§2 Σ^*-Ultrapowers

We now consider acceptable J-models $N = \langle J_\alpha^A, B\rangle$ (i.e. J_α^A is acceptable + N is amenable). Let $\kappa < \alpha$ be an uncountable cardinal in N. Then N is suitable for κ by acceptability. (Since if $a \subset \kappa$, $a \in N$, then κ is p.r. closed and $\langle J_\kappa^A, a\rangle$ is amenable. Hence $J_\kappa^a = \bigcup_{\nu<\kappa} J_\nu^a \in \mathrm{Def}(J_\kappa^A, a) \subset J_\alpha^A$.)

Call F an *extender* *at* κ, λ *on* N iff F is an extender on $\mathcal{P}(\kappa) \cap N$ of length λ. For such F we have defined the ultrapower $\pi : N \xrightarrow[F]{} M$. Making use of fine structure, we now define the so called Σ^*-*ultrapower* $\pi : N \xrightarrow[F]{*} M$.

The intention is that π will

be $\Sigma_0^{(n)}$ - preserving whenever $\omega\rho_N^n > \kappa$. Under certain conditions (e.g. if F is weakly amenable) it will be $\Sigma_1^{(n)}$ - preserving for $\omega\rho_N^n > \kappa$. If F is both weakly amenable and Σ_1 - amenable it will in in fact be Σ^* - preserving.

In keeping with our earlier definition we first define:

Def Let N be an acceptable J-model. $\pi : N \xrightarrow[F]{*} M$ iff

(a) M is transitive

(b) $\pi : N \xrightarrow[\Sigma_0^{(n)}]{} M$ for $\omega\rho_N^n > \kappa$, where

(c) $\kappa = \mathrm{crit}(\pi)$

(d) $F = \langle \lambda \cap \pi(X) \mid X \in \mathbb{P}(\kappa) \cap M \rangle$, where $\kappa < \lambda \leq \pi(\kappa)$, λ is p.r. closed and;

(e) N = the closure of $\mathrm{rng}(\pi) \cup \lambda$ under Σ_0 fcns + under good $\Sigma_1^{(n)}$ fcns for $\omega\rho_N^{n+1} > \kappa$.

Two immediate conclusions are:

<u>Lemma 1.1</u> Let $\omega\rho_N^1 \leq \kappa$. Then $\pi : N \to_F M$ iff $\pi : N \to_F^* M$.

<u>Lemma 1.2</u> Let $\omega\rho_N^\omega > \kappa$. If $\pi : N \to_F^* M$, then $\pi : N \to_{\Sigma^*} M$.

For the simple ultrapower $\pi : N \to_F M$, we had: $M = \{\pi(f)(\nu) | f \in N \wedge f : \kappa \to N \wedge \nu < \lambda$

This is no longer true, in general, for the $*$-ultrapower, but we can formulate a similar condition

Set: $\Gamma = \Gamma(\kappa, N) =_{Df}$ the set of $f : \kappa \to N$ s.t. $f \in N$ or f is a good $\underline{\Sigma}_1^{(n)}(N)$ map, where $\omega\rho_N^{n+1} > \kappa$.

Let $f : \kappa \to N$ be a good $\underline{\Sigma}_1^{(n)}(N)$ map and let $y = F_p(x)$, $y = G_q(x)$, be functionally absolute $\Sigma_1^{(n)}$ definitions ~~wrt~~ in the parameters p, q resp. Since $\omega\rho_N^{n+1} > \kappa$, The $\Pi_0^{(n+1)}$ statements: $\alpha = \mathrm{dom}\, G_q$, $\alpha = \mathrm{dom}\, F_p$, $\bigwedge \nu < \alpha\ F_p(\nu) = G_q(\nu)$ hold in N. Hence the corresponding statements hold of $\pi(\alpha), \pi(p), \pi(q)$ in M – i.e. $y = F_{\pi(p)}(x)$, $y = G_{\pi(q)}(x)$ define the same $\Sigma_1^{(n)}(M)$ map of $\pi(\alpha)$ to M.

This means we can extend π to elements of Γ by setting:

$\pi(f)$ = that f' defined by $F_{\pi(p)}$,

where $y = F_p(x)$ is a functionally absolute good def. of f in the parameter p. Clearly then:

<u>Lemma 1.3</u> $M = \{\pi(f)(\nu) \mid f \in \Gamma, \nu < \lambda\}$.

We also have Łoś Theorem:

<u>Lemma 1.4</u> $M \models \varphi(\pi(f_1)(\nu_1), \ldots, \pi(f_m)(\nu_m)) \longleftrightarrow$
$\longleftrightarrow \vec{\nu} \in F(\{\vec{\xi} \mid N \models \varphi(f_1(\xi_1), \ldots, f_m(\xi_m))\})$.

if φ is $\Sigma_0^{(n)}$ for $\omega\rho_N^n > \alpha$.

Thus, if $\pi : N \xrightarrow[F]{*} M$, then π, M are uniquely determined by the extender F. The question, whether to a given F such π, M exist, can be answered as before by an "ultra-power construction". It will again turn out that π, M exist iff a certai "term model" $\mathbb{D}$ is well founded.

Let $N = \langle J_\alpha^A, B \rangle$ be an acceptable J-model. Let $\kappa < \alpha$ and let F be an extender at κ, λ on N. Define a term model $\mathbb{D} = \mathbb{D}^*(N, F)$ by: $\mathbb{D} = \langle D, \cong, \tilde{\in}, \tilde{A}, \tilde{B} \rangle$, where:

$D = \{\langle \alpha, f \rangle \mid f \in \Gamma, \alpha < \lambda\}$, where

$\Gamma = \Gamma(\kappa, N)$.

