§ 3 Extendability

Def Let M be acceptable. Let F be an extender on M at κ, ν. M is *$*$-extendable by F* iff there are π, N s.t. $\pi : M \longrightarrow^*_F N$. (We also write "extendable" to mean "$*$-extendable".) We call M *Σ_0-extendable by F* iff there are π, N s.t. $\pi : M \longrightarrow_F N$.

Def Let $\bar{M}, M$ be acceptable. Let $\bar{F}, F$ be extenders on $\bar{M}, M$ resp. Let $\bar{F}$ be at $\bar{\kappa}, \bar{\nu}$ and F at κ, ν.

$\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$ means:

(a) $\pi : \bar{M} \longrightarrow_{\Sigma_0} M$ and $\pi(\bar{\kappa}) = \kappa$

(b) $g : \bar{\nu} \longrightarrow \nu$

(c) Let $\pi(\bar{X}) = X$, $\alpha_1, \ldots, \alpha_n < \bar{\nu}$, $\beta_i = g(\alpha_i)$.

Then $\vec{\alpha} \in \bar{F}(\bar{X}) \longleftrightarrow \vec{\beta} \in F(X)$.

<u>Lemma 1</u> Let $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$ (and $\pi : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ for all n s.t. $\omega\rho^n_{\bar{M}} > \bar{\kappa}$. Let M be extendable by F. Then $\bar{M}$ is extendab[le] by $\bar{F}$. Moreover, if $\bar{\sigma} : \bar{M} \longrightarrow^*_{\bar{F}} \bar{N}$ and $\sigma : M \longrightarrow^*_{F} N$, then there is a unique π' s.t. $\pi' : \bar{N} \longrightarrow_{\Sigma_0^{(n)}} N$ (for all n s.t. $\omega\rho^n_{\bar{M}} > \bar{\kappa}$), $\pi'\bar{\sigma} = \sigma\pi$ and $\pi' \upharpoonright \bar{\nu} = g$. π' is defined by:

$$\pi'(\bar{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha)) \quad \text{for } \alpha < \bar{\nu}$$

and $f \in \Gamma(\bar{\kappa}, \bar{M})$.

proof.

We first note that if $\pi' : \bar{N} \longrightarrow_{\Sigma_0^{(n)}} N$ s.t. $\pi'\bar{\sigma} = \sigma\pi$, $\pi' \upharpoonright \bar{\nu} = g$, then π' does satisfy the above defining condition + is therefore unique.

We now show the existence of $\bar{N}, \bar{\sigma}$.

Let $\mathbb{D} = \mathbb{D}^*(\bar{M}, \bar{F})$. For Σ_0 fmlae φ we then have:

$\mathbb{D} \models \varphi(\langle\alpha_1, f_1\rangle, \ldots, \langle\alpha_n, f_n\rangle) \longleftrightarrow$

$\longleftrightarrow \langle\alpha_1, \ldots, \alpha_n\rangle \in \bar{F}(\{\vec{\xi} \mid \bar{M} \models \varphi(\vec{f}(\vec{\xi}))\})$

$\longleftrightarrow \langle g(\alpha_1), \ldots, g(\alpha_n)\rangle \in F(\{\vec{\xi} \mid M \models \varphi(\pi(\vec{f})(\vec{\xi}))\})$

$\longleftrightarrow N \models \varphi(\sigma\pi(f_1)(g_1(\alpha_1)), \ldots, \sigma\pi(f_n)(g(\alpha_n)))$.

In particular: $\langle\alpha, f\rangle \in \langle\alpha', f'\rangle$ in $\mathbb{D}$ iff $\sigma\pi(f)(g(\alpha)) \in \sigma\pi(f')(g(\alpha'))$.

