

§1 Remarks on the gap two problem

Consider a first order language \mathcal{L} with predicates \in, A, B and axioms:
 $\aleph = \aleph^- + A$ is an infinite cardinal +
 $+ B = A^+$ is the largest cardinal.

By a (τ, τ^{++}) -model of \mathcal{L} we understand a model

$$\mathcal{M} = \langle |\mathcal{M}|, A_{\mathcal{M}}, B_{\mathcal{M}}, \in_{\mathcal{M}}, \dots \rangle$$

s.t. $\overline{\overline{|\mathcal{M}|}} = \tau^{++}$ and $\overline{A_{\mathcal{M}}} = \tau$.

Note The usual notion of (τ, τ^{++}) -model requires only that $\overline{\overline{|\mathcal{M}|}} = \tau^{++}$ and $\overline{A_{\mathcal{M}}} = \tau$.
 If we added a predicate F and the axiom $F: \text{On} \leftrightarrow V$, then the two notions would become equivalent for models of this theory.

Note If \mathcal{M} is a (τ, τ^{++}) -model, then $\overline{B_{\mathcal{M}}} = \tau^+$. To see this note that, letting $<_{\mathcal{M}}$ be the natural ordering of $\text{On}_{\mathcal{M}}$ in \mathcal{M} , then for all $x \in B$, $\overline{\{z \mid z <_{\mathcal{M}} x\}} \leq \tau$. (This is because either $\{z \mid z <_{\mathcal{M}} x\} = \emptyset$ or else

$\mathcal{M} \models \forall f: A \xrightarrow{\text{onto}} x$. Let $\mathcal{M} \models f: A \xrightarrow{\text{onto}} x$.

Set $\tilde{f} = \{ \langle z, y \rangle \mid \mathcal{M} \models z = f(y) \}$. Then

$\tilde{f}: A \xrightarrow{\text{onto}} \{ z \mid z <_{\mathcal{M}} x \}$, 1 Hence

$B = \bigcup_{x \in B} \{ z \mid z <_{\mathcal{M}} x \}$ has cardinality

$\leq \tau^+$. Suppose $\overline{B} \leq \tau$. Then, by the

above argument, for each $x \in \text{On}_{\mathcal{M}}$

we have $\{ z \mid z <_{\mathcal{M}} x \} \leq \tau$. But

$\text{On}_{\mathcal{M}} = \bigcup_{x \in \text{On}_{\mathcal{M}}} \{ z \mid z <_{\mathcal{M}} x \}$. Hence

$\overline{\text{On}_{\mathcal{M}}} \leq \tau^+ < \tau^{++}$. Contr!

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Lemma 1 Let \mathcal{M} be a $(\tau; \tau^{++})$ model,

let b be an initial segment of

$\langle \text{On}_{\mathcal{M}}, <_{\mathcal{M}} \rangle$ s.t. $\text{cf}(b) = \tau^+$. Then

b has a supremum in $\langle \text{On}_{\mathcal{M}}, <_{\mathcal{M}} \rangle$.

(i.e. there is $z \in \text{On}_{\mathcal{M}}$ s.t.

$$z \leq_{\mathcal{M}} u \iff \bigwedge x \in b \ x \leq_{\mathcal{M}} u$$

for all $u \in \text{On}_{\mathcal{M}}$).

prf. of Lemma 1. Suppose not,

Since $\overline{\{z \mid z <_{\mathcal{M}} x\}} \leq \tau^+$ for $x \in O_{\mathcal{M}} \mathcal{M}$

and $\overline{O_{\mathcal{M}} \mathcal{M}} = \tau^{++}$, we have:

$cf(O_{\mathcal{M}} \mathcal{M}) = \tau^{++}$ in $\langle \mathcal{M} \rangle$. Hence b is

a proper segment of $O_{\mathcal{M}} \mathcal{M}$. Let

$x \in O_{\mathcal{M}} \mathcal{M} \setminus b$. Let $f \in \mathcal{M}$ s.t.

