§1 Extenders

<u>Def</u> $U \subset \mathcal{P}(\kappa)$ is <u>suitable</u> for κ iff

(a) κ is p.r. closed

(b) Let $A_1, \ldots, A_n \in U$; $\nu_1, \ldots, \nu_m \in \kappa$. Let $B \subset On$ be p.r. in predicates $A_1, \ldots, A_n$ and the parameters $\nu_1, \ldots, \nu_m < \kappa$. Then $B \cap \kappa \in U$.

It follows that if U is suitable, then:

(i) $\kappa \in U$

(ii) $A, B \in U \longrightarrow A \cap B, A \cup B, A \setminus B \in U$

(iii) $\nu < \kappa \longrightarrow \nu, \{\nu\} \in U$.

If we set: $U^m = \{A^{(m)} \mid A \in U\}$, where $A^{(m)} = \{\langle \xi_1, \ldots, \xi_m \rangle \mid \prec \xi_1, \ldots, \xi_m \succ \in A\}$, then it follows from (b) that if $A_1, \ldots, A_n \in U$ and $B \subset On^n$ is p.r. in $A_1, \ldots, A_n$ + parameters from κ, then $B \cap \kappa^n \in U^n$.

Def Let $U \subset \mathcal{P}(\kappa)$ be suitable for κ. Let $\kappa < \lambda$, where λ is p.r. closed. $F : U \longrightarrow \mathcal{P}(\lambda)$ is an extender on U with length λ iff whenever $B_{A_1,\dots,A_n} \subset On$ is uniformly p.r. in $A_1,\dots,A_n \in U$ and parameters from κ then $F(\kappa \cap B_{\vec{A}}) = \lambda \cap B_{F(\vec{A})}$.

We get:

(i) $F(\nu) = \nu$, $F(\{\nu\}) = \{\nu\}$ for $\nu < \kappa$

(ii) $F(\kappa) = \lambda$, $F(\emptyset) = \emptyset$

(iii) $F(A \cap B) = F(A) \cap F(B)$, $F(A \cup B) = F(A) \cup F(B$

$F(A \setminus B) = F(A) \setminus F(B)$

(iv) $A \subset B \leftrightarrow F(A) \subset F(B)$

(v) $\kappa \cap F(A) = A$, since

$\nu \in A \leftrightarrow \{\nu\} \subset A \leftrightarrow \{\nu\} \subset F(A) \leftrightarrow \nu \in F($

We can extend F to F^n on U^n by setting: $F^n(A^n) = F(A)^n$.

We write $F(A)$ for $F^n(A)$ when $A \in U^n$. Then:

(vi) $F(\kappa^n \cap B_{\vec{A}}) = \lambda^n \cap B_{F(\vec{A})}$ if $B_{\vec{A}} \subset On^n$ is unif. p.r. in $A_1 \dots A_n \in U$ and parameters from κ.

(vii) $F(\nu_1 \times \dots \times \nu_n) = \nu_1 \times \dots \times \nu_n$;

$F(\{\langle \nu_1, \dots, \nu_n \rangle\}) = \{\langle \nu_1, \dots, \nu_n \rangle\}$

for $\nu_1, \dots, \nu_n < \kappa$.

(~~Note that $F^n \upharpoonright \{\langle \nu_1, \dots, \nu_n \rangle\} = \mathrm{id}$ for $\nu_1, \dots, \nu_n < \kappa$.~~)

<u>Def</u> Let F be an extender on U of length λ. $\vec{F} = \langle F_\alpha \mid \alpha < \lambda \rangle$ is the <u>associated hypermeasure</u> where $F_\alpha = \{X \mid \alpha \in F(X)\}$.

<u>Def</u> $\vec{F} = \langle F_\alpha \mid \alpha < \lambda \rangle$ is a <u>hypermeasure</u> iff it is associated with an extender — i.e. $F_\alpha \subset U$ for $\alpha < \lambda$ and F is an extender, where $F(X) = \{\alpha \mid X \in F_\alpha\}$

<u>Def</u> $M = \langle |M|, \in, A_1, \dots, A_n \rangle$ is <u>suitable</u> iff M is transitive, rud closed and $M \models \bigwedge x \bigvee \alpha \bigvee f\ f : \alpha \xrightarrow{\text{onto}} x$.

<u>Def</u> M is <u>suitable</u> for κ iff M is suitable, $\kappa \in M$ is p.r. closed and $J_\kappa^{A_1, \dots, A_n} \in M$ whenever $A_1, \dots, A_n \in \mathcal{P}(\kappa) \cap M$. (Hence $U = \mathcal{P}(\kappa) \cap M$ is suitable).

<u>Def</u> Let M be suitable.

$\pi : M \to_F N$ iff

(a) N is transitive

(b) $\pi : M \to_{\Sigma_0} N$ cofinally

(c) $\kappa = \mathrm{crit}(\pi)$, where M is suitable for κ

(d) $F = \langle \lambda \cap \pi(X) \mid X \in \mathbb{P}(\kappa) \cap M \rangle$, where $\kappa < \lambda \leq \pi(\kappa)$, λ is p.r. closed, and

(e) $N =$ the Σ_0 closure of $\mathrm{rng}(\pi) \cup \lambda$.

