§ 3 Iterability (a longer sketch)

We now prove a somewhat more general case of § 1 Thm 1, which we believe contains all the new ideas needed for a full proof. We again let N be an arbitrary mouse like and robust premouse satisfying ZFC^-. We let $\mathcal{I} = \langle\langle P_i\rangle, \langle \nu_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a countable putative iteration of a countable P_0, where $\delta_0 : P_0 \prec N$.

We prove:

Main Claim One of the following holds:

(a) $lh(\mathcal{I}) = i+1$ and there is $\delta : P_i \prec N$ s.t. $\delta\pi_{0i} = \delta_0$.

(b) $\mathcal{I}$ has a maximal branch b, which is of limit length, and there is $\delta : P_b \prec N$ with $\delta\pi_b = \delta_0$.

We shall suppose (b) to fail and prove (a). We again follow Steel's proof closely.

Def Fix $n^* : lh(\mathcal{I}) \xrightarrow{1\text{-}1} \omega$. Set:
$n(i) = \min\{n^*(j) \mid i \leq_T j\}$,

Def i survives at j ($i \text{ surv } j$) iff $i \leq j$, $n(i) = n(j)$ and $n(l) \geq n(i)$ for $l \in (i, j)$.

The following facts were established by Steel:

Fact 1

(a) $(n(i) = n(j) \wedge i < j) \to i \leq_T j$

(b) Let b be a branch of limit length in $\mathcal{T}$.
b is maximal $\longleftrightarrow \sup n''b = \omega$;

Fact 2 Let i surv j and $l \notin (i, j)_T$ but $l \in (i, j)$. Then $n(i) < n(h)$.

Fact 3

(a) $(i$ surv h surv $j) \to i$ surv j

(b) $(i$ surv $j \wedge i \leq_T h \leq_T j) \to i$ surv h surv j

(c) Let b be a branch of limit length.
b is maximal iff for all $i \in b$ there is $j \in b$ s.t. $i < j$ and i does not survive at j.

We are assuming that (b) fails in the Main Claim. As before, this says that a certain relation R is well founded;

Def $D = \{\langle i, \delta\rangle \mid \delta : P_i \prec N \wedge \delta\pi_{0i} = \delta_0\}$
$R \subset D^2$ is then defined by:
$\langle j, \delta'\rangle \, R \, \langle i, \delta\rangle \longleftrightarrow (i \leq_T j$, i does not survive at j, and $\delta'\pi_{ij} = \delta$.

As before, we set:

Def $r(z)$ = the rank of z in R;
$r = r(\langle 0, \delta_0\rangle)$.

Def Let $i \leq j \leq lh(\mathcal{T})$.
$c(i,j) = \{h \mid j < h < lh(\mathcal{T}) \wedge h$ is a successor ordinal $\wedge T(h) \leq i \wedge T(h)$ survives at $h\}$

Steel shows:

Fact 4

(a) $j \leq j' \to c(i,j) \supset c(i,j')$

(b) $i \leq i' \to c(i,j) \subset c(i',j)$

(c) $c(i,j)$ is finite (in fact, if $h, k \in c(i,j)$ and $h < k$, then $T(k) < T(h)$ and $n(k) < n(h)$).

Def Let $i < \gamma \leq lh(\mathcal{T})$
i is a break point at γ iff whenever $i < h \leq \gamma$ s.t. $T(h) \leq i$, then $T(h)$ does not survive at h. (In other words, $c(i,i)^{\mathcal{T}|\gamma} = \emptyset$.)

Following Steel we define the concept of enlargement, which will play a central role in the proof.

Def The standard world $\bar{W}$ is defined as before. The standard enhanced world is now $W = \langle W_{\omega\mathfrak{r}+\omega}, \theta, a\rangle$, where $\mathfrak{r}, \theta, a, \bar{W}_\nu$ are defined as before. The concepts world and enhanced world are defined as before.

<u>Def</u> Let $1 \le \gamma \le lh(\mathcal{I})$. By an <u>enlargement</u> of $\mathcal{I}|\gamma$ <u>wrt.</u> $\mathcal{I}, \delta_0, N$, we mean a sequence $\mathbb{E} = \langle \mathbb{E}_i \mid i < \gamma \rangle$ s.t. $\mathbb{E}_i = \langle W_i, N_i, \delta_i \rangle$ and:

(a) $W_i = \langle |W_i|, \theta^i, a_i \rangle$ is an enhanced world.

Let e code $\mathcal{I}$ as before.

(b) $N_i, \delta_i \in \bar{W}_i$, $\delta_i : P_i \prec N_i$, where

$\langle \bar{W}_i, N_i, \delta_i \pi_{0i}, e \rangle \equiv \langle \bar{W}, N, \delta_0, e \rangle$.

(c) $\langle \bar{W}_i, N_i, \delta_i \pi_{hi}, e \rangle \equiv \langle \bar{W}_h, N_h, \delta_h, e \rangle$ for $h \leq_T i$.

(d) $\delta_i \upharpoonright \gamma_h = \delta_h \upharpoonright \gamma_h$ and $J^{E_i}_{c_h} = J^{E_h}_{c_h}$, where

$c_i = \sup \delta_i `` \gamma_i$ and $E_i = E^{N_i}$.

(e) $P_i =_{\mathrm{Df}} \mathrm{dom}(\theta^i) \geq \omega_1 \tilde{r}_i + |c(i,\gamma)|$, where r_i is defined in $\bar{W}_i$ from $N_i, \delta_i \circ \pi_{0i}$ as r in $\bar{W}$ from N, δ_0, and $\tilde{r}_i = r_i(\langle i, \delta_i \rangle)$.