Hence $\mathbb{D}$ is well founded and $\bar{\sigma} : \bar{M} \xrightarrow[\bar{F}]{*} \bar{N}$ exists. Now let φ be $\Sigma_0^{(n)}$, where $\omega\rho_{\bar{M}}^n > \kappa$. Let $\langle\alpha_1, f_1\rangle, \ldots, \langle\alpha_n, f_n\rangle \in \mathbb{D}$. Then $[\alpha_i, f_i] = \bar{\sigma}(f_i)(\alpha_i)$ and:

$\bar{N} \models \varphi(\bar{\sigma}(\vec{f})(\vec{\alpha})) \longleftrightarrow$

$\longleftrightarrow \vec{\alpha} \in \bar{F}(\{\vec{\xi} \mid \bar{M} \models \varphi(\vec{f}(\vec{\xi}))\})$

$\longleftrightarrow g(\vec{\alpha}) \in F(\{\xi \mid M \models \varphi(\pi(\vec{f}), \xi)\})$

$\longleftrightarrow N \models \varphi(\sigma\pi(\vec{f})(g(\vec{\alpha})))$.

This proves the existence of π!

QED (Lemma 1

Def $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow^* \langle M, F \rangle$ iff

(a) $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$

(b) Let $\bar{\alpha} <$ length$(\bar{F})$, $\alpha = g(\bar{\alpha})$. Then $\bar{F}_{\bar{\alpha}}$ is $\Sigma_1(\bar{M})$ in a parameter $\bar{p}$ and F_α is " " $p = \pi(\bar{p})$ by the same definition. (Hence $\bar{F}$ is Σ_1-amenable wrt. $\bar{M}$

Lemma 2 Assume:

(a) $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow^* \langle M, F \rangle$

(b) $\pi : \bar{M} \longrightarrow_{\Sigma^*} M$

(c) $\bar{F}, F$ are weakly amenable

(d) F is Σ_1 amenable wrt M,

Then $\pi' : \bar{N} \longrightarrow_{\Sigma^*} N$, where $\bar{N}, N, \pi'$ are as above.

The proof stretches over several sublemmas.

Note (d) is unnecessary.

This doesn't use: F is Σ_1-amenable w.r.t. M

<u>Lemma 2.1</u> Assume (a) and $\pi : \bar{M} \xrightarrow[\Sigma_0^{(n)}]{} M$, where $\omega\rho_{\bar{M}}^{n+1} \le \bar{u} < \omega\rho_{\bar{M}}^{n}$, $\omega\rho_{M}^{n+1} \le u < \omega\rho_{M}^{n}$, $\bar{F}$ is at $\bar{u}, \bar{v}$ and F is at u, v. Let $\bar{R}(\vec{z}, x)$ be $\Sigma_1^{(n)}(\bar{N})$ and $R(\vec{z}, x)$ be $\Sigma_1^{(n)}(N)$ by the same definition. Let $\bar{x} \in \bar{N}$, $x = \pi'(\bar{x})$. Set: $\bar{P} = \{\vec{z} \mid \bar{R}(\bar{\sigma}(\vec{z}), \bar{x})\}$, $P = \{\vec{z} \mid R(\sigma(\vec{z}), x)\}$. There is $\bar{q} \in \bar{M}$ s.t. $\bar{P}$ is $\Sigma_1^{(n)}(\bar{M})$ in $\bar{q}$ and P is $\Sigma_1^{(n)}(M)$ in $q = \pi(\bar{q})$ by the same definition.

proof.