$\mathcal{M} \models f: B \xrightarrow{\text{onto}} x$. Then

$\tilde{f} = \{\langle u, \sigma \rangle \mid \mathcal{M} \models u = f(\sigma)\}$ is a map of

B onto $\tilde{x} = \{z \mid z <_{\mathcal{M}} x\}$. For $z \in B$

let $\mathcal{M} \models f_z = f \upharpoonright z$ + let \tilde{f}_z, \tilde{z}

have the obvious definitions.

Then $\tilde{f}_z: \tilde{z} \xrightarrow{1-1} \tilde{x}$. Since $x \neq \sup b$

in $\langle \mathcal{M} \rangle$, there is $u \in \tilde{x} \setminus b$. Hence

there is $z \in B$ s.t. $u \in \text{rng}(\tilde{f}_z)$.

However:

(1) $\text{rng}(\tilde{f}_z) \cap b$ is bounded in b ,

since $\overline{\text{rng}(\tilde{f}_z)} = \tilde{z} \leq \tau$ and

$cf(b) = \tau^+$ in $\langle \mathcal{M} \rangle$.

Pick $d \in b$ s.t. $\text{rng}(\tilde{f}_z) \cap b \subset d$.

Then there is a unique $q_z \in \text{On}_M$ s.t.

$$(2) M \models q_z = \min \{ \sigma \in \text{rng}(f_z) \mid \sigma > d \}.$$

It follows immediately that

$$(3) q_z = \min(\text{rng}(f_z) \setminus b) \text{ in } \langle M, \leq \rangle.$$

(Note that def. of q_z does not depend on d ; it would be the same for any $d' \in b$ s.t. $\text{rng}(f_z \upharpoonright b) \cap d' = \emptyset$.)

Obviously;

$$(4) z <_M z' \in B \rightarrow q_z \leq_M q_{z'}$$

We now define $\langle M_u \mid u \in B \setminus z \rangle$ s.t.

$M \models M_u \in \omega$ as follows:

Pick a $d = d_u \in b$ s.t. $b \cap \text{rng}(f_u) \cap d = \emptyset$.

Working in M define a map

$$q_u = q_f : [z, u] \rightarrow \text{On} \text{ by}$$

$$q_f(w) = \min \{ \sigma \in \text{rng}(f_w) \mid \sigma > d \}.$$

Then in fact $M \models q_w = q_f(w)$ for

$$z \leq_M w \leq_M u.$$

In M let $a = a_u = \{ q_f(w) \mid z \leq w \leq u \}$,

Then $M \models a$ is finite, since

$g(w) \leq g(w')$ for $w' \leq w$.

Let $\Omega \models m_u = \bar{a}$; Then

(5) $\Omega \models m_u \in \omega$; hence $m_u \in A$.

(6) $z \leq_u u \leq_{\Omega} u' \in B \rightarrow$

$$\rightarrow m_u \leq_{\Omega} m_{u'}$$

since $\Omega \models g_u = g_{u'} \upharpoonright [z, u]$ & hence

$$\Omega \models a_u < a_{u'}$$

(7) Let $z \leq_u u \in B$. There is $u' \in B$

$$\text{s.t. } \Omega \models a_u \neq a_{u'}$$

(hence $m_u <_{\Omega} m_{u'}$).

pr b.

$$g_u = g(u) = \min a \text{ in } \Omega.$$

Let $p \in \Omega_m \setminus b$ s.t. $p \leq_{\Omega} g_u$

(This must exist, since otherwise $g_u = \sup b$ in Ω .)

Then $\Omega \models \forall u' \in B \cdot p \in \text{rng } f \upharpoonright u'$.

Let $p \in \text{rng } f_{u'}$, where $u \leq_{\Omega} u' \in B$.