(f) $C^{E_h}_{c_h, \infty} \models \varphi(\delta_h \upharpoonright \gamma_h, t_h, e)$ in $\bar{W}_h \longleftrightarrow$

$C^{E_i}_{c_h, \infty} \models \varphi(\delta_i \upharpoonright \gamma_h, t_h, e)$ in $\bar{W}_i$ for $h \leq i$,

where T_i = the complete theory of $\langle \bar{W}_i, N_i, \delta_i, e \rangle$

and $t_i = \langle T_h \mid h \leq i \rangle$.

We shall need, however a stronger version of (f). In order to formulate this, we first define:

<u>Def</u> For $h \leq i$ set:

$S^{h,i}_m$ = the set of Σ_1 fmlae φ s.t.

$C^{E_i}_{c_i, \infty} \models \varphi(\delta_i \upharpoonright \gamma_h, t_h, e, \langle S^{h,i}_l \mid l < m \rangle)$ in W_i

Set: $S^i_m = S^{i,i}_m$. We strengthen

Then (f) says: $S^{h,i}_0 = S^h_0$. We strengthen this to:

(g') $S^{h,i}_m = S^h_m$ for $h \leq i$, $m \leq |c(h,\gamma)|$.

We again work in $V[G]$, where G collapses the standard enhanced world W to ω. We assume (b) of the Main Claim to be false in V and prove:

<u>Thm 1</u> Let $\gamma < lh(\mathcal{I})$. Then $\mathcal{I}|\gamma+1$ has an enlargement $\mathbb{F}$. Moreover, if $i < \gamma$ is a breakpoint at γ and $\mathbb{E}$ an enlargement of $\mathcal{I}|i+1$, then $\mathbb{F}$ can be so chosen that $\mathbb{F}|i+1 = \mathbb{E}$ and $W_\gamma^{\mathbb{F}} \cap On < W_i^{\mathbb{E}} \cap On$.

From this we prove (a), thus showing the Main Claim to be true:

<u>Case 1</u> $lh(\mathcal{I}) = \gamma+1$. Then Then $\mathcal{I}$ has an enlargement $\mathbb{F}$. But

$\bar{W}_\gamma^{\mathbb{F}} \models \delta_\gamma : P_\gamma \prec N_\gamma$. But

$\langle \bar{W}_\gamma^F, N_\gamma, \delta_\gamma \pi_{0\gamma} \rangle \equiv \langle \bar{W}, N, \delta_0 \rangle$

Hence there is $\delta' \in \bar{W}$ s.t. $\delta' \pi_{0\gamma} = \delta_0$. QED

<u>Case 2</u> $lh(\mathcal{I})$ is a limit ordinal θ.

Define $n_l < \omega$, $j_l < \theta$ $(l < \omega)$ by:

$n_0 = 0$, $n_{l+1} = \min\{n^*(j) \mid j > j_l\}$

$j_l =$ that j s.t. $n^*(j) = n_l$.

Then j_l is a breakpoint in θ, hence in j_{l+1}. Applying Thm 1 we define a sequence $\mathbb{E}_l$ $(l < \omega)$ s.t. $\mathbb{E}_l$ is an enlargement of $\mathcal{I}|(j_l+1)$ and:

$W_{j_{l+1}} \cap On < W_{j_l} \cap On$ $(l < \omega)$.

Contr!

QED (Main Claim)

It remains to prove Thm 1.

We first show:

Lemma 2 Let $\mathbb{E}$ be an enlargement of $\mathcal{I}|(i+1)$. Let $\eta = T(i+1)$, where η does not survive at $i+1$. There is an enlargement $\mathbb{F}$ of $\mathcal{I}|(i+2)$ s.t. $\mathbb{F}|(i+1) = \mathbb{E}$ and $W^{\mathbb{F}}_{i+1} \cap On < W^{\mathbb{F}}_{i} \cap On$.

prf.

We imitate the construction in §2.

We imitate the construction in §2. As before there is $g : \lambda_i \to \delta_i(\kappa_i)$ satisfying (1)(a) + (1)(b) of §1. We again define $\sigma : P_{i+1} \prec N_\eta$ by $\sigma(\pi_{\eta, i+1}(f)(\alpha)) = \delta_\eta(f)(g(\alpha))$ and observe that $\sigma \in W_\eta$. Since η does not survive at $i+1$, we have: $\langle i+1, \sigma \rangle R_\eta \langle \eta, \delta_\eta \rangle$, where R_η is defined in W_η from $N_\eta, \delta_\eta \pi_{0\eta}$ as R was defined in W from N, δ_0. We again let $\bar{\pi} = \pi_\eta(\langle i+1, \sigma \rangle) < \pi_\eta$, and pick $\alpha > \bar{\tau} = (\theta^+_{\bar{\pi}})_{W_\eta}$ s.t. $\alpha \in W_\eta$ and $C^{E_\eta}_{\bar{\tau}, \alpha}$ is admissible. The theory $T = T_{\bar{\tau}, \alpha}$ in the infinitary language of $C^{E_\eta}_{\bar{\tau}, \alpha}$ is as before and we observe that $T_{\bar{\tau}, \alpha}$ is consistent. Note that for each $\xi < \delta_i(\kappa_i) = \delta_\eta(\kappa_i)$ the statement:

$\forall \tau > \xi \, \forall \alpha \, (\bar{c} < \tau < \alpha \wedge C^{E\eta}_{\bar{c},\alpha}$ is admissible
$\wedge \, T_{\tau,\alpha}$ is consistent)

holds in W_η. This has the form:

(1) $C^{E\eta}_{\bar{c},\infty} \models \varphi(\xi, g''\delta_i, T_0, e)$ in W_η

where φ is Σ_1 + T_0 = the complete theory of $\langle \bar{W}_\eta, N_\eta, \sigma, e \rangle$.