$\langle \alpha, f \rangle \cong \langle \beta, g \rangle \longleftrightarrow \langle \alpha, \beta \rangle \in F(\{\langle \xi, \zeta \rangle \mid f(\xi) = g(\xi$

$\quad \tilde{\in} \qquad$ " $\qquad$ " $\qquad \in$ "

$\tilde{A} \langle \alpha, f \rangle \longleftrightarrow \alpha \in F(\{\xi \mid A f(\xi)\})$

$\tilde{B} \quad$ " $\qquad$ " $\quad B$ "

Łoś Thm for Σ_0 fmlae reads:

__Lemma 2.1__ Let $\langle \alpha_i, f_i \rangle \in D$ $(i = 1, \dots, n)$,

$$\mathbb{D} \models \varphi(\langle \alpha_1, f_1 \rangle, \dots, \langle \alpha_n, f_n \rangle) \longleftrightarrow$$

$$\longleftrightarrow \vec{\alpha} \in F(\{\vec{\xi} \mid N \models \varphi(f_1(\xi_1), \dots, f_n(\xi_n))\}),$$

if φ is Σ_0.

The proof is by induction on φ making use of the following:

__Lemma 2.2__ Let $R(y^m, x^{j_1}, \dots, x^{j_r})$ be $\Sigma_1^{(m)}(N)$, where $\omega\rho_N^m > \kappa$ and $j_1, \dots, j_r \leq m$. Let $n \geq m$ s.t. $\omega\rho_N^{n+1} > \kappa$. Let $f_1, \dots, f_r$ be good $\underline{\Sigma}_1^n(N)$ maps, where f_i is into H^{j_i} and $\mathrm{dom}(f_i) = \kappa$. There is a good $\underline{\Sigma}_1^{(m)}(N)$ map $g \in \Gamma$ s.t.

$$\bigvee y^m R(y^m, f_1(\xi_1), \dots, f_r(\xi_r)) \longleftrightarrow$$

$$\longleftrightarrow R(g(\langle \xi_1, \dots, \xi_r \rangle), f_1(\xi_1), \dots, f_r(\xi_r))$$

for all $\xi_1, \dots, \xi_r < \kappa$.

(__Note__: $m = j_1 = \dots = j_r = 0$ is the case used in the proof of Lemma 2.1.)

proof of Lemma 2.2

By [MO] §1.2 Lemma 5.3 there is a $\Sigma_1^{(m)}$ fcn F to H^m s.t.

$$\bigvee y^m R(y^m, x_1, \ldots, x_n) \longleftrightarrow$$
$$\longleftrightarrow R(F(x_1, \ldots, x_n), x_1, \ldots, x_n).$$

Set $G(\nu) \simeq F(f_1((\nu)^0_n), \ldots, f_m((\nu)^{n-1}_n))$, where $\nu = \langle (\nu)^0_n, \ldots, (\nu)^{n-1}_n \rangle$. Then G is a good $\underline{\Sigma}_1^{(m)}$ fcn to H^m and $\mathrm{dom}(G) \subset \kappa$. Hence $\mathrm{dom}(G) \in N$, since $\omega\rho_N^{m+1} > \kappa$. Let $d = \mathrm{dom}(G)$ and set: $g(\nu) = \begin{cases} G(\nu) & \text{if } \nu \in d \\ 0 & \text{if } \nu \notin d \end{cases}$

for $\nu < \kappa$. Then $g : \kappa \to H^m$ is a good $\underline{\Sigma}_1^{(m)}$ function. [To see this set $F'(x_1, \ldots, x_n, z) \simeq \begin{cases} F(x_1, \ldots, x_n) & \text{if } z \in d \\ 0 & \text{if not.} \end{cases}$

Then F' is $\underline{\Sigma}_1^{(m)}$ to H^m and $g(\nu) \simeq F'(f_1((\nu)^0_n), \ldots, f_m((\nu)^{n-1}_n), \nu)$.]
The conclusion is immediate.

QED (Lemma 2.2)

It is obvious that if $\pi : N \xrightarrow[F]{*} M$, then $\langle \alpha, f \rangle \tilde{\in} \langle \beta, g \rangle \longleftrightarrow \pi(f)(\alpha) \in \pi(g)(\beta)$ for $\langle \alpha, f \rangle, \langle \beta, g \rangle \in D$ and that the map $\langle \alpha, f \rangle \longmapsto \pi(f)(\alpha)$ is onto M. Thus:

Lemma 2.3 If $\tilde{\in}$ is not well founded, then there are no π, M s.t. $\pi : N \xrightarrow[F]{*} M$.

From now on assume that $\tilde{\in}$ is well founded. We shall show that π, M exist. By Lemma 2.1 $\tilde{=}$ is an equality relation for $\mathbb{D}$ and satisfies extensionality. It follows that there is a structure preserving map $[\ ] : \mathbb{D} \xrightarrow{\sim} M$ onto a transitive M s.t.

$$\begin{array}{ccc} [x] \in [y] & \longleftrightarrow & x \tilde{\in} y \\ = & & \tilde{=} \end{array}.$$

Define $\pi : N \longrightarrow M$ by: $\pi(x) = [\langle 0, \mathrm{const}_x$ where const_x is the constant fcn x on κ. By Łoś Thm we conclude: $\pi : N \longrightarrow_{\Sigma_0} M$.