Then $g_{u'} \leq_{\Omega} p < g_u$. Hence

$$\Omega \models a_{u'} = \text{rng } (g_{u'}) \neq \text{rng } (g_u) = a_u$$

Hence $m_{u'} >_{\Omega} m_u$. QED (7)

Now select $\langle z_\xi \mid \xi < \tau^+ \rangle$ s.t.

$z_\xi \in B$ by: $z_0 = z$;

$z_{\xi+1} \succ_{M_\xi} z_\xi$ s.t. $m_{z_{\xi+1}} > m_{z_\xi}$;

z_λ s.t. $z_\lambda \succ z_\xi$ for all $\xi < \lambda$

(Lim (λ)).

[z_λ exists since $\text{cf}(B) = \tau^+ \text{ in } \aleph_\alpha$]

Then $m_{z_\xi} < m_{z_{\xi'}}$ for $\xi < \xi'$.

Hence $\langle m_{z_\xi} \mid \xi < \tau^+ \rangle$ injects τ^+ into A , where $\bar{A} = \tau < \tau^+$.

Contradiction! QED (Lemma 1)

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Now let L^* be L together with a new predicate C and the additional axiom:

$C = \langle C_\lambda \mid B < \lambda \wedge \text{Lim}(\lambda) \rangle$ is a \square_B -sequence.

We shall give a model of set theory satisfying $\text{GCH} + \Delta^+$ s.t.

L^* has no (ω_1, ω_2) -model.

(Moreover, \square_{\aleph_2} will hold for

$\kappa > \omega_1$. Hence $(\kappa, \kappa^{++}) \not\vdash (\omega_1, \omega_2)$

for $\kappa > \omega_1$.) Since Δ^+ holds,

there will be a Kurepa tree in the

model. This will show that the

gap 2 conjecture can fail at

(ω_1, ω_2) even in the presence of a

Kurepa tree.

The absence of a (ω_1, ω_2) -model for L^* means, of course, that \square

fails. Since \square holds whenever ω_2

is not Mahlo in L , we shall force

over a ground model containing

a Mahlo cardinal κ . The forcing

has two stages. At the first stage

we do ordinary collapsing to turn κ into ω_2 . The resulting generic extension has neither a Kurepa tree nor a (ω_1, ω_2) -model for \mathcal{L}^* . We then force to reinstate the principle \diamond^+ , but without adding a (ω_1, ω_2) -model of \mathcal{L}^* .

In the following let N be a countable transitive model of $ZFC + GCH + \diamond^+$ + There is a Mahlo cardinal.

Let κ be a Mahlo cardinal in N .

Let $S = \langle S_\alpha \mid \alpha < \omega_1 \rangle$ be a fixed \diamond -sequence in N . In the first stage of our forcing we use the normal conditions for collapsing to make κ become ω_2 :

Def Let $\omega_1 \leq \tau < \mu$.

\mathbb{P}_τ^μ = the set of maps p with $\text{dom}(p) \subseteq \omega \times [\tau, \mu)$, $\bar{p} \leq \omega$, and $p(i, \nu) < \nu$ for $\langle i, \nu \rangle \in \text{dom}(p)$

$p \leq q \iff p \supseteq q$ for $p, q \in \mathbb{P}_\tau^\mu$.

We also set: $\mathbb{Q}^\mu = \mathbb{P}_{\omega_1}^\mu$.

The properties of this forcing are well known:

(a) \mathbb{C}_τ^M is ω_1 -distributive

(b) (Assume GCH) If $\mu > \omega_1$ is regular, then \mathbb{C}_τ^M satisfies the μ -CC (i.e., every antichain has cardinality $< \mu$),

(c) $\mathbb{C}_\tau^M \Vdash \aleph_{\bar{\alpha}} < \aleph_{\bar{\beta}} \iff \bar{\alpha} < \bar{\beta} \leq \omega_1$ ($\bar{\alpha} < \mu$)

It follows that if $\mu > \omega_1^N$ is regular in N and G is \mathbb{C}_τ^M -generic over N , then $\omega_1^{N[G]} = \omega_1^N$ and $\omega_2^{N[G]} = \mu$.