But just as before:

(2) $C^{E\eta}_{\bar{c},\delta_\eta(\kappa_i)} \prec_{\Sigma_1} C^{E\eta}_{\bar{c},\infty}$ in W_η.

Since (1) holds for all $\xi < \delta_\eta(\kappa_\eta)$, we conclude:

(3) $C^{E\eta}_{\bar{c},\delta_\eta(\kappa_i)} \models \bigwedge \xi \, \varphi(\xi, g''\delta_i, T_0, e)$.

Hence there are arbitrarily large $\tau < \delta_\eta(\kappa_i)$ s.t. for an $\alpha < \delta_\eta(\kappa_i)$, $C^{E\eta}_{\bar{c},\alpha}$ is admissible and $T_{\tau,\alpha}$ is consistent. The same is obviously true in W_i, since:

$$(C^{E\eta}_{\bar{c},\delta_\eta(\kappa_i)})_{W_\eta} = (C^{E_i}_{\bar{c},\delta_i(\kappa_i)})_{W_i}$$

Using the definition of $S^i_n = S^{i,i,\ldots}_n$ and the fact that W_i is a ZF-model, it is easily seen that:

$$(4)\ C^{E_i}_{c_i,\infty} \models \bigvee \tau \bigwedge n<\omega \bigwedge \varphi \in S^i_n \ C^{E_i}_{c_i,\tau} \models \varphi(\delta_i \upharpoonright \gamma_i, t_i, e, \langle s^i_l \mid l<n\rangle)$$

Hence by (1)(b) of § 2:

$$(5)\ C^{E_i}_{\bar{c},\delta_i(\varkappa_i)} \models \bigvee \tau \bigwedge n<\omega \bigwedge \varphi \in S^i_n \ C^{E_i}_{\bar{c},\tau} \models \varphi(g \upharpoonright \gamma_i, t_i, e, \langle s^i_l \mid l<n\rangle)$$

Since the $\varphi \in S^i_n$ are Σ_1 formulae, it is clear that if (5) holds for some $\tau < \delta_i(\varkappa_i)$, then for all larger such $\bar{\tau}$. Hence:

$$(6)\ C^{E_i}_{\bar{c},\delta_i(\varkappa_i)} \models \text{There are } \alpha, \bar{\tau} \text{ s.t. } C^{E_i}_{\bar{c},\alpha} \text{ is admissible} \wedge \bar{c} < \bar{\tau} < \alpha \wedge T_{\bar{\tau},\alpha} \text{ is consistent} \wedge \bigwedge n<\omega \bigwedge \varphi \in S^i_n \ C^{E_i}_{\bar{c},\tau} \models \varphi(g \upharpoonright \gamma_i, t_i, e, \langle s^i_l \mid l<n\rangle),$$

This statement has the form:

$$(7)\ C^{E_i}_{\bar{c},\delta_i(\varkappa_i)} \models \psi(g''\gamma_i, T_0, t_i, e, \langle s^i_l \mid l<\omega\rangle)$$

where ψ is Σ_1.

Hence, by (1)(b) of § 1 we have in $\bar{W}_i$:

$$(8)\ C^{E_i}_{c_i,\infty} \models \psi(\delta_i''\gamma_i, T_0, t_i, e, \langle s^i_l \mid l<\omega\rangle).$$

This says that there is $\alpha \in W_i$ s.t. $C^{E_i}_{c_i,\alpha}$ is admissible and there is $\dot{\tau} \in (c_i, \alpha)$ s.t.

- $\tilde{T}_{\bar{\tau},\alpha}$ is consistent, where $\tilde{T}_{\bar{\tau},\alpha}$ is like $T_{\bar{\tau},\alpha}$ except that (B)(ii) is replaced by: $\dot{\sigma} \upharpoonright \underline{\gamma_i} = \underline{\delta_i \upharpoonright \gamma_i}$.

- $C^{E_i}_{c_i,\infty} \models \bigwedge m < \omega \bigwedge \varphi \in S^i_m$

$$C^{E_i}_{c_i,\tau} \models \varphi(\delta_i \upharpoonright \gamma_i, t_i, e, \langle s^i_\ell \mid \ell < n \rangle).$$

Note, however, that if $\varphi \notin S^i_n$, then

$$C^{E_i}_{c_i,\infty} \models \neg\varphi(\delta_i \upharpoonright \gamma_i, t_i, e, \langle s^i_\ell \mid \ell < n \rangle)$$

and hence:

$$C^{E_i}_{c_i,\tau} \models \varphi(\delta_i \upharpoonright \gamma_i, t_i, e, \langle s^i_\ell \mid \ell < n \rangle),$$

since φ is Σ_1. By induction ~~of~~ on n we then get in W_i:

(9) S^i_m has the same definition in $C^{E_i}_{c_i,\tau}$ as in $C^{E_i}_{c_i,\infty}$.

Now let $\mathfrak{A}$ be a good model of $\tilde{T}_{\bar{\tau},\alpha}$.