<u>Lemma 2.4</u> Let $\bar{H} = H^n_N$, $H = \bigcup \pi''\bar{H}$, where $\rho^n_N = \min\{\rho^n_N \mid \omega\rho^n_N > \kappa\}$.

Then $\pi \upharpoonright \bar{H} : \bar{H} \longrightarrow_F H$

proof.

By the def. of Γ, whenever $f \in \Gamma$ and $\mathrm{rng}(f) \subset u \in \bar{H}$, then $f \in H$.

Hence $H = \{[\langle \alpha, g\rangle] \mid g \in \bar{H}, g : \kappa \to \bar{H}, \alpha < \lambda$

The conclusion follows easily.

QED (2.4)

In particular it follows that $\kappa = \mathrm{crit}(\pi)$, $[\langle \alpha, f\rangle] = \pi(f)(\alpha)$ for $f \in \bar{H}$, $f : \kappa \to \bar{H}$, $\alpha < \lambda$.

We now prove:

Lemma 3

(a) M is an acceptable J-model

(b) $\pi : N \xrightarrow{}_{\Sigma_0^{(n)}} M$ if $\omega\rho_N^n > \kappa$

(c) $\pi : N \xrightarrow{}_{\Sigma_2^{(n)}} M$ if $\omega\rho_N^{n+1} > \kappa$

(d) Let φ be $\Sigma_0^{(n)}$ for an n s.t. $\omega\rho_N^n > \kappa$ or $\Sigma_1^{(n)}$ for an n s.t. $\omega\rho_N^{n+1} > \kappa$. Then:

$$M \models \varphi([\alpha_1, f_1], \ldots, [\alpha_m, f_m]) \longleftrightarrow$$
$$\longleftrightarrow \vec{\alpha} \in F(\{\vec{\xi} \mid N \models \varphi(f_1(\xi_1), \ldots, f_m(\xi_m))\}).$$

The proof stretches over several sublemmas. We first verify (b)-(d) in the sense of a "pseudo interpretation" of the $\Sigma_h^{(n)}$ fmlae in M. Using this we verify (a). We then show that our pseudo interpretation is sufficiently correct for (b)-(d) to hold.

For $\omega p^n_N > \kappa$ set:

$$\Gamma_n = \begin{cases} \{f\in\Gamma \mid rng(f)\subset H^n_N\} & \text{if } \omega p^{n+1}_N > \kappa \\ \{f\in\Gamma \mid rng(f)\in H^n_N\} & \text{if} \\ \text{if } \omega p^{n+1} \le \kappa < \omega p^n \text{ in } N \end{cases}$$

<u>Remark</u> $\Gamma_n = \{f\in\Gamma \mid f\in H^n_N\}$ if $\omega p^{n+1} \le \kappa < \omega p$ in N.

<u>Remark</u> By the remarks following the proof of [MO] §1.2 Lemma 5.1, Γ_n may (but need not) be regarded as a set of fcns <u>to</u> H^n - i.e. defined by $\varphi(y^n, x$

Set: $H_n = \{[\langle\alpha, f\rangle] \mid \langle\alpha, f\rangle \in D \wedge f\in\Gamma_n\}$. It is obvious that:

(1) H_n is transitive.

We say that $M \models \varphi(\vec{x})$ in the <u>pseud</u> <u>interpretation</u> iff φ holds when v^i is taken as ranging over H_i for $\omega p^i_N > \kappa$. We then get a pseudo Łoś Theorem:

<u>Lemma 3.1</u> Let φ be a $\Sigma_0^{(n)}$ fmla for an n s.t. $\omega\rho_N^n > \kappa$ or a $\Sigma_1^{(n)}$ fmla for an n s.t. $\omega\rho_N^{n+1} > \kappa$. In the sense of the pseudo interpretation we have:

$$M \models \varphi([\alpha_1, f_1], \ldots, [\alpha_n, f_n]) \longleftrightarrow$$
$$\longleftrightarrow \vec{\alpha} \in F(\{\vec{\xi} \mid N \models \varphi(f_1(\alpha_1), \ldots, f_n(\alpha_n))\}).$$

proof. By ind. on n and for given n by ind. on φ using Lemma 2.2.

QED (Lemma 3.1)

<u>Cor 3.2</u> $\pi : N \longrightarrow_{\Sigma_0^{(n)}} M$ for $\omega\rho_N^n > \kappa$ in the pseudo interpretation.

<u>Cor 3.3</u> $\pi : N \longrightarrow_{\Sigma_2^{(n)}} M$ for $\omega\rho_N^{n+1} > \kappa$,

in the pseudo interpretation.

pf.

Let $M \models \bigvee x^n \varphi(x^n, \pi(\vec{z}))$, where φ is $\Pi_1^{(n}$

Let $\langle \alpha, g \rangle \in D$, $g \in \Gamma_n$ s.t. $M \models \varphi([\alpha, g], \pi(\vec{z}))$;

By Lemma 3.1, $\alpha \in F(\{\xi \mid N \models \varphi(g(\xi), \vec{z})\}$

Hence $\bigvee_\xi \ N \models \varphi(g(\xi), \vec{z})$. Hence $N \models \bigvee x^n \varphi(x^n, \vec{z})$.

QED (3.3

By Lemma 2.1, Cor 3.3 we have:

Cor 3.4 M is an acceptable J-model.

proof.

If $\omega\rho_N^1 \le \kappa$, then $\pi : N \xrightarrow[\Sigma_0]{} M$ cofinally by Lemma 2.4. If $\omega\rho_N^1 > \kappa$, then $\pi : N \xrightarrow[\Sigma_2^{(0)}]{} M$ in the pseudo interpretation, hence $\pi : N \xrightarrow[\Sigma_2]{} M$ since $H_0 = M$.