Let $\bar{x} = \bar{\sigma}(\bar{f})(\bar{\alpha}) = [\bar{\alpha}, \bar{f}]$. Then $x = \pi'(\bar{x}) = \sigma(f)(\alpha)$, where $\alpha = g(\bar{\alpha})$, $f = \pi(\bar{f})$. Pick $\bar{r} \in \bar{M}$ s.t. $\bar{F}_{\bar{\alpha}}$ is $\Sigma_1(\bar{M})$ in $\bar{r}$ and F_α is $\Sigma_1(M)$ in $r = \pi(\bar{r})$ by the same def. If $\bar{f} \in \bar{M}$, set $\bar{p} = \bar{f}$. Otherwise let $\bar{f}$ be a good $\Sigma_1^{(n-1)}(\bar{M})$ fcn by a functionally absolute definition in the parameter $\bar{p}$. By §2 Lemma 7 we then have: $\bar{P}$ is $\Sigma_1^{(n)}(\bar{M})$ in $\langle \bar{r}, \bar{p} \rangle$ uniformly in the def. of $\bar{R}$, the def. of $\bar{F}_{\bar{\alpha}}$ from $\bar{r}$ and the def. of $\bar{f}$ from $\bar{p}$. But then the same thing must be true of P — i.e. P is $\Sigma^{(n)}(M)$ in $\langle r, p \rangle$ uniformly in the def. of R, the def. of F_α from r, and the def. of $f = \pi(\bar{f})$ from p. Since these definitions are unchanged, P is $\Sigma_1^{(n)}(M)$ in $\langle r, p\rangle$ by the same definition. QED (Lemma 2.1)

Recall now that $R(\vec{z}^m, \vec{x}^{m-1}, \dots, \vec{x}^0)$ is $\Sigma_h^{(n)}$ iff $R_{\vec{x}} = \{\vec{z}^m \mid R(\vec{z}^m, \dots, \vec{x}^0)\}$ is uniformly $\Sigma_h(\langle H^m, Q^1_{\vec{x}}, \dots, Q^q_{\vec{x}}\rangle)$, where $Q^i_{\vec{x}} = \{\vec{w}^m \mid Q^i(\vec{w}^m, \vec{x})\}$ and Q^i is $\Sigma_1^{(m-1)}$ for $i = 1, \dots, q$. Using this we get:

Doesn't need Σ_1-amenability of $\bar{F}$

<u>Lemma 2.2</u> Assume that the assumptions of Lemma 2.1 hold, that $\bar{F}, F$ are weakly amenable, and that F is Σ_1-amenable wrt. M. Let $m \geq n$, $h \geq 0$, let $\bar{R}(\vec{z}^m, \vec{x})$ be $\Sigma_h^{(m)}(\bar{N})$ and let $R(\vec{z}^m, \vec{x})$ be $\Sigma_h^{(m)}(N)$ by the same def. Let $x_1, \dots, x_r \in N$. There is $\bar{p} \in M$ s.t. $R_{\vec{x}}$ is $\Sigma_h^{(m)}(\bar{M})$ in $\bar{p}$ and $R_{\pi(\vec{x})}$ is $\Sigma_h^{(m)}(M)$ in $p = \pi(\bar{p})$ by the same definition.

pf. of Lemma 2.2. (Ind. on m).
$\overline{R}_{\vec{x}}$ is uniformly $\Sigma_n(H^m_{\bar{N}}, \overline{Q}_{\vec{x}})$ where $\overline{Q}$ is $\Sigma_1^{(m-1)}(\bar{N})$. Moreover $R_{\vec{x}}$ is uniformly $\Sigma_n(H^m_N, Q_{\vec{x}})$ by the same def., where Q is $\Sigma_1^{(m-1)}(N)$ by the same def. Since $H^m_N = H^m_M$ and $H^m_{\bar{N}} = H^m_{\bar{M}}$, it suffices to note that there is $\bar{p} \in \bar{M}$ s.t. $\overline{Q}_{\vec{x}}$ is $\Sigma_1^{(m-1)}(\bar{M})$ in $\bar{p}$ and $Q_{\pi'(\vec{x})}$ is $\Sigma_1^{(m-1)}(M)$ in $p = \pi(\bar{p})$ by the same definition. If $m = n+1$ this follows by Lemma 2.1 and $\bar{\sigma} \upharpoonright H^m_{\bar{M}} = \mathrm{id}$, $\sigma \upharpoonright H^m_M = \mathrm{id}$. If $m > n+1$, it follows by the induction hypothesis. QED (Lemma 2.2)

We can now prove Lemma 2. If $\omega\rho^\omega_{\bar{M}} > \bar{\kappa}$ the result is immediate by Lemma 1, so let $\omega\rho^{n+1}_{\bar{M}} \le \bar{\kappa} < \omega\rho^n_{\bar{M}}$. Then $\omega\rho^{n+1}_M \le \kappa < \omega\rho^n_M$, since π is $\Sigma_1^{(n+1)}$-preserving and $\pi(\bar{\kappa}) = \kappa$.