Set: $W_{i+1} = W'_i$, $N_{i+1} = N'_i$, $\delta_{i+1} = \sigma'_i$.

(a)–(e) follow as before in §2,

By (9) we have: $S^i_m = S^{i,i+1}_m$ ($m < \omega$)

From this (g) follows. QED (Lemma 2)

As pendant to Lemma 2 we now prove:

<u>Lemma 3</u> Let $\mathbb{E}$ be an enlargement of $\mathcal{J}|(i+1)$. Let $h = T(i+1)$ survive at $i+1$. There is an enlargement $\mathbb{F}$ of $\mathcal{J}|(i+2)$ s.t.

(a) $\mathbb{F}|h = \mathbb{E}|h$, (b) $W^{\mathbb{F}}_{i+1} = W^{\mathbb{E}}_h$,

(c) $\delta_h = \delta_{i+1}\pi_{oh}$.

proof of Lemma 3.

We first note:

(10) $c(j,i) = c(h,i)$ for $h \le j \le i$

proof. Otherwise there is $k \ge i$, $j' \in (h,i]$ s.t. $j' = T(k+1)$ and j survives at $k+1$. Hence $k > i$, since $T(k+1) = j' \ne h = T(i+1)$. Hence $j' < i+1 < k+1$ and $n(i+1) < n(j')$. Hence j' does not survive at $k+1$. Contr! QED(10)

But $|c(h,i)| \ge 1$, since h survives at $i+1$ & hence $i+1 \in c(h,i)$. From now on let:

(11) $|c(j,i)| = n+1$ for $h \le j \le i$.

Define $g : \lambda_i \to \delta_i(\kappa_i) = \delta_h(\kappa_h)$ and $\sigma : P_{i+1} \prec N_h$ exactly as before. (Hence $\sigma \upharpoonright \lambda_i = g$, $\sigma \pi_{h,i+1} = \delta_h$.) We shall form an enlargement $\mathbb{E}'$ of $\mathcal{Y}|(i+2)$ s.t. $\mathbb{E}'|h = \mathbb{E}|h$, $W'_{i+1} = W_h$, $\delta'_{i+1} = \sigma$.

This means, however, that we must redefine $W'_\ell, N'_\ell, \delta'_\ell$ for $h \le \ell \le i$, since we need: $\delta'_\ell \upharpoonright \gamma_\ell = \delta'_{i+1} \upharpoonright \gamma_\ell = \sigma \upharpoonright \gamma_\ell = g \upharpoonright \gamma_\ell$, whereas $c_\ell = \sup \delta_\ell{}''\lambda_\ell \ge c_h > \delta_i(\kappa_i) \ge \sup \sigma{}''\lambda_i$.

Set: $\bar{\pi} = \omega\hat{\pi}_\ell + n$ (where $p_\ell \ge \omega\tilde{\pi}_\ell + n + 1$)

Let $\bar{\tau} = On \cap (W_\ell)_{\bar{\pi}}$. Let $\alpha > \bar{\tau}$, $\alpha \in W_\ell$ s.t. $C^{E_\ell}_{c_\ell,\alpha}$ is admissible.

Let $T = T^\ell_{\bar{\tau},\alpha}$ be the theory in the infinitary language of $C^{E_\ell}_{c_\ell,\alpha}$

consisting of:

<u>Predicate</u>: $\dot{\in}$

<u>Constants</u>: $\underline{x}$ $(x \in C^{E_\ell}_{c_\ell, \alpha})$, $\dot{W}$, $\dot{N}$, $\dot{\delta}$

<u>Axioms</u>: (A) as before, and

(B) (i) $\dot{N}, \dot{\delta} \in \dot{\bar{W}}$ and $\langle \dot{\bar{W}}, \dot{N}, \dot{\delta}, \underline{e} \rangle \models \underline{T}_\ell$,

where T_ℓ = the complete theory of

$\langle \bar{W}_\ell, N, \delta_\ell, e \rangle$

(ii) $\dot{\delta} \restriction \gamma_\ell = \underline{\delta_\ell \restriction \gamma_\ell}$

(iii) $p\dot{W} = \omega \dot{r} + \underline{m}$, where r' is defined in $\dot{\bar{W}}$ from $\dot{N}, \dot{\delta} \circ \underline{\pi}_{0\ell}$ as r was defined in $\bar{W}$ from N, δ_0 and $\dot{r} = r'(\langle \underline{\ell}, \dot{\delta} \rangle)$ in $\dot{\bar{W}}$.

Then $T^\ell_{\tau, \alpha}$ is consistent, since $\langle W_\ell, (W_\ell)_{\omega \tilde{r}_\ell + m}, N_\ell, \delta_\ell \rangle$ is a model. Since $\tau \equiv On \cap \bar{W}_\ell$, it follows that there are arbitrarily large $\tau \in \bar{W}_\ell$ s.t. for some $\alpha > \tau$, $\alpha \in \bar{W}_\ell$, $C^{E_\ell}_{c_\ell, \alpha}$ is admissible, and $T^\ell_{\tau, \alpha}$ is consistent (cf. the argument in the proof of Lemma 2). In particular, we can pick τ large enough that

(12) $\bigwedge \varphi \in S^\ell_m \; C^{E_\ell}_{c_\ell, \tau} \models \varphi(\delta_\ell \restriction \gamma_\ell, t_\ell, e, \langle s^\ell_k \mid k < m \rangle)$

for all $m \leq M = |c(\ell, i)| - 1$.

