.

proof

Suppose not. Let $j$ be the least counter example. Then $j = l+1$ for an $l \geq i$.

Since $e_{lj}(\kappa_l + k) = e_{lj}(\kappa_l) + k$

for $k < \delta_l$, $h_j = e_{lj}(k)$ for a $k < h_l$.

But then there is $j' > j$ s.t.

$e_{l j'}(k) = e_{j j'}(h_j) < \kappa_{j'}$.

Contradiction! QED (Claim).

Now let $\tilde{h} = e_j(h_j)$ for $j \geq i_0$. It follows easily that $\tilde{h} = \sup_{i<\delta} h(i)$. Hence $\tilde{h} \in \tilde{\Delta}$, and

$$e_j(h_j + l) = \tilde{h} + l \quad \text{for } h_j + l < \delta_j.$$

But if $l < h_j$, then there is $k > j$ s.t. $e_{jk}(l) < h_j$. Hence $e_j(l) = e_k e_{jk}(l) < h_j$. Hence $\operatorname{rng} e_j \subset \tilde{\Delta}$. Hence $\Delta = \bigcup_j \operatorname{rng}(e_j)$ is well founded.

QED (Thm)

We now turn to the proof of Theorem 4.
$\tilde{I}$ is a normal iteration of $M$ of length $\tilde{\eta}$.
Hence it has a well founded cofinal branch $\tilde{b}$.
By Lemma 4.4 $b = \{\gamma(i) \mid i \in \tilde{b}\}$ is then a cofinal branch in $T = T_\gamma$. Using this branch we define $\langle \delta_\xi \mid \xi \in b \rangle$, $\Delta$, $\langle \tilde{e}^\xi \mid \xi \in b \rangle$ as above.
By Lemma 4.6 $\Delta$ is an ordinal. We set:
$e^\xi(i) = \mathrm{lub}\{\tilde{e}^\xi(j) \mid j < i\}$ for $i \leq \delta_\xi$.

<u>Claim</u> there is a normal iteration $I$ of length $\Delta$ s.t. $I, \langle e^\xi \mid \xi \in b \rangle$ is the good limit of $\langle (I^\xi \mid \delta_\xi) \mid \xi \in b \rangle$, $\langle (e^{\xi,\mu} \restriction (\delta_\xi + 1) \mid \xi \leq \mu \text{ in } b \rangle$.

(It will then follow easily that the desired extension of $S$ is the structure gotten by setting: $T``\{\eta\} =: b$, $I^\eta =: I$, $e^{\xi,\eta} =: e^\xi$.)

Following the procedure outlined in (I) we inductively construct $I_{(\gamma)}$ for $\gamma \leq \Delta$, showing that $I_{(\gamma)}$, $\langle e^\xi_{(\gamma)} \mid \xi \in b \rangle$ is a good limit of $\langle I^\xi_{(\gamma)} \mid \xi \in b \rangle$, $\langle e^{\xi,\mu}_{(\gamma)} \mid \xi \leq \mu \text{ in } b \rangle$.

(Here $\delta^\xi_{(\gamma)} = \{i < \delta^\xi \mid \tilde{e}^\xi(i) < \gamma\}$, $I^\xi_{(\gamma)} = I^\xi \mid \delta^\xi_{(\gamma)}$,
$e^{\xi,\mu}_{(\gamma)} = e^{\xi,\mu} \restriction \delta^\xi_{(\gamma)}(+1)$, $e^\xi_{(\gamma)} = e^\xi \restriction \delta^\xi_{(\gamma)}(+1)$)

<u>Case 1</u> $\gamma \leq \tilde{\eta} = \mathrm{lh}(\tilde{I}) = \sup_{\xi \in b} \Delta_\xi$.