<u>Claim</u> $\pi' : \bar{N} \longrightarrow_{\Sigma_1^{(m)}} N$ für $m \geq n$.
Let $\bar{R}(x_1, \ldots, x_r)$ be $\Sigma_1^{(m)}(\bar{N})$ and $R(x_1, \ldots, x_r)$ be $\Sigma_1^{(m)}(N)$ by the same definition. Fix $\bar{x}_1, \ldots, \bar{x}_r \in \bar{N}$ and let $x_i = \pi'(\bar{x}_i)$ $(i = 1, \ldots, r)$. By Lemma 2.1 (if $n = m$) or Lemma 2.2 (if $n < m$) $\bar{R}(\bar{x}_1, \ldots, \bar{x}_r)$ is expressible by a $\Sigma_1^{(m)}(\bar{M})$ condition on a parameter $\bar{p}$ and $R(x_1, \ldots, x_r)$ by the same $\Sigma_1^{(m)}(M)$ condition on $p = \pi(\bar{p})$. Since π is $\Sigma_1^{(m)}$-preserving, we get:

$$\bar{R}(\bar{x}_1, \ldots, \bar{x}_r) \longleftrightarrow R(x_1, \ldots, x_r).$$

QED (Lemma 2)

We now note some obvious corollaries of the proof:

<u>Cor 2.3</u> Let the assumptions of Lemma 2.1 hold and assume $\pi : \bar{M} \longrightarrow_{\Sigma_1^{(n)}} M$. Then $\pi' : \bar{N} \longrightarrow_{\Sigma_1^{(n)}} N$.

<u>Cor 2.4</u> Let the assumptions of Lemma 2.2 hold and assume $\pi : \bar{M} \longrightarrow_{\Sigma_h^{(m)}} M$. Then $\pi' : \bar{N} \longrightarrow_{\Sigma_h^{(m)}} N$.

Recall that we called $\pi : \bar{M} \to M$ <u>strong</u> $\Sigma_\ell^{(m)}$-preserving iff π is $\Sigma_\ell^{(m)}$-preserving and $\pi``H^m_{\bar{M}} = rng(\pi) \cap H^m_M$ (or equivalently: $\pi^{-1}\omega\rho^m_M \subset \omega\rho^m_{\bar{M}}$). Strongness holds automatically for $\ell \geq 1$ but may fail for $\ell = 0$. If $\omega\rho^{n+1}_{\bar{M}} \leq \bar{\kappa} < \omega\rho^n_{\bar{M}}$, $\pi(\bar{\kappa}) = \kappa$ + π is strongly $\Sigma_0^{(n+1)}$-preserving, then $\omega\rho^{n+1}_M \leq \kappa < \omega\rho^n_M$. Moreover, if π, π' are as in Cor 2.4 + π is strongly $\Sigma_h^{(m)}$-preserving, then so is π', since $H^m_M = H^m_N$ and $\pi' \restriction H^m_{\bar{N}} = \pi \restriction H^m_{\bar{M}}$. Hence:

<u>Cor 2.5</u> Assume:

(a) $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow^* \langle M, F \rangle$

(b) $\bar{F}, F$ are weakly amenable

(c) F is Σ_1 amenable wrt. M

(d) $\pi : \bar{M} \longrightarrow_{\Sigma_h^{(m)}} M$ strongly, where $\omega\rho^m_{\bar{M}} \leq \bar{\kappa} = crit(\bar{F})$ and $h < \omega$.

Then $\pi' : \bar{N} \longrightarrow_{\Sigma_h^{(m)}} N$ strongly.