(Note that $(C^{E_\ell}_{c_\ell, \infty})_{\bar{W}_\ell} \prec_{\Sigma_1} (C^{E_\ell}_{c_\ell, \infty} | W_\ell)$.

Hence in $\bar{W}_\ell$:

(13) $C^{E\ell}_{c_\ell,\infty} \models \bigvee\alpha \bigvee\tau \, (C^{E\ell}_{c_\ell,\alpha}$ is admissible $\wedge$
$\wedge\, c_\ell < \tau < \alpha \wedge T^\ell_{\tau,\alpha}$ is consistent $\wedge$
$\bigwedge m \leq n \bigwedge \varphi \in S^\ell_m \;\; C^{E\ell}_{c_\ell,\tau} \models \varphi(\delta_\ell \restriction \gamma_\ell, t_\ell, e, \langle S^\ell_k | k < m\rangle)$.

This has the form:

(14) $C^{E\ell}_{c_\ell,\infty} \models \Psi(\delta_\ell{}''\gamma_\ell, t_\ell, e, \langle S^\ell_k \,|\, k \leq n\rangle)$

(Note that $t_\ell(\ell) = T_\ell$).

But since $S^\ell_k = S^{\ell,i}_k$ for $k \leq n+1$, we conclude:

(15) $C^{Ei}_{c_\ell,} \models \Psi(\delta_i{}''\gamma_\ell, t_\ell, e, \langle S^\ell_k \,|\, k \leq n\rangle)$,

But then by (1)(b) of §1:

(16) $C^{Ei}_{\bar{c},\delta_i(\kappa_i)} \models \Psi(g''\gamma_\ell, t_\ell, e, \langle S^\ell_k \,|\, k \leq n\rangle)$

We note that:

(17) If $\varphi \notin S^\ell$ and φ is Σ_1, then
$C^{Ei}_{c_i,\infty} \models \neg\varphi(\delta_i \restriction \gamma_\ell, t_\ell, e, \langle S^\ell_k \,|\, k < m\rangle)$
for $m \leq n$, since $S^\ell_m = S^{\ell,i}_m$ for $m \leq n+1$.

By (1)(b) of §1 it follows that:
(18) If $\varphi \notin S^\ell_m$ and φ is Σ_1, then
$C^{Ei}_{\bar{c},\delta_i(\kappa_i)} \models \neg\varphi(g \restriction \gamma_\ell, t_\ell, e, \langle S^\ell_k \,|\, k < m\rangle)$
for $m \leq n$.

By (16) there are $\tau, \alpha < \delta_i(\kappa_i)$ s.t.

(19) $\bar{\tau} < \tau < \alpha$, $C^{E_i}_{\bar{\tau},\alpha}$ is

$\bigwedge \varphi \in S^l_m \; C^{E_i}_{\bar{\tau},\tau} \models \varphi(g \restriction \gamma_l, t_l, e, \langle s^l_k \mid k < n \rangle)$

for $m \leq n$, and the theory $\tilde{T} = \tilde{T}^l_{\tau,\alpha}$ is consistent in the infinitary language of $C^{E_i}_{\bar{\tau},\alpha}$, where $\tilde{T}^l_{\tau,\alpha}$ is like $T^l_{\bar{\tau},\alpha}$ except that in (B)(ii) we replace $\underline{\delta_i \restriction \gamma_l}$ by $\underline{g \restriction \gamma_l}$.

By (18) we have:

(20) If $\varphi \in S^l_m$ and φ is Σ_1, then

$C^{E_i}_{\bar{\tau},\tau} \models \neg\varphi(g \restriction \gamma_l, t_l, e, \langle s^l_k \mid k \leq m \rangle)$

for $m \leq n$.

Recall that:

$$(C^{E_i}_{\bar{\tau},\delta_i(\kappa_i)})_{W_i} = (C^{E_h}_{\bar{\tau},\delta_h(\kappa_i)})_{W_h} \prec_{\Sigma_1} (C^{E_h}_{\bar{\tau},\infty})_{W_h},$$

Hence by (19), (20):

(21) $\langle s^l_m \mid m \leq n \rangle$ has the same definition in $C^{E_h}_{\bar{\tau},\tau}$ as in $C^{E_h}_{\bar{\tau},\infty}$ in the parameters

$\sigma \restriction \gamma_l = g \restriction \gamma_l, t_l, e$.

Now let $\mathfrak{A}_l$ be a good model of $\tilde{T}^l_{\bar{\tau},\alpha}$ for $h \leq l \leq i$. Set:

$W'_l = \dot{W}^{\mathfrak{A}_l}$, $N'_l = \dot{N}^{\mathfrak{A}_l}$, $\delta'_l = \dot{\delta}^{\mathfrak{A}_l}$,

for $h \leq l \leq i$. For $l < h$ set:

$\langle W'_l, N'_l, \delta'_l \rangle = \langle W^E_l, N^E_l, \delta^E_l \rangle$.

Finally set:

$W'_{i+1} = W_h$, $N'_{i+1} = N_h$, $\delta'_{i+1} = \sigma$, It is easily verified that

$$\mathbb{F} = \langle \langle W'_\ell, N'_\ell, \delta'_\ell \rangle \mid \ell < i+2 \rangle$$

is an enlargement of $\mathbb{I} | (i+2)$.

We use (21) to show: $s^\ell_m = s^{\ell, i+1}_m$ for $h \leq \ell \leq i$, $m \leq n \geq c(\ell, i+1)$.