QED (3.4)

Set $\omega\rho_n = On \cap H_n$. . .

Cor 3.5 Let $M = \langle J_{\alpha'}^{A'}, B' \rangle$. Then $H_n = J_{\rho_n}^{A'}$.

proof.

If $\omega\rho_N^{n+1} > \kappa$, we have:

$$\pi \upharpoonright H_n : J^{A}_{\rho_N^n} \xrightarrow[\Sigma_2]{} \langle H_n, A' \cap H_n \rangle$$

If $\omega\rho^{n+1} \le \kappa < \omega\rho^n$ in N, then by Lemma 2.4 we have.

$\pi \upharpoonright H_n : H_N^n \xrightarrow[F]{} H_n$ + hence

$\pi \upharpoonright H_n : J^A_{\rho_N^n} \xrightarrow[\Sigma_0]{} \langle H_n, A' \cap H_n \rangle$ cofinally,

QED (Cor 3.5)

Thus it remains only to prove:

Lemma 3.5 $\rho_n = \rho^n_M$ for $\omega\rho^{n+1} > \kappa$ and $\rho_n \leq \rho^n_M$ for $\omega\rho^n > \kappa$.

proof. By ind. on n.

$n = 0$ is immediate, so assume $n > 0$.

We first show: $\rho_n \leq \rho^n_M$. Let $A \subset \omega\rho^n$ be $\Sigma_1^{(n-1)}(M)$. It suffices to show:

Claim $\langle H_n, A \rangle$ is amenable.

Let $z \in H_n$. **Claim** $z \cap A \in H_n$.

Let $Ax \longleftrightarrow A'(x, [\alpha, f])$ where A' is $\Sigma_1^{(n-1)}(M)$. Let $\bar{A}'$ be $\Sigma_1^{(n-1)}(N)$ by the same definition. Let $z = [\beta, g]$ where $g \in \Gamma_n$. Define $k : \kappa \to H_n$ by:

$k(\xi) = g((\xi)_1) \cap \{x \mid \bar{A}'(x, f((\xi)_0))\}$.

Then $k \in \Gamma_n$. Set $w = [\langle \alpha, \beta \rangle, k]$.

Then $w \in H_n$ and by Łos Thm (Lemma 2.1): $w = z \cap A$.

[To see this note that for $\xi_0, \xi_1, \xi_2 < \kappa$:

$\xi_0 = \langle \xi_1, \xi_2 \rangle \to h(\xi_0) = g(\xi_2) \cap \{x \mid \bar{A}'(x, f(\xi_1))\}$.

Hence $\langle \langle \alpha, \beta \rangle, \alpha, \beta \rangle \in \{\vec{\xi} \in \lambda^3 \mid \xi_0 = \langle \xi_1, \xi_2 \rangle\} =$

$= F(\{\vec{\xi} \in \kappa^3 \mid \xi_0 = \langle \xi_0, \xi_2 \rangle\} \subset$

$\subset F(\{\vec{\xi} \in \kappa^3 \mid h(\xi_0) = g(\xi_2) \cap \{z \mid \bar{A}(z, f(\xi_1)\})$.]

QED ($\omega\rho^n \leq \omega\rho^n_M$).

We now show $\omega\rho^n_M \leq \omega\rho_m$ if $\omega\rho^{n+1}_N > \kappa$. Let $\bar{A}$ be $\Sigma_1^{(n-1)}(N)$ in p and let A be $\Sigma_1^{(n-1)}(M)$ in $\pi(p)$ by the same definition. Assume $\bar{A} \cap \omega\rho^n_N \notin N$

Claim $A \cap \omega\rho_m \notin M$

Suppose not. Let $A \cap \omega\rho_m = [\alpha, f]$. The statement: $A \cap \omega\rho_m = x$ is expressed by $\bigwedge z^n (z^n \in A \leftrightarrow z^n \in x)$ which is Π_1^n in $\pi(p)$ and x. Hence by Łos Thm: $\alpha \in F(\{\xi \mid \bar{A} \cap \omega\rho^n_N = f(\xi) \cap \omega\rho^n_N\})$. Hence $\bar{A} \cap \omega\rho^n_N = f(\xi) \cap \omega\rho^n_N \in N$ for some ξ.

Contra! QED (Lemma 3)

As a corollary of the proof of $\omega\rho^n_M \leqq \omega\rho_n$ for $\omega\rho^{n+1}_N > \kappa$ we note:

<u>Corollary 3.6</u> Let $\omega\rho^{n+1}_N > \kappa$. Then $\pi '' P^n_N \subset P^n_M$.

<u>Corollary 3.7</u> $\pi : N \xrightarrow[F]{*} M$.

proof.
(a) — (d) in the def. of $\pi : N \xrightarrow[F]{*} M$ are satisfied. We prove (e). By (a) - (d) we know that $\pi(f)$ is defined for $f \in \Gamma$. By Lemma 2.4 we have $[\alpha, id] = \alpha$ for $\alpha < \lambda$.
By Łoś Thm:

$$\mathbb{D} \models \langle \alpha, f \rangle = f(\langle \alpha, id \rangle)$$

for $\langle \alpha, f \rangle \in \mathbb{D}$. Hence $[\alpha, f] = \pi(f)(\alpha$
Hence $M = \{\pi(f)(\alpha) \mid \langle \alpha, f \rangle \in D\}$.