We now prove some theorem on Σ_0-extendability. An analogue of Lemma 1 is:

<u>Lemma 3</u> Let $\langle\pi, g\rangle : \langle\bar{M}, \bar{F}\rangle \to \langle M, F\rangle$, $\pi : \bar{M} \to_{\Sigma_0} M$. Let M be Σ_0-extendable by F. Then $\bar{M}$ is Σ_0-extendable by $\bar{F}$. Moreover, if $\bar{\sigma} : \bar{M} \to_{\bar{F}} \bar{N}$, $\sigma : M \to_F N$, there is a unique $\pi' : \bar{N} \to_{\Sigma_0} N$ s.t. $\pi'\bar{\sigma} = \sigma\pi$ and $\pi' \upharpoonright \bar{\nu} = g$. π' is defined by: $\pi'(\bar{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$ for $\alpha < \bar{\nu}$, $f \in \bar{M}$, $f : \bar{\kappa} \to \bar{M}$.

prf.
Uniqueness is trivial as before. Let $\mathbb{D} = \mathbb{D}(\bar{M}, \bar{F})$ (the term model for the Σ_0 ultrapower). By Łoś Theorem for Σ_0 ultrapowers:

$\mathbb{D} \models \varphi(\langle\alpha_1, f_1\rangle, \ldots, \langle\alpha_n, f_n\rangle) \longleftrightarrow$

$\longleftrightarrow \langle\alpha_1, \ldots, \alpha_n\rangle \in \bar{F}(\{\vec{\xi} \mid \bar{M} \models \varphi(\vec{f}(\vec{\xi}))\})$

$\longleftrightarrow \langle g(\alpha_1), \ldots, g(\alpha_n)\rangle \in F(\{\vec{\xi} \mid M \models \varphi(\vec{f}(\vec{\xi}))\})$

$\longleftrightarrow N \models \varphi(\sigma\pi(f_1)(\alpha_1), \ldots, \sigma\pi(f_n)(\alpha_n))$,

The rest of the proof is as before.

QED (Lemma 3)

The same proof yields:

<u>Lemma 3.1</u> Let $\langle \pi, g\rangle : \langle \bar{M}, \bar{F}\rangle \to \langle M, F\rangle$, $\pi : \bar{M} \to_{\Sigma_0} M$. Let M be $*$-extendable by F. Then $\bar{M}$ is Σ_0-extendable by $\bar{F}$. Moreover, if $\bar{\sigma} : \bar{M} \to_{\bar{F}} \bar{N}$ and $\sigma : M \xrightarrow{*}_{F} N$, there is a unique $\pi' : \bar{N} \to_{\Sigma_0} N$ s.t. $\pi'\bar{\sigma} = \sigma\pi$ and $\pi' \upharpoonright \bar{\nu} = g$. π' is defined by:
$\pi'(\bar{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$ for $\alpha < \bar{\nu}$, $f \in \bar{M}$, $f : \bar{\kappa} \to \bar{M}$.

<u>Lemma 4</u> Let $\langle \pi, g\rangle : \langle \bar{M}, \bar{F}\rangle \xrightarrow{*} \langle M, F\rangle$ and $\pi : \bar{M} \to_{\Sigma_1} M$. Let $\bar{\sigma} : \bar{M} \to_{\bar{F}} \bar{N}$, $\sigma : M \to_{F} N$ and let $\pi' : \bar{N} \to N$ be as in Lemma 3. Then $\pi' : \bar{N} \to_{\Sigma_1} N$.

(<u>Note</u> This improves Lemma 3. The corresponding improvement of Lemma 3.1 is false.)