We also note that (a), (b) are satisfied by $\mathbb{C}_\tau^M \times \mathbb{C}_\tau^{M'}$, and that $\mathbb{C}_\tau^M = \mathbb{C}_\alpha^M \times \mathbb{C}_\tau^\alpha$ for $\omega_1 \leq \alpha < \mu$.

We force with \mathbb{C}^κ , where κ is a Mahlo cardinal in N . It is known that \square then becomes false in the resulting model. We improve this to:

Lemma 2 Let κ be Mahlo in N and let G be \mathbb{C}^κ -generic over N . Then L^* has no (ω_1, ω_2) -model in $N[G]$.

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prf. of Lemma 2.

Suppose not. Let \mathcal{M} be an (ω_1, ω_2) -
-model of L^* in $N[G]$. Let

$$\mathcal{M} = \langle |\mathcal{M}|, \varepsilon_{\mathcal{M}}, A_{\mathcal{M}}, B_{\mathcal{M}}, C_{\mathcal{M}}, in \rangle.$$

We can assume w.l.o.g. that $\bar{\kappa} = \kappa$
and hence that $|\mathcal{M}| \subset \kappa$ in $N[G]$.

Let $\mathcal{M} = \mathcal{M}^{\circ G}$ and let $\theta > \kappa$ be
regular in N s.t. $\bar{\kappa} \in H_{\theta}$ in N .

Set $H = H_{\theta}$. Since G is \mathbb{C}^{ω} -generic
over the ZFC-model H and the
above properties of \mathcal{M} are absolute
in $H[G]$, there is $p \in G$ s.t.

(1) $p \Vdash^H (\bar{\kappa} \text{ is an } (\omega, \omega_2)\text{-model of } L^* \wedge |\bar{\kappa}| \subset \bar{\kappa})$

Fix Skolem functions for H and set:

$X_{\alpha} =$ the smallest $X \subset H$ s.t.

$$\alpha \cup \{p, \bar{\kappa}, \kappa\} \in X$$

for $\alpha < \kappa$. Set:

$$C = \{ \alpha < \kappa \mid \kappa \cap X_{\alpha} = \alpha \}.$$

Then C is club in κ . By Mahloner

there is a regular $\tau \in C$. Let
 $X = X_{\tau}$ and set: $\sigma: \bar{H} \xrightarrow{\sim} X$,

where \bar{H} is transitive.

Then $\sigma: \bar{H} \prec H$, $\tau = \text{crit}(\sigma)$, $\sigma(\tau) = \kappa$,
and $\sigma(\mathbb{Q}^\tau) = \mathbb{Q}^\kappa$. Clearly:

$$(2) \varphi \Vdash_{\mathbb{Q}^\tau}^{\bar{H}} \varphi(t_1, \dots, t_m) \iff \varphi \Vdash_{\mathbb{Q}^\kappa}^H \varphi(t_1, \dots, t_m),$$

for $\varphi \in \mathbb{Q}^\tau$, $t_1, \dots, t_m \in \bar{H}$, since $\sigma(\varphi) = \varphi$.

Let $\bar{G} = G \cap \mathbb{Q}^\tau$. By (2):

(3) There is a unique $\tilde{\sigma}: \bar{H}[\bar{G}] \prec H[G]$
defined by $\tilde{\sigma}(t^{\bar{G}}) = t^G$.

Hence:

(4) $\tilde{\sigma} \upharpoonright \bar{H} = \sigma$, since
 $\tilde{\sigma}(x) = \tilde{\sigma}(x^{\bar{G}}) = \sigma(x)^G = \sigma(x)$ for $x \in \bar{H}$.