QED (3.7)

((Cor 3.8. $[\langle \alpha, f \rangle] = \pi(f)(\alpha)$), We now show that the preservation properties of π can be improved if we make stronger assumptions on N.

Lemma 4.1 Let F be weakly amenable Let $\omega\rho_N^{n+1} \leq \kappa < \omega\rho_N^n$. Then

$\pi : N \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally.

(Recall "$\pi : N \longrightarrow_{\Sigma_i^{(n)}} M$ cofinally means that π is $\Sigma_i^{(n)}$ preserving and $\omega\rho_M^n = \sup \pi '' \omega\rho_N^n$.)

proof.
$n = 0$ is immediate. Assume $n > 0$.

Claim 1 There is $B \subset \omega\rho_M$ s.t. B is $\underline{\Sigma}_1^{(n)}(M)$ in the pseudo interpretation and $B \cap \omega\rho_N^{n+1} \notin M$.

prf.
Let $\bar{B}$ be $\Sigma_1^{(n)}(N)$ s.t. $\bar{B} \cap \omega\rho_N^{n+1} \notin M$.

Let $\bar{B}$ be $\Sigma_1^{(n)}(N)$ in $\bar{p}$ + let $B \subset \omega\rho_n$ have the same $\Sigma_1^{(n)}$ def in $p = \pi(\bar{p}$ in the pseudo interpretation.

Then $\bar{B} \cap \omega\rho_N^{n+1} = B \cap \omega\rho_N^{n+1} \notin M$,

since $\mathcal{P}(\kappa) \cap N = \mathcal{P}(\kappa) \cap M$ and $\kappa \geq \omega\rho_N^{n+1}$. QED (Claim 1).

It remains only to show:

<u>Claim 2</u> There is $D \in \underline{\Sigma}_1^{(n-1)}(M)$ s.t. $D \subset \omega\rho_n$ and $D \notin M$.

Pf.
Let B be as in Claim 1. Then B is $\Sigma_1(\langle H_n, D' \rangle)$ where D' is $\underline{\Sigma}_1^{(n-1)}(M)$.
Hence $D' \notin M$, since $B \notin M$.
Since $H_n = J_{\rho_n}^{A}$, there is f. p.r. in A s.t. f maps $\omega\rho_n$ onto
Set: $D = \{\nu \mid f(\nu) \in D'\}$. Then D is $\underline{\Sigma}_1^{(n-1)}(M)$ and $D \notin M$, since $D' \notin M$.

QED (Lemma 4.1)

Claim 1 in the proof of Lemma 4.1 Then gives us:

Cor 4.2 Let F be weakly amenable and $\omega\rho_N^{n+1} \leq \kappa < \omega\rho_N^{n}$. Then

$$\omega\rho_M^{n+1} \leq \omega\rho_N^{n+1}.$$

The proof of Lemmas 4.1, 4.2 did not use the full strength of weak amenability but merel. the fact that $\mathbb{P}(\omega\rho)\cap N = \mathbb{P}(\omega\rho)\cap M$ where $\rho = \rho_N^{n+1} \leq \kappa < \omega\rho_N^{n}$. Thus:

Cor 4.3 Let $\rho = \rho_N^{n+1}$ s.t. $\omega\rho + N < \kappa < \omega\rho_N^{n}$. Then the conclusions of 4.2, 4.3 hold.

proof.
$\omega\rho + N = \omega\rho + M$, since $\pi : N \xrightarrow[\Sigma_1]{} M$

and $\pi \upharpoonright \kappa = id$. Hence $\mathbb{P}(\omega\rho) \cap N =$
$= \mathbb{P}(\omega\rho) \cap M$ by acceptability.

QED (Cor 4.3)

<u>Lemma 4.4</u> Let $\omega\rho^{n+1} \leq \kappa < \omega\rho^{n}$ in N.
If $R^n_N \neq \emptyset$, then

(a) $\pi : N \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally

(b) $\pi'' R^n_N \subset R^n_M$

prf.
$n = 0$ is trivial. Let $n > 0$.

<u>Claim 1</u> M is the closure of $\omega\rho_n \cup \{r\}$ under good $\Sigma_1^{(n-1)}$ fcns.

prf.
Let $x \in M$, $x = \pi(f)(\alpha)$ where $\langle \alpha, f \rangle \in D$ and f is a good $\Sigma_1^{(n-1)}(N)$ fcn. Then $\alpha < \lambda \leq \pi(\kappa)$ and $\pi(f)$ is a good $\Sigma_1^{(n-1)}(M)$ fcn in a parameter $\pi(q)$. Let $q = \bar{G}(\xi, \bar{r})$ where $\bar{G}$ is a good $\Sigma_1^{(n-1)}(N)$ fcn

and $\xi < \omega\rho^n_N$. Let G be $\Sigma_1^{(n-1)}(M)$ by the same functionally absolute definition. Then $x = \pi(f)(\alpha, G(\pi(\xi), r_{\,}$ $= H(\alpha, \pi(\xi), r)$, where H is good, $\alpha < \lambda \leq \pi(h) < \omega\rho_n$ and $\pi(\xi) < \omega\rho_n$.

QED (Claim 1).

Clearly $r_k \in H^k_M$ for $k < n$. It follows that $H^{n-1}_M = h_{M^{n-1}, r \upharpoonright n-1}(\omega\rho_n \cup \{r_{n-}$ Hence there is a $\underline{\Sigma}_1^{(n-1)}(M)$ fcn f mapping $\omega\rho_n$ partially onto $\omega\rho^{n-1}_M$. Hence $\omega\rho^n_M = \omega\rho_n$, since otherwise $>$ holds and $R = \{\langle\nu, \tau\rangle \mid f(\nu) < f(\tau)\} \in H^n_M$. But $\omega\rho^n_M$ is admissible in R & hence $\mathrm{otp}(R) < \omega\rho^n_M \leq \omega\rho^{n-1}_M$. Contr! This proves (a).