QED (Lemma 3)

This proof shows more than we have stated: For $h \leq \ell \leq i$ set: $\tau_\ell =$ the smallest τ s.t. (19) holds. Then $\vec{\tau} = \langle \tau_\ell \mid h \leq \ell \leq i \rangle$ is definable in $\overline{W}_h$ from t_h + hence $\vec{\tau} \in W_h$. Clearly, $\sigma, t_h \in W_h$. But these are all countable sets, hence $\sigma \restriction \delta_i, t_h, \vec{\tau} \in C^{E_h}_{\bar{c}, \delta_h}(\kappa_i)$.

We note:

<u>Corollary 3.1</u> Let $\vec{\tau} = \langle \tau_\ell \mid h \leq \ell \leq i \rangle$ be as above. Then $\tilde{T} = \tilde{T}(\vec{\tau}, t_i, \sigma)$ is a consistent theory in $C^{E_h}_{\bar{c}, \delta_h}(\kappa_i)$, where $\tilde{T}$ is as follows:

<u>Predicate</u> $\dot{\in}$

<u>Constants</u> $\dot{W}, \dot{N}$,

<u>Axioms</u> (A) ZFC^*, $\wedge v (v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$ for all x,

$\dot{W} = \langle \dot{W}_\ell \mid \underline{h} \leq \ell \leq \underline{i} \rangle$, $\dot{W}_{\underline{\ell}} \cap [\underline{\tau_\ell}]^{\underline{\omega}} = \underline{[\tau_\ell]^\omega}$,

$\dot{N} = \langle \dot{N}_\ell \mid \underline{h} \leq \ell \leq \underline{i} \rangle$, $\dot{\delta} = \langle \dot{\delta}_\ell \mid \underline{h} \leq \ell \leq \underline{i} \rangle$,

$\dot{W}_{\underline{\ell}}$ is an enhanced world, $On \wedge \dot{W}_{\underline{\ell}} = \underline{\tau_\ell}$

(B) (i) $\dot N_\ell, \dot\delta_\ell \in \dot W_\ell$, $\underline{\dot\delta_\ell} : \underline{P_\ell} \prec \dot N_\ell$,
$\langle \dot{\underline{W}}_\ell, \underline{\dot N}_\ell, \underline{\dot\delta}_\ell, \underline{e} \rangle \models \underline{t}_i(\underline{\ell})$; moreover
$\langle \underline{\dot W_\ell}, \underline{\dot N_\ell}, \underline{\dot\delta_\ell \pi_{\eta\ell}}, \underline{e} \rangle \models \underline{t}_i(\underline{\eta})$ for $\eta \leq_T \ell$

(ii) $\underline{\dot\delta_\ell} \restriction \underline{\gamma_\ell} = \underline{\sigma \restriction \gamma_\ell}$

(iii) $p^{\dot W_\ell} = \omega \cdot \dot\gamma_\ell + \underline{n}$ $\quad (n = |c(h,i)| - 1)$.

proof.
$C^{E_h}_{\bar c, \delta_h(\kappa_i)}$ is admissible and
$\langle H, \langle W_\ell^{\mathbb{F}} \mid h \leq \ell \leq i \rangle, \langle N_\ell^{\mathbb{F}} \mid h \leq \ell \leq i \rangle,$
$\langle \sigma^{\mathbb{F}_\ell} \mid h \leq \ell \leq i \rangle \rangle$ is a model of $\tilde T$, where
$\mathbb{F}$ is as in ~~Lemma~~ Lemma 3, $\vec\tau = \langle \tau_\ell \mid h \leq \ell \leq i \rangle$ is chosen as above and $H = H_\mu^{V[G]}$, where $\mu > \delta_h(\kappa_i)$ is regular in $V[G]$. QED (Cor 3.1)

If $\tilde t = \langle \tilde T_\ell \mid \ell \leq i \rangle$ is any sequence of subsets of ω s.t. $\tilde T_\ell = T_\ell$ for $\ell \leq h$, then $\tilde T(\vec\tau, \tilde t, \sigma)$ is the same theory on $C^E_{\bar c, \delta_h(\kappa_i)}$ with $\tilde t$ in place of t_i in (B)(i). Finally, if $\tilde\sigma \in W_h$ is any function s.t. $\tilde\sigma : P_{i+1} \prec N_h$ and $\tilde\sigma \pi_{h,i+1} = \delta_h$, then $\sup \tilde\sigma''\gamma_i < \delta_h(\kappa_i)$ and we let $\tilde T(\vec\tau, \tilde t, \tilde\sigma)$ be the above theory on $C^{E_h}_{\tilde c, \delta_h(\kappa_i)}$, where $\tilde c = \sup \tilde\sigma''\gamma_i$. $\tilde T(\vec\tau, \tilde t, \tilde\sigma)$ makes sense for any $\vec{\tau} = \langle \tau_\ell \mid h \le \ell \le i \rangle \in C_{\delta_h}(\kappa_i)$.

We have therefore shown:

<u>Lemma 4</u> Let $\mathbb{E}$ be an enlargement of $\mathcal{Y}|(i+1)$, $\mathbb{E} = \langle \langle W_\ell, N_\ell, \delta_\ell \rangle \mid \ell \le i \rangle$. There exist $\sigma \in W_h$, $\vec{\tau} \in C_{\delta_h}(\kappa_i)$, $\tilde{t} \in C_{\delta_h}(\kappa_i)$ s.t.

