Appendix to §7

All theorems up to the statement of solidity (Lemma 6) go through for arbitrary premice. Lemma 6 was proven only for basic mice. The reason for assuming basicness was that, in certain key steps, this gave us unique iterability, which in turn enabled us to apply the Dodd-Jensen lemma. In the meantime, however, Neeman and Steel have proven a lemma about arbitrary mice which in many ways has the force of Dodd-Jensen, without requiring unique iterability. In the following we state and prove the Neeman-Steel lemma and then show how to prove Lemma 6 for arbitrary mice. (We recall that a mouse is defined to be an iterable premouse.)

The Neeman-Steel Lemma

<u>Lemma</u> Let N be a countable iterable mouse. Let $\langle \eta_i \mid i<\omega \rangle$ ennumerate On_N. There is an iteration strategy S s.t. whenever N' is an S-iterate of N with map π and $\sigma : N \to_{\Sigma^*} N'$, Then:

(a) N' is a simple iterate of N with simple iteration map π

(b) $\pi(\vec{\eta}) \leq \sigma(\vec{\eta})$ lexiaographically.*

<u>Cor</u> Call such strategies $\vec{\eta}$-true. Let S, S' be $\vec{\eta}$-true strategies + let N' be an S-it. of N with map π + an S'-it. of N with map π'. Then $\pi = \pi'$.

*) i.e. $\pi(\eta_i) < \sigma(\eta_i)$ if i is least s.t. $\pi(\eta_i) \neq \sigma(\eta_i)$.

<u>pf. of Lemma.</u>

Fix a strategy.

<u>Claim</u> There is an S-iterate N' of N and a map $\sigma : N \xrightarrow[\Sigma^*]{} N'$ s.t. whenever N^* is an S-iterate of N' with map π, and $\sigma^* : N \xrightarrow[\Sigma^*]{} N'$, then N^* is a simple iterate of N' and $\pi\sigma(\vec{\eta}) \leq \sigma^*(\vec{\eta})$ lexicographically.

We first note that the Claim proves the Theorem. Let $\bar{S}$ be the strategy induced by $\sigma : N \to N'$. Let $\bar{N}$ be an $\bar{S}$ iterate of N with map $\bar{\pi}$ and let $\bar{\sigma} : N \xrightarrow[\Sigma^*]{} \bar{N}$. Let $\pi : N' \to N^*$, $\sigma' : \bar{N}' \xrightarrow[\Sigma^*]{} N^*$ be the result of copying $(\sigma'\bar{\pi} = \pi\sigma)$. Then $\sigma'\bar{\sigma} : N \xrightarrow[\Sigma^*]{} N^*$. Hence N^* is a simple iterate of N'.

Hence $\bar{N}$ is a simple iterate of N. But $\pi\sigma(\vec{\eta}) \leq \sigma'\bar{\sigma}(\vec{\eta})$ lexicographic. Hence $\sigma'\bar{\pi}(\vec{\eta}) \leq \sigma'\bar{\sigma}(\vec{\eta})$ and $\bar{\pi}(\vec{\eta}) \leq \bar{\sigma}(\vec{\eta})$ lexicographically.

QED (Claim $\to$ Thm).

To obtain N, σ, we define a sequen. N_i, σ_i, π_i s.t.

(a) $\sigma_i : N \xrightarrow[\Sigma^*]{} N_i$

(b) N_0 is an S-iterate of N with map π_0

(c) N_{i+1} is an S-iterate of N_i with map π_{i+1}.

<u>Case 1</u> $i=0$. Set:

$M' R M$ iff M is S-iterable and M' is a nonsimple S-iterate of M.

R is obviously well founded. Let N_0 be R-minimal s.t. there

is $\sigma_0 : N \xrightarrow[\Sigma^*]{} N_0$. Let π_0 be the iteration map from N to N_0.
(Thus whenever N' is an S-iterate of N_0 and $\sigma : N \xrightarrow[\Sigma^*]{} N'$, then N' is a simple iterate of N_0.)

<u>Case 2</u> $i = h+1$. Set:
$\langle M', \xi' \rangle \, R \, \langle M, \xi \rangle$ iff M is S-iterable, M' is a simple S-iterate of M with map π and $\xi' < \pi(\xi)$.
Then R is well founded.
Choose $\langle N', \xi' \rangle$ R minimal s.t.
(a) N' is a simple S-it. of N_i with map
(b) There is $\sigma : N \xrightarrow[\Sigma^*]{} N'$ s.t.

$$\sigma(\eta_j) = \pi \sigma_h(\eta_j) \text{ for } j < h$$

and $\sigma(\eta_h) = \xi'$.