(b) Then follows from:

<u>Claim 2</u> Let $n < m < \omega$. M is the closure of $\omega\rho_M^m \cup \{r \upharpoonright (m-1)\}$ under good $\Sigma_1^{(m)}$ fcns.

This follows from Claim 1. It is apparent from the def. of good $\Sigma_1^{(n)}$ fcn that the $\Sigma_1^{(n)}$ fcns can be charactized as the smallest class s.t.

(a) Each $\Sigma_1^{(i)}$ map to H^i is good $(i \leq n)$

(b) If $G(x_1^{j_1}, \dots, x_n^{j_n})$ is a $\Sigma_1^{(i)}$ map to H^i $(i, j_1, \dots, j_n \leq n)$ and F_h is a good map to H^{j_h} $(h = 1, \dots, n)$, then $G(\vec{F}(\vec{z}))$ is good.

By induction on good $\Sigma_1^{(n)}$ G it follows that for all $\vec{x}$ there are a good $\Sigma_1^{(m)}$ G' and $\vec{z} \in H^m$ s.t. $G(\vec{x}) = G'(\vec{x}, \vec{z})$. In particular, if $x \in M$, $x = G(\alpha, r_0, \dots, r_{m-1})$, $\alpha < \omega\rho_M^n$, then there is

$z \in H^m_M$ s.t. $x = G'(\alpha, z, r_0, \ldots, r_{m-1})$ (using $r_m, \ldots, r_{n-1} \in H^m$). But $\omega\rho^n$ is p.r. closed + $z = f(\xi)$ for a $\xi < \omega\rho^n$. Hence $x = G''(\alpha, \xi, r \upharpoonright (m-1))$ for a $\xi < \omega\rho^n$, where G'' is a good $\Sigma_1^{(m-1)}$ fcn.

QED (Lemma 4.4)

We now investigate the consequences of Σ_1 amenability.

Lemma 5.1 Let F be Σ_1-amenable. Let $\omega\rho_N^{n+1} \leq \kappa < \omega\rho_N^n$. Then

$$\pi : N \xrightarrow[\Sigma_0^{(n)}]{} M \text{ cofinally.}$$

proof. At $n = 0$ this is immediate. Otherwise $\omega\rho_N^1 > \kappa$ + it follows that F is weakly amenable, since if $\langle X_i \mid i < \kappa \rangle \in N$, then $Y = \{ i \mid X_i \in F_\alpha \}$ is $\Sigma_1(N)$. and $Y \subset \kappa < \omega\rho_N^1$. QED (5.1)

Lemma 5.2 Let F be Σ_1-amenable. Let $\omega\rho^{n+1} \leq \kappa < \omega\rho^n$ in N. Let $B \subset \kappa$ be $\underline{\Sigma}_1^{(n)}(M)$. Then B is $\underline{\Sigma}_1^{(n)}(N)$.

Proof.

Let B be $\Sigma_1^{(n)}(M)$ in $[\beta, f] = \pi(f)(\beta)$. Let $Bz \longleftrightarrow \bigvee x^n\, B'(x^n, z, \pi(f)(\beta))$, where B' is $\Sigma_0^{(n)}(M)$. Then $H_M^n = \bigcup \pi''H_N^n$ and

$$Bz \longleftrightarrow \bigvee u \in H_N^n\ \bigvee x \in \pi(u)\ B'(x, z, \pi(f)(\beta))$$

$$\longleftrightarrow \quad " \quad D(\pi(u), z, \pi(f)(\beta)),$$

where D is $\Sigma_0^{(n)}(M)$. Let $\bar{D}$ be $\Sigma_0^{(n)}(N)$ by the same definition. Then:

$$Bz \leftrightarrow \bigvee u \in H_N^n\ \beta \in F(\{\xi \mid \bar{D}(u, z, f(\xi))\})$$

$$\longleftrightarrow \bigvee u^n \bigvee x^n (\underbrace{x^n = \{\xi \mid \bar{D}(u, z, f(\xi))\}}_{\underline{\Sigma}_0^{(n)}} \wedge \underbrace{\beta \in F(x^n}_{\underline{\Sigma}_1^{(0)}}$$

for $z < \kappa$. QED(5.2).

Since κ is p.r. closed, we can replace κ by J_κ^A in Lemma 5.2, where $M = \langle J_\alpha^A, D\rangle$. (Hence $J_\kappa^A = J_\kappa^{\bar{A}}$ where $N = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{D}\rangle$). We then conclude;

Lemma 5.3 Let F, n be as above. Then $\underline{\Sigma}_1^{(n)}(M) \cap \mathcal{P}(J_\kappa^A) = \underline{\Sigma}_1^{(n)}(N) \cap \mathcal{P}(J_\kappa^{\bar{A}})$, where $M = \langle J_\alpha^A, D \rangle$, $N = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{D} \rangle$.

. . . .

Until further notice assume:

(*) F is Σ_1-amenable wrt. N and one of the following holds:

(a) F is weakly amenable

(b) $\omega\rho^+ < \kappa$ where $\rho = \rho_N^n$, $\omega\rho_N^n \leq \kappa$.