As a preliminary to proving Lemma 4 we prove the following analogue of Lemma 2.1:

<u>Lemma 4.1</u> Let $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \xrightarrow{*} \langle M, F\rangle$ and $\pi : \overline{M} \to_{\Sigma_0} M$. Let $\overline{\sigma} : \overline{M} \to_{\overline{F}} \overline{N}$, $\sigma : M \to_F N$ and let $\pi' : \overline{N} \to N$ be as in Lemma 4. Let $\overline{R}(\vec{z}, x)$ be $\Sigma_1(\overline{N})$ and $R(\vec{z}, x)$ be $\Sigma_1(N)$ by the same definition. Let $\overline{x} \in \overline{N}$, $x = \pi'(\overline{x})$. Set $\overline{P} = \{\vec{z} \mid \overline{R}(\overline{\sigma}(\vec{z}), \overline{x})\}$, $P = \{\vec{z} \mid R(\sigma(\vec{z}), x)\}$. There is $\overline{q} \in \overline{M}$ s.t. $\overline{P}$ is $\Sigma_1(\overline{M})$ in $\overline{q}$ and P is $\Sigma_1(M)$ in q by the same definition.

proof.

Let $\overline{x} = [\langle \overline{\alpha}, \overline{f}\rangle] = \overline{\sigma}(\overline{f})(\overline{\alpha})$. Then $x = \pi'(\overline{x}) = \sigma(f)(\alpha)$ where $f = \pi(\overline{f})$, $\alpha = g(\overline{\alpha})$. Pick $\overline{r} \in \overline{M}$ s.t. $\overline{F}_{\overline{\alpha}}$ is $\Sigma_1(\overline{M})$ in $\overline{r}$ and F_α is $\Sigma_1(M)$ in $r = \pi(\overline{r})$ by the same definition. By §1 Lemma 9 $\overline{P}$ is $\Sigma_1(\overline{M})$ in $\langle \overline{r}, \overline{f}\rangle$ uniformly in the def. of $\overline{R}$ and the def. of $\overline{F}_{\overline{\alpha}}$ from $\overline{r}$. Similarly for P. Hence P is $\Sigma_1(M)$ in $\langle r, f\rangle$ by the same definition.

QED (Lemma 4.1)

We now prove Lemma 4. Let $\bar{R}$ be $\Sigma_1(\bar{N})$ and R be $\Sigma_1(N)$ by the same def. Let $x_1, \ldots, x_n \in \bar{N}$. There is $\bar{p} \in \bar{M}$ s.t. $\bar{R}(x_1, \ldots, x_n)$ is expressible in $\bar{M}$ by a Σ_1 condition on $\bar{p}$ and $R(\pi'(x_1), \ldots, \pi'(x_n))$ is expressible in M by the same Σ_1 condition on $p = \pi(\bar{p})$. Since π is Σ_1 - preserving, we conclude $\bar{R}(x_1, \ldots, x_n) \longleftrightarrow R(\pi'(x_1), \ldots, \pi'(x_n))$.

QED (Lemma 4)

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We can strengthen the notion $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$ in another direction by setting:

<u>Def</u> Let $\bar{M}, M$ be acceptable. Let $\bar{F}$ be an extender on $\bar{M}$ at $\bar{\kappa}, \bar{\nu}$ + F on M at κ, ν.

$\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{\ \ }^{0} \langle M, F \rangle$ means

(a) $\pi : \bar{M} \xrightarrow[\Sigma_0]{} M$

(b) $g : \bar{\nu} \longrightarrow \nu$

(c) Let $\bar{X} = \langle X_i \mid i < \bar{\kappa} \rangle \in \bar{M}$, $X = F(X)$, $\alpha_1, \ldots, \alpha_n < \bar{\nu}$, $\beta_i = g(\alpha_i)$. Then $\{ l \mid \vec{\alpha} \in \bar{F}(\bar{X}_l) \} \in \bar{M}$ and

$$\pi(\{ l \mid \vec{\alpha} \in \bar{F}(\bar{X}_l) \}) = \{ l \mid \vec{\beta} \in F(X_l) \}.$$

(Hence $\pi(\bar{\kappa}) = \kappa$ and $\bar{F}$ is weakly amenable.)

Clearly $\longrightarrow^{0}$ implies $\longrightarrow$.