Set $N_i = N'$, $\sigma_i = \sigma$, $\pi_i = \pi$.

Now let $\langle \pi_{ij} \mid i \leq j < \omega \rangle$ be commutativ s.t. $\pi_{ii} = id \upharpoonright N_i$, $\pi_{i,i+1} = \pi_{i+1}$, $\pi_{ij}\pi_{hi} = \pi_{hj}$. Then $\pi_{ij} : N_i \xrightarrow[\Sigma^*]{} N_j$.

Set: $N', \langle \pi_i' \rangle = \lim\limits_{i \leq j < \omega} (N_i, \pi_{ij})$.

It is easily seen by ind. on i that:

(*) If N^* is an S-iterate of N_i with map π and $\sigma : N \xrightarrow[\Sigma^*]{} N^*$,

Then

(a) N^* is a simple it. of N_i

(b) $\pi \sigma_i (\langle \eta_j \mid j < i \rangle) \leq \sigma(\langle \eta_j \mid j < i \rangle)$

lexicographically.

Define $\sigma : N \to N'$ by: $\sigma(\eta_i) = \pi_j' \sigma_j(\eta_i)$ for $j > i$. (This is easily seen to be a definition).

Then $\sigma : N \xrightarrow[\Sigma^*]{} N'$ has the desired property by (*). QED

Note "Mouse" means "iterable premouse". We could, in fact, have proven the Neeman-Steel lemma for smoothly iterable premice, replacing "iterate" by "smooth iterate" and "iteration strategy" by "smooth iteration strategy". We could then prove Lemma 6 for smoothly iterable premice. However, this would be superfluous, since in §9 we show that smoothly iterable premice are iterable without making any use of §7.

. . .

We now modify the proof of Lemma 6 for arbitrary mice. By a Löwenheim-Skolem argument, if there is a counterexample, then there is a countable one. Hence our counterexample M of minimal length is countable. Let $\langle \eta_i \mid i<\omega \rangle$ be an enumeration of On_M. Case 1 is as before. From the failure of Case 1, however, we cannot infer that M is uniquely iterable. As before, we coiterate $\langle M, W, \alpha \rangle$, M to get $\mathcal{J}^W, \mathcal{J}^Q$, and also form the iteration $\mathcal{J}^M = \sigma(\mathcal{J}^W)$. In the iterations $\mathcal{J}^M$, $\mathcal{J}^Q$ of M, we apply the strategy given by the Neeman-Steel lemma. (This dictates the strategy for $\mathcal{J}^W$.) Dodd-Jensen was used only in cases 2.1.1, 2.1.2, 2.1.3, and 2.1.5. In all but the last, the same proof goes through literally, using Neeman-Steel in place of Dodd-Jensen. The proof (1) in 2.1.5 need a trivial reformulation: We first observe that $\pi^M_{0\theta}(\vec{\gamma}) \leq \sigma_\theta \pi^Q_{0\theta}(\vec{\gamma})$ lexicographically; hence: $\pi^{W_\theta}_{-1,\theta}(\vec{\gamma}) = \sigma_\theta^{-1} \pi^M_{0\theta}(\vec{\gamma}) \leq \pi^Q_{0,\theta}(\vec{\gamma})$ lexicographically. Similarly $\pi^Q_{0,\theta}(\vec{\gamma}) \leq \pi^W_{-1,\theta}(\vec{\gamma})$ lexicographically. QED (2.1.5)

A slight change is needed in the proof of Case 2.2.4, since we made use of the assumption that M is basic in proving the <u>Claim</u> $A \in M$. We modify the proof as follows: Let $i = 0$, $\nu_0 = \alpha$. Since $\nu_j > \alpha$ for $j > 0$, we know: $A \in M_1$. If $M_0 = M_1$, we are done. If not, then $\pi : M \xrightarrow[E_\alpha]{*} M_1$, where $\pi = \pi^Q_{01}$.

Then $A \in H =_{\text{df}} H^{M_1}_{\pi(\alpha)}$, where

(*) $\pi \upharpoonright H^M_\tau : H^M_\tau \xrightarrow[E_\alpha]{} H$ & $\tau = \kappa_0^{+M}$.

Since $\alpha = \lambda_0^{+W}$, we have $\alpha < \sigma(\alpha) = \lambda_0^{+M}$. Hence $J^{EM}_{\sigma(\alpha)} \models ZFC^-$ and $E_\alpha \in J^{EM}_{\sigma(\alpha)}$. Hence $H \in J^{EM}_{\sigma(\alpha)}$ by (*). Hence $A \in H \subset M$. QED (2.2.4)

Case 2.2.5.1 must be modified similarly.