In particular,

(5) $\tilde{\sigma} \upharpoonright \bar{\tau} = \text{id}$.

By (3) we have $\tilde{\sigma} \upharpoonright \bar{\nu} : \bar{\nu} \prec \nu$, but
 $\tilde{\sigma} \upharpoonright \bar{\nu} = \text{id}$ by (5). Hence

(6) $\bar{\nu} \prec \nu$.

Let $\bar{\nu} = \langle \bar{\nu}_1, \bar{E}_{\bar{\nu}}, \bar{A}_{\bar{\nu}}, \bar{B}_{\bar{\nu}}, \bar{C}_{\bar{\nu}, m} \rangle$

(7) $\bar{E}_{\bar{\nu}}$ is an end extension of $E_{\bar{\nu}}$

prf.

Let $x \in \bar{\nu}$. Claim $\bar{E}_{\bar{\nu}} \text{''} \{x\} = E_{\bar{\nu}} \text{''} \{x\}$.

The case $\bar{\nu} \Vdash x = \emptyset$ is trivial by (6).

So assume $\bar{\nu} \Vdash x \neq \emptyset$. There is

$f \in \bar{M}$ s.t. $\bar{M} \models f: B \xrightarrow{\text{onto}} x$. Thus

$\tilde{f}: B_{\bar{M}} \xrightarrow{\text{onto}} \in_{\bar{M}} \{x\}$, where

$B_{\bar{M}} = \gamma = \omega_1^N$ in \bar{H} . Let $g \in \bar{H}$ s.t.

$g: \gamma \xrightarrow{\text{onto}} \in_{\bar{M}} \{x\}$. Then

$\tilde{\sigma}(g): \gamma \xrightarrow{\text{onto}} \in_{\bar{M}} \{x\}$. But

$\tilde{\sigma}(g)(\nu) = \tilde{\sigma}(g(\nu)) = g(\nu)$ by (5). Hence

$\tilde{\sigma}(g) = g$ and $\in_{\bar{M}} \{x\} = \text{rng}(g) = \in_{\bar{M}} \{x\}$.

QED (7)

The same proof shows:

(8) $A_{\bar{M}} = A_M$; $B_{\bar{M}} = B_M$; $\langle_{\bar{M}}$ is

an end extension of $\langle_{\bar{M}}$ (where $\langle_M = \in_M \cap \text{On}_M^2$).

Thus $\text{On}_{\bar{M}}$ is an initial segment

of On_M in $N[G]$. Moreover

(9) $\text{cf}(\text{On}_{\bar{M}}) = \text{cf}(\sigma) = \omega_1$ in $N[G]$.

Hence there is $u \in \text{On}_M$ s.t.

(9.1) $u = \sup \text{On}_{\bar{M}}$ in \langle_M .

Note W.l.o.g. we take the axiom:

$C = \langle C_\lambda \mid B < \lambda \wedge \text{Lim}(\lambda) \rangle$ is a \square_B^- sequence.

as meaning:

(a) C_λ is cut in λ and $\text{otp}(C_\lambda) \leq B$

(b) $\forall \gamma \in C_\lambda$ and $\text{Lim}(\gamma)$, then

$$C_\gamma = \gamma \cap C_\lambda.$$

(This is the usual formulation of \square_B^- with the additional condition that, whenever $\xi \in C_\lambda$ is a successor point in C_λ , then ξ is a successor ordinal. \forall is trivial that if there is a \square_B^- sequence, then there is one with the additional condition.)

Def For $x \in \text{On}_{\bar{\mu}}$ set $\bar{\mu} \models \text{Lim}(x)$ set:

$$C_x = \{y \mid \bar{\mu} \models y \in C_x\}, \text{ for particular}$$

set: $D = C_u$, where $u = \sup \text{On}_{\bar{\mu}}$,

$$\text{Set } \Gamma = \{x \in \text{On}_{\bar{\mu}} \mid \bar{\mu} \models \text{Lim}(x)\}$$

$$\text{and } \tilde{x} = \langle_{\bar{\mu}} \{x\} \text{ for } x \in \text{On}_{\bar{\mu}}.$$

Then D has the properties:

(10) (a) $D \cap \Gamma$ is cofinal in $\text{On}_{\bar{\mu}}$

(b) $\forall x \in D \cap \Gamma$, then $D \cap \tilde{x} = C_x$.