~~Set~~: $I_{(\gamma)} =: \tilde{I} \mid \gamma$, Note that
$\delta^\xi_{(\gamma)} = \{i < \delta^\xi \mid e^\xi(\gamma) < \gamma\}$. But, letting
$\tilde{\kappa}_\tau = \mathrm{crit}(\tilde{e}^\tau)$ for $\tau \in b$, we have:
$\tilde{\kappa}_\tau = t_\xi$ for $\tau = T(\xi+1)$, $\xi + 1 \in b$.
(by the proof of 4.6)

Case 1.1 $\gamma < \tilde{\eta}$.

Then $\gamma < \tilde{\kappa}_\tau$ for some $\tau$. Hence $\tilde{e}^\tau \upharpoonright \gamma = \mathrm{id}$. Hence $e^\xi(i) = e^{\xi,\tau}(i)$ for $i < \delta^\xi_{(\gamma)}$. But $e^{\xi,\tau} \upharpoonright \delta^\xi_{(\gamma)}$ inserts $I^\xi_{(\gamma)} = I^\xi | \delta^\xi_{(\gamma)}$ into $I^\tau | \gamma$, where $e^\tau \upharpoonright \gamma = \mathrm{id}$.

Hence $e^{\xi,\tau} \upharpoonright \delta^\xi_{(\gamma)} = e^\xi \upharpoonright \delta^\xi_{(\gamma)}$ inserts $I^\xi_{(\gamma)}$ into $I_{(\gamma)} = \tilde{I} | \gamma$.

QED (Case 1.1)

Case 1.2 $\gamma = \tilde{\eta}$

$I_{(\tilde{\eta})} = \bigcup_{\gamma < \tilde{\eta}} I_{(\gamma)} = \bigcup_{\gamma < \tilde{\eta}} \tilde{I} | \gamma = \tilde{I}$ is a good limit of $\langle I^\xi_{(\tilde{\eta})} \mid \xi \in b \rangle$, $\langle e^{\xi,\mu}_{(\tilde{\eta})} \mid \xi \leq \mu$ in $b \rangle$, where

$\delta^\xi_{(\tilde{\eta})} = \{ i < \delta^\xi \mid \tilde{e}^\xi(i) < \tilde{\eta} \} = \{ i < \delta^\xi \mid \forall \mu > \xi$ in $b\ (\tilde{e}^{\xi,\mu}(i) < \tilde{\kappa}_\mu \}$

QED (Case 1.2)

If $\Delta = \tilde{\eta}$, then $\delta^\xi_{(\tilde{\eta})} = \delta^\xi$ and we are done.

If $\Delta \neq \tilde{\eta}$, the next case is:

Case 2 $\gamma = \tilde{\eta} + 1$

Then $\tilde{\eta} \in \mathrm{rng}(\tilde{e}^\xi)$ for sufficient $\xi \in b$. Let

$\tilde{\eta}_\xi \simeq : (\tilde{e}^\xi)^{-1}(\tilde{\eta})$.

By the theorem ~~at~~ the end of I we must show:

Claim If $h \leq_T \tilde{\eta}_\xi$, then $e^\xi(h) \in \tilde{b}$.

We know: $\tilde{\kappa}_\tau = \mathrm{crit}(\hat{e}^{\tau,\xi+1}) = t_\xi$ for $\tau = T(\xi+1)$ s.t. $\xi+1 \in b$. Set:

$\tilde{\lambda}_\tau = \hat{e}^{\tau,\xi+1}(\tilde{\kappa}_t) = \lambda_\xi + 1$.

Then $\hat{\imath}_\mu = \mathrm{lub}\{ \lambda_\xi \mid \xi < \mu \} \geq \tilde{\lambda}_\xi$ for $\xi < \tau$ in $b$.

for $\mu \in b$. Moreover,

$(\tilde{\kappa}_\sigma, \tilde{\lambda}_\tau) = (t_\zeta, s_\zeta]$ is in limbo at $\mu$ for all $\mu > \tau$ in $b$ (when $\tau = T(\zeta+1)$, $\zeta+1 \in b$).