For countable mice we can weaken the assumption of iterability to <u>countable iterability</u> (i.e. there is an iteration strategy S s.t. every countable S-iteration can be continued) if the following assumption is satisfied:

(*) $\bigwedge A \subset \omega_1 \quad A^{\#}$ exists.

All proofs will go through as long as we know:

(1) The coiteration of two countable countably iterable premice terminates after countably many steps.

(2) If N, M, Q are countable, $\langle N, M, \lambda \rangle$ is ~~good~~ and countably iterable and Q is countably iterable, then the coiteration of $\langle N, M, \lambda \rangle$ with Q will terminate after countably many steps.

We get:

<u>Fact</u> Assume (*) $\wedge A \subset \omega^+$ $A^{\#}$ exists.
Then (1), (2) hold.

proof.
It suffices to display the proof of (1)
Our earlier proof in §4 that the coiteration of two countable mice terminates in countably many steps proceeded by contradiction, assuming only that the coiteration could be continued to length $\omega_1 + 1$.
Thus it suffices only to show:

<u>Claim</u> Let S be a countable iteration strategy for a countable mouse Q.
Let $\mathcal{I} = \langle\langle Q_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be an S-iteration of length ω_1. Then $\mathcal{I}$ has a cofinal branch b (which is necessarily unique and well founded by the regularity of ω_1).

prf.
Let $A \subset \omega_1$ code $\mathcal{I}$. Let $X \prec H_\tau$ be countable, where $\tau > \omega_1$ is regular

and $A \in H_\tau$. Let $\sigma : H \xrightarrow{\sim} X$, $\sigma(\bar{A}) = A$, $\sigma(\alpha) = \alpha$. Then $\bar{A} = \alpha \cap A$, $\sigma(\mathcal{T}|\alpha) = \mathcal{T}$, $\sigma(\bar{A}^{\#}) = A^{\#}$. Call a well founded branch b in $\mathcal{T}$ <u>economical</u> iff $\sup\{\nu_i \mid i \in b\} = \sup\{\nu \mid E_\nu^{Q_b} \neq \emptyset\}$. Then $\bar{b} = \{j \mid i \leq_T \alpha\}$ is non economical, since $\nu_\alpha \geq \sup\{\nu_i \mid i \in \bar{b}\}$ where $Q_\alpha = Q_{\bar{b}}$, $E_{\nu_\alpha}^{Q_\alpha} \neq \emptyset$. Since Q is basic, it follows by §6 that $\bar{b}$ is the unique cofinal non economical branch in $\mathcal{T}|\alpha$.

Let $\mathrm{rn}(Q_{\bar{b}}) < \beta < \omega_1$ where β is admissible in $\bar{A}$. Let $\mathcal{L}$ be the infinitary language over $L_\beta[\bar{A}]$ with:

<u>Constants</u> $\underline{x}$ $(x \in L_\beta[\bar{A}])$, $\mathring{Q}$, $\mathring{b}$.

<u>Axioms</u>: ZF^-, $\bigwedge v\,(v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z})$ for $x \in L_\beta[\bar{A}]$, $\mathring{b}$ is a cofinal non-economical branch in $\underline{\mathcal{T}|\alpha}$, $\mathring{Q} = Q_{\mathring{b}}$, $\mathrm{rn}(\mathring{Q}) < \underline{\gamma}$, $\mathring{Q}$ is transitive

(where $\mathrm{rn}(Q_{\bar{b}}) < \gamma < \beta$).

$\mathcal{L}$ is consistent, since it has the model $\langle L_{\omega_1}[\bar{A}], Q_{\bar{b}}, \bar{b}\rangle$. But if $\mathfrak{A}$ is any model of $\mathcal{L}$ (s.t. (w.l.o.g.) $\underline{x}^{\mathfrak{A}} = x$ for $x \in L_\beta[\bar{A}]$), Then $\mathring{b}^{\mathfrak{U}}$ is a well founded non economical branch through $\mathcal{J}|\alpha$. Hence $\mathring{b}^{\mathfrak{U}} = \bar{b}$. By the completeness theorem for admissible sets we conclude:

$$i \in \bar{b} \longleftrightarrow \mathcal{L} \vdash \underline{i} \in \mathring{b}.$$

Hence $\bar{b} \in L[\bar{A}]$. But then $\sigma \upharpoonright L_{On \wedge \bar{H}}[\bar{A}]$ extends to $\tilde{\sigma} : L[\bar{A}] \prec L[A]$, using the indiscernables given by $\bar{A}^{\#}, A^{\#}$. Hence $b = \tilde{\sigma}(\bar{b})$ is a cofinal well founded branch in $\mathcal{J}$. QED(Fact)