(i) $\sigma : P_{i+1} \prec N_h$ and $\sigma \pi_{h,i+1} = \delta_h$

(ii) $\vec{\tau} = \langle \tau_\ell \mid h \le \ell \le i \rangle$

(iii) $\tilde{t} = \langle \tilde{t}_\ell \mid \ell \le i \rangle$ s.t. $\tilde{t} \upharpoonright (h+1) = t_h$ and $\tilde{t}_\ell \subset \omega$ for $h < \ell \le i$.

(iv) $\tilde{T}(\vec{\tau}, \tilde{t}, \sigma)$ is consistent in the infinitary language of $C^{E_h}_{\tilde{c}, \delta_h(\kappa_i)}$, where $\tilde{c} = \sup \sigma `` \gamma_i$.

(v) Set: S^ℓ_m = the set of Σ_1 fmlae φ s.t. $C^{E_h}_{\tilde{c}, \tau_\ell} \models \varphi(\sigma \upharpoonright \gamma_\ell, \tilde{t} \upharpoonright (\ell+1), e, \langle S^\ell_k \mid k < m \rangle)$ for $m \le n = |c(h,i)| - 1$.

Set: $\tilde{S}^\ell_m$ = the set of Σ_1 fmlae φ s.t. $C^{E_h}_{\tilde{c}, \delta_h(\kappa_i)} \models \varphi(\sigma \upharpoonright \gamma_\ell, \tilde{t} \upharpoonright (\ell+1), e, \langle \tilde{S}^\ell_k \mid k < m \rangle)$ for $m \le n$. Then $S^\ell_m = \tilde{S}^\ell_m$ for $m \le n$, $h \le \ell \le i$.

We call $\langle \sigma, \vec{\tau}, \tilde{t} \rangle$ satisfying Lemma 4 an <u>enlarger</u> wrt $\mathbb{E} = \mathbb{E}|(i+1)$.

It is clear that if $\langle \sigma, \vec{\tau}, \tilde{t} \rangle$ is an enlarger and we set:

$\mathbb{F}|h = \mathbb{E}|h$

$W_\ell^{\mathbb{F}} = \dot{W}_{\underline{\ell}}^{\mathfrak{A}}$, $N_\ell^{\mathbb{F}} = \dot{N}_{\underline{\ell}}^{\mathfrak{A}}$, $\delta_\ell^{\mathbb{F}} = \dot{\delta}_{\underline{\ell}}^{\mathfrak{A}}$

$(h \leq \ell \leq i)$, where $\mathfrak{A}$ is a good model of $\tilde{\mathrm{T}}(\vec{\tau}, \tilde{t}, \sigma)$, and:

$W_{i+1}^{\mathbb{F}} = W_h$, $N_{i+1}^{\mathbb{F}} = N_h$, $\delta_{i+1}^{\mathbb{F}} = \sigma$,

Then $\mathbb{F}$ is an enlargement of $\mathcal{Y}|(i+2)$ with $\tilde{t} = t_i^{\mathbb{F}}$.

We call any such $\mathbb{F}$ an <u>enlargement given by</u> $\langle \sigma, \vec{\tau}, \tilde{t} \rangle$.

It is clear, however, that being an enlarger ~~wrt.~~ $\mathbb{E}$ depends only on t_h and is expressible in $\langle W_h, N_h, \delta_h \rangle$ in t_h. For <u>any</u> $t^* = \langle T_\ell^* \mid \ell \leq h \rangle$ s.t. $T_\ell^* < \omega$ ~~and~~ for $\ell \leq h$, we say: $\langle \vec{\tau}, \tilde{t}, \sigma \rangle$ is an <u>enlarger of</u> $\mathcal{Y}|(i+1)$ <u>wrt.</u> t^* iff the above hold. Thus Lemma 4 can be

<u>Corollary 4.1</u> Let $\mathbb{E}$ be an enlargement of $\mathcal{T}|(i+1)$. Let $t^* = t_h^{\mathbb{E}}$. Then
$\langle \bar{W}_h, N_h, \delta_h \rangle \models$ There is an enlarger of $\mathcal{T}|(i+2)$ wrt. t^*.
Moreover, if $\langle \vec{\tau}, \tilde{t}, \sigma \rangle$ is such an enlarger, then it gives rise to an enlargement $\mathbb{F}$ of $\mathcal{T}|(i+2)$ with: $\mathbb{F}|h = \mathbb{E}|h$, $\tilde{t} = t_i^{\mathbb{F}}$, $\langle W_{i+1}^{\mathbb{F}}, N_{i+1}^{\mathbb{F}}, \delta_{i+1}^{\mathbb{F}} \rangle = \langle W_h^{\mathbb{E}}, N_h^{\mathbb{E}}, \sigma \rangle$.

We now apply this machinery to prove Thm 1. We proceed by induction on $\gamma < lh(\mathcal{T})$.

<u>Case 1</u> $\gamma = 0$ Then $\langle\langle W, N, \delta \rangle\rangle$ is an enlargement of $\mathcal{T}|1$.

<u>Case 2</u> $\gamma = j+1$. Let $h = T(j+1)$

<u>Case 2.1</u> h does not survive at $j+1$.

Let $\mathbb{E}$ be an enlargement of $\mathcal{T}|(j+1)$. Let $\mathbb{E}$ be an enlargement at $j+1$, then If $i \leq j$ is a breakpoint at $j+1$, then either $i = j$ or $i < j$, so by the induction hyp. we may assume $W_j^{\mathbb{E}} \cap On \leq W_i^{\mathbb{E}} \cap On$. By Lemma 2 we can then extend $\mathbb{E}$ to an enlargement $\mathbb{E}'$ of $\mathcal{T}|(j+2)$ s.t.: $W_{j+1}^{\mathbb{E}'} \cap On < W_j^{\mathbb{E}'} \cap On$.