(i.e. if $i \in (\tilde{\kappa}_\tau, \tilde{\lambda}_\tau)$ and $j \geq \hat{\imath}_\mu$, then $i \not<_{T\mu} j$.)

---

For $\gamma \in b$, set:

$i_\gamma =$ the least $i \in \tilde{b}$ s.t. $\gamma = \delta(i)$.

~~Let $i = i_\gamma$. Then~~ $\hat{s}_\gamma \leq i \leq s_\gamma$.

If $j <_{\tilde{T}} i$, then $j <_{T\gamma} i$, since $\tilde{I}|s_\gamma+1 = I^\gamma|s_\gamma+1$. But then $j \notin (\tilde{\kappa}_\tau, \tilde{\lambda}_\tau]$ for $\tau < \gamma$ in $b$, since ~~the~~ $(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau)$ is in limbo at $\gamma$. Thus we have shown:

(1) $b \cap \bigcup_{\tau \in b} (\tilde{\kappa}_\tau, \tilde{\lambda}_\tau) = \emptyset$.

We know that $\tilde{\kappa}_\gamma = t_\zeta \leq s_\gamma$ if $\gamma = T(\zeta+1)$ and $\zeta+1 \in b$, by Lemma 4.7. Moreover $s_\gamma \leq s_\zeta$. Hence

(2) $i = i_\gamma \leq \tilde{\kappa}_\gamma$

since otherwise $i \in (\tilde{\kappa}_\gamma, \tilde{\lambda}_\gamma) = (t_\zeta, s_\zeta]$.

We consider two cases:

Set: $A = \{\tau \in b \mid i_\tau < \tilde{\kappa}_\tau\}$

<u>Case 2.1</u> $A$ is cofinal in $\eta$.

We first note:

(3) Let $\hat{i}_\mu \leq j \leq \tilde{e}^{\xi,\mu}(i)_\mu$.
Then $j \in \text{rng}(\tilde{e}^{\xi,\mu})$.

prf:
Suppose not. Let $\mu$ be the least counterexample.
Then $\mu > \xi$. If $\mu$ is a limit, there is
$\gamma$ s.t. $\xi \leq \gamma < \mu$ and $j = \tilde{e}^{\gamma,\mu}(j')$.
But then $j \geq \tilde{\alpha}_\gamma$, since otherwise
$j = j' < \tilde{\alpha}_\gamma < \tilde{\lambda}_\gamma \leq \hat{\lambda}_\mu$. But $\tilde{\alpha}_\gamma \geq \hat{i}_\gamma$.
Hence $j' \in \text{rng}(e^{\xi\gamma})$ and $j \in \text{rng}(e^{\xi,\mu})$,
Contr! Now let: $\mu = \gamma+1$, $\tau = T(\gamma+1)$.
Then $j \geq \hat{\alpha}_\mu = \alpha_\gamma + 1 = \tilde{\lambda}_\tau$. But ~~then~~:
$\tilde{e}^{\tau,\mu}(\tilde{\alpha}_\tau + h) = \tilde{\lambda}_\tau + h = \alpha_\gamma + 1 + h$. Let
$\tilde{e}^{\tau,\mu}(i) = \tilde{\lambda}_\tau + k$, $j = \tilde{\lambda}_\tau + h$ (hence $h < k$). Then
$e^{\tau\mu}(j') = j$ where $j' = \tilde{\alpha}_\tau + h < \tilde{\alpha}_\tau + k = \tilde{e}^{\xi,\tau}(i)$

Hence $j' \geq \tilde{\alpha}_\tau \geq \hat{i}_\tau$. Hence $j' \in \text{rng}(e^{\xi,\tau})$
and $j = e^{\tau\mu}(j') \in \text{rng}(e^{\xi,\mu})$.