Def Any $D \subset \Omega_{\bar{\mu}}$ satisfying (10) is called devilish.

We have shown that there is a devilish set in $N[G]$. But $N[G] = N[\bar{G}][G']$ where $G' = G \cap \mathbb{C}_{\tau}^n$ is \mathbb{C}_{τ}^n -generic over $N[\bar{G}]$. Hence there is $g \in G'$ st.

$$(11) g \notin \mathbb{C}_{\tau}^n \text{ over } N[\bar{G}] \text{ (there is a devilish set),}$$

Now let $G_0 \times G_1$ be $\mathbb{C}_{\tau}^n \times \mathbb{C}_{\tau}^n$ -generic over $N[\bar{G}]$ with $g \in G_i$ ($i=0,1$). We know

that $\mathbb{C}_{\tau}^n \times \mathbb{C}_{\tau}^n$ is ω_1 -distributive. Hence ω_1 is absolute in $N[\bar{G}][G_0 \times G_1] =$

$= N[\bar{G}][G_0][G_1]$. Let $D_i \in N[\bar{G}][G_i]$ be devilish. We derive a contradiction.

We first note:

$$(12) \Gamma \cap D_0 \cap D_1 \text{ is bounded in } \Omega_{\bar{\mu}}.$$

Proof.

Suppose not. Then

$$D_0 = D_1 = \bigcup_{x \in \Gamma \cap D_0 \cap D_1} C_x. \text{ Hence}$$

$$D = D_i \in N[\bar{G}][G_i] \text{ for } i=0,1.$$

By the product lemma, $D \in N[\bar{G}]$.

But $\bar{\mu} \models \text{otp}(C_x) < \aleph_1$, hence

$$\bar{\mu} \models \bar{C}_x \leq A, \text{ for } x \in \Gamma \cap D.$$

By this we get:

Claim $cf(D) \leq \omega_1$ in $\langle \bar{\omega} \rangle$ (in $N[\bar{\sigma}]$)

prf. Suppose not. Let $\langle x_i \mid i < \omega_2 \rangle$ be a monotone sequence in D . Then $\langle x_i \mid i < \omega_1 \rangle$ is bounded in $On_{\bar{\omega}}$ and there is $w \in On_{\bar{\omega}}$ s.t. $w = \sup_{i < \omega_1} x_i$ in $\langle \bar{\omega} \rangle$

by Lemma 1. But then $x_i \in D \cap \bar{w} = C_w$ for $i < \omega_1$. Hence $\bar{C}_w \geq \omega_1$, where $w \in \Gamma \cap D$. Contr! QED

Since D is unbounded in $On_{\bar{\omega}}$ it follows that: $cf(On_{\bar{\omega}}) = \omega_1$ in $\langle \bar{\omega} \rangle$. But

in $N[\bar{\sigma}]$ we have: $cf(On_{\bar{\omega}}) = \tau = \omega_2$.

Contr!

QED (12)

Now let $x_0 \in On_{\bar{\omega}}$ s.t. $(D_0 \cap D_1 \cap \Gamma) \setminus \bar{x}_0 = \emptyset$.

In $\bar{\omega}$ we define for each $z \in D_0 \cap \Gamma$ an m_z s.t. $\bar{\omega} \models m_z \in \omega$, (hence

$m_z \in A$). We do this as follows:

Pick a $z' \in D_1 \cap \Gamma$ s.t. $z <_{\bar{\omega}} z'$.