(Thus $\mathcal{P}(\omega\rho) \cap N = \mathcal{P}(\omega\rho) \cap M$, where $\rho = \rho_N^n$, $\omega\rho_N^n \leq \kappa$. Note that (a) holds whenever $\omega\rho_N^1 \leq \kappa$.)

Lemma 6.1 Assume (*). Let $\omega\rho_N^m \leq \kappa$. Then

(i) $H_N^m = H_M^m$

(ii) $\underline{\Sigma}_1^{(m)}(N) \cap \mathcal{P}(H_N^m) = \underline{\Sigma}_1^{(m)}(M) \cap \mathcal{P}(H_M^m)$.

proof (By ind. on m).

We first prove (i). It suffices to show $\omega\rho_N^m = \omega\rho_M^m$, since $\omega\rho_M^m$ is a cardinal in M and $H_M^m = J^A_{\rho_M^m}$ where $M = \langle J_\alpha^A, D \rangle$ and similarly for N, where $N = \langle J^{\bar{A}}_{\bar{\alpha}}, \bar{D} \rangle$ and $J^{A}_{\kappa} = J^{\bar{A}}_{\kappa}$. Let $m = h+1$. Then $\rho^m_N \geq \rho^m_M$, since there is $B \subset \omega\rho^m_N$ s.t. B is $\underline{\Sigma}^{(h)}_1(N)$ and $B \notin N$. Hence B is $\underline{\Sigma}^{(h)}_1(M)$, since $\pi : N \xrightarrow[\Sigma^{(h)}_1]{} M$ & $\pi \restriction \kappa = \mathrm{id}$. But $B \notin M$, since $\mathcal{P}(\omega\rho) \cap M = \mathcal{P}(\omega\rho) \cap N$ $(\rho = \rho^m_N)$.

We show: $\rho^m_N \leq \rho^m_M$. Suppose not. Then there is $B \subset \omega\rho^m_M < \omega\rho^m_N$, $B \in \underline{\Sigma}^{(h)}_1(M)$ s.t. $B \notin M$. But then $B \in \underline{\Sigma}^{(h)}_1(N)$ by Lemma 5.2 if $\kappa < \omega\rho^h_N$ and otherwise by the induction hypothesis. But $B \notin N$ since $\mathcal{P}(\kappa) \cap N \subset M$. Contr! This proves (i).

To prove (ii) let $H = H^m_N = H^m_M$ and let $B \subset H$. B is $\underline{\Sigma}^{(m)}_1(M)$ iff B is $\Sigma_1(\langle H, D \rangle)$ for a $D \in \underline{\Sigma}^{(h)}_1(M)$. But this is equivalent to saying B is $\Sigma_1(\langle H, D \rangle)$ for a $D \in \underline{\Sigma}^{(h)}_1(N)$ (by 5.2 or the induction hypothesis), which is in turn equivalent to: $B \in \underline{\Sigma}^{(m)}_1(N)$.

QED (Lemma 6.1)

Cor 6.2 $\pi : N \longrightarrow_{\Sigma^*} M$

pf. We show: $\pi : N \longrightarrow_{\Sigma_1^{(n)}} M$. For $\omega\rho^n_N > u$
For $\omega\rho^n_N > u$ use Lemma 5.1. For $\omega\rho^n_N < u$ employ ind. on n, using Lemma 6.1 (a).

QED (6.2)

Cor 6.3 $\pi '' P^m_N \subset P^m_M$ for $\omega\rho^m_N \leq u$.

proof
Let $\bar{p} \in P^m_N$, $p = \pi(\bar{p})$. It suffices to show

Claim $A^{h, p \upharpoonright h}_M \notin M$ for $m \geq h \geq 1$.

We proceed by ind on $m - h$. If $m = h$, then $A^{h,\bar{p} \upharpoonright h}_N = A^{h, p \upharpoonright h}_M$, since $\pi : N \longrightarrow_{\Sigma^*} M$, $\pi \upharpoonright H^m = \mathrm{id}$. But $A = A^{h, \bar{p} \upharpoonright h}_N \notin N$; hence $A \notin M$, since $\mathbb{P}(H) \wedge M = \mathbb{P}(H) \cap N$, where $H = H^m_N = H^m_M$. Now let it hold for $h + 1 \leq m$. Then $A^{h+1, p \upharpoonright (h+1)}_M = B \wedge H^{h+1}_M$, where B is $\Sigma_1(\langle H^h_M, A^{h, p \upharpoonright h} \rangle)$ in p_h. Hence $A^{h, p \upharpoonright h} \notin M$, since otherwise $\langle H^h_M, A^{h, p \upharpoonright}$ $\in M$.

QED (Cor 6.3)

Cor 6.4 $\pi''P^*_N \subset P^*_M$

prf. Lemma 3 if $\omega\rho^\omega_N > \omega$; otherwise Cor 6.3

Cor 6.5 $\mathbb{P}(\kappa) \cap \underline{\Sigma}_1^{(m)}(N) = \mathbb{P}(\kappa) \cap \underline{\Sigma}_1^{(m)}(M)$.

prf.
If $\omega\rho_N^{n+1} > \kappa$, then $\omega\rho_M^{n+1} > \kappa$ and:
$\mathbb{P}(\kappa) \cap \underline{\Sigma}_1^{(m)}(N) = \mathbb{P}(\kappa) \cap N = \mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap \underline{\Sigma}_1^{(m)}(M)$.
Now let $\omega\rho_N^{n+1} \leq \kappa$. Set: $H = J_\kappa^{\bar{A}} = J_\kappa^{A}$
(where $N = \langle J_\alpha^{\bar{A}}, \bar{D}\rangle$, $M = \langle J_\alpha^{A}, D\rangle$). By induction on n we prove:

Claim $\mathbb{P}(H) \cap \underline{\Sigma}_1^{(m)}(N) = \mathbb{P}(H) \cap \underline{\Sigma}_1^{(m)}(M)$

If $\kappa < \omega\rho_N^m$, it follows by Lemma 5.2.
Now let $\omega\rho_N^m \leq \kappa$, $m = h+1$. $A \subset H$ is $\underline{\Sigma}_1^{(m)}$ iff $A\vec{x} \leftrightarrow \langle H, \vec{R}_{\vec{x}}\rangle \models \varphi$,
where φ is Σ_1 and $R_{i,\vec{x}}(\vec{z}^m) \leftrightarrow$
$\leftrightarrow R_i'(\vec{x}, \vec{z}^m)$ and R_i' is a $\Sigma_1^{(h)}$
relation on H. The claim follows by the induction hypothesis and
$H_N^m = N_M^m$. QED (Cor 6.5)

Finally we note a stronger form of Lemma 5.1:

<u>Lemma 7</u> Let F be Σ_1-amenable. Let $\omega\rho_N^{n+1} \leq \nu < \omega\rho_N^{n}$. Let $R(x_1, \dots, x_p, y_1, \dots, y_q)$ be $\Sigma_1^{(n)}(M)$

Note: This doesn't require Σ_1-amenability as long as r is given.

Let $f_1, \dots, f_q \in \Gamma(\nu, N)$. Let $\alpha_1, \dots, \alpha_q < \nu$. $\tilde{R} = \{\vec{x} \mid R(\pi(x_1), \dots, \pi(x_p), \pi(f_1)(\alpha_1), \dots, \pi(f_n)(\alpha_q))$ is $\underline{\Sigma}_1^{(n)}(N)$. Moreover, if p_i is s.t. $f_i = p_i \in N$ or f_i is a good $\Sigma_1^{(n-1)}(N)$ function in p_i $(i = 1, \dots, q)$ and r is s.t. $F_{\langle\alpha_1, \dots, \alpha_q\rangle}$ is $\Sigma_1(N)$ in r, then $\tilde{R}$ is $\Sigma_1^{(n)}(N)$ in $\vec{p}, r$ (uniformly in the $\Sigma_1^{(n)}$ def. of R, the functionally absolute def. of f_i from p_i $(i=1, \dots, q)$ and the Σ_1 def. of $F_{\vec{\alpha}}$ from r

The proof is a virtual repetition of that of Lemma 5.2.

(<u>Remark</u>: The proof uses only that $F_{\langle\alpha_1, \dots, \alpha_n\rangle}$ is $\underline{\Sigma}_1^{(n)}(N)$ in r).

Lemma 8 Let $\langle N_i \mid i<\theta\rangle, \langle \pi_{ij} \mid i\le j<\theta\rangle$ be s.t. N_0 is acceptable and:

(a) N_i is transitive

(b) $\pi_{ij} : N_i \to N_j$, $\pi_{ij}\pi_{hi}=\pi_{hj}$, $\pi_{ii}=id$;
$N_\lambda, \langle \pi_{i\lambda} \mid i<\lambda\rangle =$ the direct limit of
$\langle N_i \mid i<\lambda\rangle, \langle \pi_{ij} \mid i\le j<\lambda\rangle$ for limit $\lambda<\theta$

(c) If $i+1<\theta$ and N_i is acceptable, then
$\pi_{i\,i+1} : N_i \xrightarrow[F_i]{*} N_{i+1}$, where F is weakly amenable and Σ_1-amenable on N_i.

Then for all $j<\theta$:

(i) N_j is acceptable

(ii) $\pi_{ij} : N_i \xrightarrow[\Sigma^*]{} N_j$ for $i\le j$

(iii) $\pi_{ij}{}''P^*_{N_i} \subset P^*_{N_j}$ for $i\le j$

(iv) Let $\kappa_i = \mathrm{crit}(F_i)$. If $\kappa_i\le\kappa_h$ for $i\le h<j$, then $\mathcal{P}(\kappa_i)\cap\underline{\Sigma}^{(n)}(N_i) = \mathcal{P}(\kappa_i)\cap\underline{\Sigma}^{(n)}(N$
for $n<\omega$.

(v) If $\kappa_h<\omega\rho^{n+1}_{N_h}$ for $i\le h<j$, then
$\pi_{ij} : N_i \xrightarrow[\Sigma_2^{(n)}]{} N_j$ and $\pi_{ij}{}''P^n_{N_i} \subset P^n_{N_j}$,

(vi) If $\omega\rho^{n+1}_{N_h}\le\kappa_h<\omega\rho^n_h$ for $i\le h<j$,
then $\pi_{ij} : N_i \xrightarrow[\Sigma_0^{(n)}]{} N_j$ cofinally.

Lemma 8 is proven by induction on j.

<u>Note</u> When $\langle N_i \mid i<\theta\rangle$, $\langle \pi_{ij} \mid i\le j<\theta\rangle$ are as in (a), (b), Lim(θ), and $\langle N_i\rangle$, $\langle \pi_{ij}\rangle$ has a well founded direct limit, then we often write:

$$N, \langle \pi_i \mid i<\theta\rangle = \lim_{i\le j<\theta} (N_i, \pi_{ij})$$

to indicate that $N, \langle \pi_i\rangle$ is the transitivized direct limit of $\langle N_i\rangle$, $\langle \pi_{ij}\rangle$.