QED (3)

In particular, if $\tilde{e}^{\xi}(\bar{\eta}) = \eta$ and $\hat{i}_\mu \leq j \leq \tilde{e}^{\xi,\mu}(\bar{\eta})$
then $j \in \tilde{e}^{\xi,\mu}{}''\bar{\eta}$.
Hence $i_\gamma \in e^{\xi,\gamma}{}''\bar{\eta}$ for all $\gamma \geq \xi$.

But if $\mu \in A$, $\gamma < \mu$, then
$i_\gamma \leq \tilde{\alpha}_\gamma < \hat{\alpha}_\mu \leq i_\mu$.
Hence $i_\gamma < \tilde{\alpha}_\gamma < i_\mu$ for $\gamma < \mu$, $\gamma, \mu \in A$.

Hence $\sup_{\gamma \in A} i_\gamma = \sup_{\gamma \in A} \tilde{u}_\gamma = \tilde{\eta}$. At $\tilde{e}^{\xi}(\bar{\eta}) = \tilde{\eta}$, $\tilde{e}^{\xi\gamma}(h_\gamma) = i_\gamma$ for $\xi < \gamma$ in $b$, then $\sup_{\gamma \in A} h_\gamma = \bar{\eta}$. Hence $\tilde{e}^{\xi}(\bar{\eta}) = e^{\xi}(\bar{\eta})$.

Set $\bar{b} = (\tilde{e}^{\xi})^{-1} '' \tilde{b}$. Then for $\gamma \in A$ we have $(\tilde{e}^{\xi})^{-1}(i_\gamma) = (\tilde{e}^{\xi,\gamma})^{-1}(i_\gamma) = h_\gamma$. $\bar{b}$ is easily seen to be a cofinal well founded branch in $I^{\xi} | \bar{\eta}$. By uniqueness we then have: $\bar{b} = T^{\xi} '' \{\bar{\eta}\}$. Thus, if $i \leq_{T^\xi} \bar{\eta}$, then $i \in \bar{b}$ and $e^{\xi}(i) \in \tilde{b}$.

QED (Case 2.1)

<u>Case 2.2</u> Case 2.1 fails

Then there is $\tau_0$ s.t. $i_\gamma = \tilde{u}_\gamma$ for all $\gamma \geq \tau_0$ in $b$.

<u>Claim 1</u>. $e^{\tau_0 \gamma}(i_{\tau_0}) = i_\gamma$ for $\tau_0 \leq_b \gamma$

proof

Let $\gamma$ be the least counterexample. Then $\gamma > \tau_0$.

<u>Case 2.2.1</u> $\gamma = \mu + 1$. Let $\tau = T(\mu+1)$ (hence $\tau \geq \tau_0$ in $b$). Then $e^{\tau_0 \tau}(i_{\tau_0}) = i_\tau$.

But $e^{\tau\gamma}(i_\tau) = e^{\tau\gamma}(\tilde{u}_\tau) = \tilde{\lambda}_\tau = \lambda_\mu + 1 = \hat{\imath}_\gamma < \eta_\gamma$

Hence $e^{\tau\gamma}(i_\tau) = i_\gamma$. QED (Case 2.2.1)

Case 2.2.2 $\gamma$ is a limit.

Let $j = \tilde{e}^{\tau_0, \gamma}(i_{\tau_0})$.

Then $j \geq \hat{s}_\gamma$, since for $\tau_0 \leq_b \mu+1 \leq_b \gamma$ we have: $i_{\mu+1} = s_\mu + 1 \leq j$. But then $j \leq \hat{s}_\gamma$, since if $h < j$ and $h = \tilde{e}^{\zeta,\gamma}(\bar{h})$ where $\tau_0 \leq_b \zeta <_b \gamma$, then $\bar{h} < \tilde{s}_\zeta$. Hence $h = \bar{h} < i_\zeta \leq s_\zeta < \hat{s}_\gamma$. Hence $\hat{s}_\gamma \leq j < s_\gamma$ and $j = i_\gamma$. QED Claim 1

Claim 2 $\tilde{\gamma} = \tilde{e}^\zeta(i_\zeta)$ for $\tau_0 \leq_b \zeta$.

proof:
Let $h < \tilde{e}^\zeta(i_\zeta)$. Let $\zeta$ be chosen large enough that $e^\zeta(\bar{h}) = h$, where $\bar{h} < i_\zeta = \tilde{s}_\zeta$. Then $h = \bar{h} < \tilde{s}_\zeta < \tilde{\gamma}$. QED (Claim 2).