Arguing in $\bar{\omega}$, there is a sequence $\langle r^z(i) \mid i < m_z \rangle$ defined by:

$r^z(0) \simeq$ the least $r \in C_z$ s.t. $r > x_0$

$r^z(i+1) \simeq$ the least $r \in C_z$ s.t.

$\forall s (s \in C_z \wedge r^z(i) < s < r)$.

Then $\bar{\omega} \models m_z < \omega$, since otherwise

$\bar{\omega} \models m_z = \omega$ and there is σ s.t.
 $\bar{\omega} \models \sigma = \sup_{i < \omega} r^z(i)$. Then $\sigma \in \Gamma \cap D_0 \cap D_1$
 where $\sigma > \kappa_0$. Contr!

It is obvious that the definition of m_z does not depend on the C_z' chosen, since $C_z' = \tilde{z}' \cap D_1$.

Also:

$$(13) \quad z < z' \text{ in } (\Gamma \cap D_0) \rightarrow m_z \leq_{\bar{\omega}} m_{z'}$$

Moreover:

$$(14) \quad \text{If } z \in (\Gamma \cap D_0), \text{ there is } z' \geq_{\bar{\omega}} z \text{ in } \Gamma \cap D_0 \text{ s.t. } m_z < m_{z'}$$

prf.

Choose $\omega \in D_1 \cap \Gamma$ s.t. $\omega > \kappa_0, z$.

Choose $z' \in D_0 \cap \Gamma$ s.t. $z' > \omega$.

Then $\bar{\omega} \models r^z \neq r^{z'}$. Hence

$$\bar{\omega} \models m_z < m_{z'}. \quad \text{Q.E.D. (14)}$$

Now choose $z_\zeta \in D_0 \cap \Gamma$ ($\zeta < \omega_1$)

$$\text{s.t. } z_\zeta <_{\bar{\omega}} z_{\zeta+1} \text{ and } m_{z_\zeta} <_{\bar{\omega}} m_{z_{\zeta+1}}$$

$$\text{and } z_\lambda >_{\bar{\omega}} z_\zeta \text{ for all } \zeta < \lambda$$

(Fin(λ)). Then $m_{z_\zeta} < m_{z_{\zeta'}}$

for $\zeta < \zeta' < \omega_1$. Hence

$\langle m_{z_\zeta} \mid \zeta < \omega_1 \rangle$ injects ω_1

into A_{ω_2} , where $\overline{A_{\omega_2}} = \omega_1$, Contr!

QED (Lemma 2)

Our main result is:

Lemma 3 Let G be \mathbb{C}^κ -generic over N . There is a generic extension

$N[G][H]$ of $N[G]$ s.t. in $N[G][H]$

(a) $\kappa = \omega_2$; (b) \diamond^+ holds;

(c) \mathcal{L}^* has no (ω_1, ω_2) -model.

The proof depends on the construction of forcing conditions $IP \in N[G]$ s.t. (a)-(c) are forced. The actual construction of IP will be given in §2.

Here we list the salient properties of IP and derive Lemma 3 from them.

Since $G \Vdash H$ holds in N , there is an $A_0 \in N$ s.t. $A_0 \subset \kappa$ and $L_\tau[A_0] = H_\tau$

for all cardinals τ s.t. $\omega \leq \tau \leq \kappa$;

Let G be \mathbb{C}_ν^κ -generic over N . Set:

$$A_1 = \{ \langle \mu, \gamma, i \rangle \mid \forall p \in G \ p(\gamma, i) = \mu \}$$

$$A = \{ \langle \mu, i \rangle \mid (i=0, \mu \in A_0) \vee (i=1, \mu \in A_1) \}$$

Let τ be regular in N s.t. $\omega_1 \leq \tau \leq \kappa$,

Let $\bar{G} = G \cap \mathbb{C}^\tau$. Then \bar{G} is \mathbb{C}^τ -

-generic over N and