QED (Case 2.1)

Case 2.2 h survives at $j+1$.

Let $\mathbb{E}$ be an enlargement of $\mathcal{T}|(j+1)$. At $i \leq j$ is a breakpoint at $j+1$, then $h > i$. Hence we can assume:

$$W_h^{\mathbb{E}} \cap On < W_i^{\mathbb{E}} \cap On.$$

By Lemma 3, $\mathbb{E}|h$ extends to an enlargement $\mathbb{E}'$ of $\mathcal{T}|(j+2)$ s.t. $W_{j+1}^{\mathbb{E}'} = W_h^{\mathbb{E}}$. Hence $W_{j+1}^{\mathbb{E}'} \cap On < W_i^{\mathbb{E}'} \cap On$.

QED (Case 2)

Case 3 Lim (γ)

Choose $i_0 \leq_T \gamma$ s.t. i survives at γ.
Let $\mathbb{E}$ be an enlargement of $\mathcal{T}|(i_0+1)$. At $i < \gamma$ is a breakpoint at γ, we may assume i_0 chosen large enough that $i < i_0$. Hence by the ind. hyp. we may assume:

$$W_{i_0}^{\mathbb{E}} \cap On < W_i^{\mathbb{E}} \cap On.$$

We extend $\mathbb{E}|i_0$ to an enlargement $\mathbb{E}'$ of $\mathcal{T}|\gamma+1$ s.t. $W_\gamma^{\mathbb{E}'} = W_{i_0}^{\mathbb{E}}$. We proceed as follows: For $j \leq_T i_0$ we construct enlargements $\mathbb{E}_j$ of $\mathcal{T}|(j+1)$ s.t. $\mathbb{E}_j|h = \mathbb{E}_h|h$ for $h \leq_T j$ and $W_j^{\mathbb{E}_j'} = W_{i_0}^{\mathbb{E}}$, $N_j^{\mathbb{E}_j'} = N_{i_0}^{\mathbb{E}}$

For $j = i_0$, set $\mathbb{E}_0 = \mathbb{E}$. Now let $j = k+1$, $h = T(j)$. We know that $\mathbb{E}_h$ extends to an enlargement of $\mathcal{I}|j$, since h is a breakpoint at j. By Cor 4.1, there is in $W_h^{\mathbb{E}_h}$ an enlarger $\langle \vec{t}, \tilde{t}, \sigma \rangle$ of $\mathcal{I}|(j+1)$ wrt. $t_h^{\mathbb{E}_h}$. We let $\langle \vec{t}_h, \tilde{t}_h, \sigma_h \rangle$ be the least such in the sense of $L_{\eta_0}[a^{W_h}]$, where $\eta_0 = On \cap (W_h)_0$. (We recall that $C_{\eta_0} \subset L_{\eta_0}[a^{W_h}]$, where $\langle \vec{t}, \tilde{t}, \sigma \rangle \in C_{\eta_0}$.)

Let $\mathbb{E}_j$ be an enlargement given by $\langle \vec{t}_h, \tilde{t}_h, \sigma_h \rangle$. Then $W_j^{\mathbb{E}_j} = W_h^{\mathbb{E}_h} = W_{i_0}^{\mathbb{E}}$, $N_j^{\mathbb{E}_j} = N_h^{\mathbb{E}_h} = N_{i_0}^{\mathbb{E}}$, and $\tilde{t} = t_j^{\mathbb{E}_j}$.

Now let $j = \eta$, $Lim(\eta)$. Then $\mathbb{E}' = \bigcup_{j \leq_T \eta} \mathbb{E}^j|j$ is an enlargement of $\mathcal{I}|\eta$. We extend this to $\mathbb{E}_\eta$ by setting $W_\eta^{\mathbb{E}_\eta} = W_{i_0}^{\mathbb{E}}$, $N_\eta^{\mathbb{E}_\eta} = N_{i_0}^{\mathbb{E}}$, and $\delta_\eta^{\mathbb{E}_\eta} = \sigma$, where $\sigma : P_\eta \prec N = N^{\mathbb{E}_\eta}$ is defined by: $\sigma \pi_{j\eta} = \delta_j^{\mathbb{E}_j}$ for $i_0 \leq_T j \leq_T \eta$.

It is easily verified that $\mathbb{E}_\eta$ is an enlargement, as soon as we have verified that $\sigma \in W_\eta = W_\eta^{\mathbb{E}_\eta}$. Since W_η is a ZFC* model, this will follow from: $\langle \delta_i \mid i_0 \leq_T i \leq_T \eta \rangle \in W_\eta$, where $\delta_i = \delta_i^{\mathbb{E}_i}$. But $\delta_{l+1} = \sigma_{T(l+1)}$ and δ_λ is defined canonically from $\langle \delta_l \mid i_0 \leq_T l <_T \lambda \rangle$ for $\lim(\lambda)$. Hence $\langle \delta_i \mid i_0 \leq_T i \leq_T \eta \rangle$ is recursively definable in $\langle W_\eta, N_\eta, \delta_{i_0}^{\mathbb{E}}, A_\eta \rangle$ from $t_{i_0}^{\mathbb{E}}$.

QED (Thm 1)