Now let $h < i_\zeta$; $\tau_0 \leq_b \zeta$. Then $e^\zeta(i_\zeta) = i_\zeta$ (since if $\zeta = T(\mu+1)$, $\mu+1 \in b$, then $e^\zeta \restriction s_\mu + 1 = id$ and $. = t_\mu \leq s_\mu$).

QED (Claim 2)

Hence if $\zeta \geq \tau_0$, $h < i_\zeta$, then $e^\zeta(h) < e^\zeta(i_\zeta) \in \tilde{b}$. Hence $h \in \tilde{b}$.

QED (Case 2).

Case 3 $\gamma > \tilde{\eta} + 1$

The successor case is given by (I). Assume that $\gamma$ is a limit. At $\tilde{e}^3(\bar{\gamma}) = \gamma$, then
$e^3(\bar{\eta}+h) = \tilde{e}^3(\bar{\eta}+h) = \tilde{\eta} + h$ for $h \geq 0$ where $\tilde{e}^3(\bar{\eta}) = \tilde{\eta}$.
Let $b$ be a cofinal well founded branch in $I_{(\gamma)}$. Then $(e^3)^{-1}{}''b := \bar{b}$ is a cofinal well founded branch in $I^3/\bar{\gamma}$. Hence
$\bar{b} = T^3{}''\{\bar{\gamma}\}$ by uniqueness. Hence:
$h \leq_T \bar{\gamma} \rightarrow e^3(h) \in b$. QED (Case 3)

QED (Thm 4)

Def We write $\sigma_h^{ij}, \tilde{\sigma}_h^{ij}$ for the insertion maps given by $e^{ij}, \tilde{e}^{ij}$.

---

We have proven that, if $M$ is uniquely normally iterable, then it is uniquely normally insertable in the following sense:

Def Let $I$ be a normal iteration of $M$. $I$ is uniquely insertable iff it is insertable by the uniqueness strategy (i.e. Thm 4 holds). $M$ is uniquely insertable iff every normal iteration of $M$ is uniquely insertable. We have shown:

Corollary 5 If $M$ is uniquely normally iterable, then it is uniquely insertable.

We can relativize this:

Let $\Sigma$ be a successful iteration strategy for $M$ which is insertion invariant.

Def $I$ is a normal $\Sigma$-iteration of $M$ iff the branches of infinite length are picked by $\Sigma$.

Def $\mathfrak{I}$ is an $\Sigma$-inseration iff it is an inseration and each of the component iterations $I$ is an $\Sigma$-iteration of $M$.

Def An $\Sigma$-iteration of $M$ is uniquely $\Sigma$-insertable iff it is insertable by the unique strategy (for $\Sigma$-inserations). $M$ is uniquely $\Sigma$-insertable iff every normal $\Sigma$-iteration of $M$ is uniquely $\Sigma$-insertable.

. . . .

By trivial modifications of our proofs we obtain:

Corollary 6 Let $\Sigma$ be a successful insertion-invariant strategy for M. Then M is uniquely $\Sigma$-iterable.

The proofs are left to the reader.

* * * *

For future reference, we note a consequence of our proofs: By induction on $i$ it follows that, if $h \leq_T i$ and there is no truncation point in $(h, i]$, then $e^{h,i}(\eta_h) = \eta_i$. In particular, if $\mathcal{I}|i+1$ has no truncation on the main branch, then $e^{0,i}(\eta_0) = \eta_i$. This will be used later in § 5.