(Equivalently to (e):
$N = \{\pi(f)(\xi) \mid \xi < \lambda, f : \kappa \to M, f \in M\}$)

<u>Lemma 1</u> If $\pi : M \to_F N$, $\kappa = \mathrm{crit}(\pi)$, then F is an extender of length $\lambda = F(\kappa)$ on $\mathbb{P}(\kappa) \cap M$ and N, π are uniquely determined by F.

prf.

(1) $F \upharpoonright \kappa = \mathrm{id}$, $F(\kappa) = \lambda$

(2) $F(\kappa \cap B_{A_1 \ldots A_n}) = \lambda \cap B_{F(A_1) \ldots F(A_n)}$

for B unif. p.r. in $A_1, \ldots, A_n \in \mathbb{P}(\kappa) \cap M$, since $\pi(\kappa \cap B_{\vec{A}}) = \pi(\kappa) \cap B_{\pi(\vec{A})}$,

and $\lambda \cap B_{\pi(\vec{A})} = \lambda \cap B_{\lambda \cap \pi(\vec{A})} =$
$= \lambda \cap B_{F(\vec{A})}$, since λ is
p.r. closed. QED (2)

Hence F is an extender. For
Σ_0 fmlae φ we have Łoś Thm:

(4) $F(\{\vec{\xi} \in \kappa^n \mid M \models \varphi(f_1(\xi_1), \dots, f_m(\xi_m))\} =$
$= \lambda \cap \{\vec{\xi} \in \pi(\kappa)^n \mid N \models \varphi(\pi(f_1)(\xi_1), \dots, \pi(f_m)(\xi_m)\}$
$= \{\vec{\xi} \in \lambda^n \mid N \models \varphi(\pi(f_1)(\xi_1), \dots, \pi(f_m)(\xi_m))\}$
for $f_i : \kappa \to M$, $f_i \in M$ $(i = 1, \dots, m)$.

Since $N = \{\pi(f)(\xi) \mid f : \kappa \to M, f \in M, \xi < \kappa\}$
it follows that N, π are
uniquely determined by F.

QED (Lemma 1)

Def $\pi : M \to_F N$ <u>weakly</u> iff
(b) - (e) hold as above and
(a') $N = \langle |N|, \in^N, A^N \rangle$ is r.t.,
wfcore(N) is transitive and
$\lambda \subset$ wfcore(N).

[This involves a slight abuse of notation.] Clearly the same proof yields:

<u>Corollary 2</u> If $\bar{\pi} : M \to_F N$ weakly, $\kappa = \mathrm{crit}(\bar{\pi})$, then F is an extender of length $\lambda = F(\kappa)$ on $\mathbb{P}(\kappa) \cap M$ and $N, \bar{\pi}$ are uniquely determined by F (up to isomorphism).

Finally, using an "ultrapower" construction:

<u>Lemma 3</u> Let M be suitable for κ, $U = \mathbb{P}(\kappa) \cap M$ and F an extender on U of length λ. There are π, N s.t. $\pi : M \to_F N$ weakly.

prf.

Define a term model $\mathbb{D} = \mathbb{D}(M, F)$ by: $\mathbb{D} = \langle D, \hat{=}, \hat{\in}, \hat{A} \rangle$ where:

$D = \{\langle \alpha, f \rangle \mid f \in M, f : \kappa \to M, \alpha < \lambda\}$

$\langle \alpha, f \rangle \cong \langle \beta, g \rangle \longleftrightarrow \langle \alpha, \beta \rangle \in F(\{\langle \xi, \zeta \rangle \mid f(\xi) = g($

$\quad$ " $\tilde{E}$ " $\longleftrightarrow$ " (" $\in$ "

$\tilde{A}\langle \alpha, f \rangle \longleftrightarrow \alpha \in F(\{\xi \mid f(\xi) \in A\})$

By ind on Σ_0 fmlae φ we get

Łoś Thm:

(1) $\mathbb{D} \models \varphi(\langle \alpha_1, f_1 \rangle, \dots, \langle \alpha_m, f_m \rangle) \longleftrightarrow$

$\longleftrightarrow \vec{\alpha} \in F(\{\vec{\xi} \mid M \models \varphi(f_1(\xi_1), \dots, f_m(\xi_m))\})$.

(2) $\langle \alpha, f \rangle \tilde{\in} \langle \beta, id \rangle \longrightarrow$

$\longrightarrow \bigvee \gamma < \beta \quad \langle \alpha, f \rangle \cong \langle \gamma, id \rangle$

pf.:

Set $I = I_{f, id} = \{\langle \xi, \zeta \rangle \mid f(\xi) = \zeta\}$.

$\langle \alpha, \beta \rangle \in F(\{\langle \xi, \zeta \rangle \mid f(\xi) < \zeta\}) \longleftrightarrow$

$\longleftrightarrow$ " $\in F(\{$ " $\mid \bigvee \eta < \zeta \ f(\xi) = \eta\})$

$\longleftrightarrow$ " $\in \{\langle \xi, \zeta \rangle \mid \bigvee \eta < \zeta \ \langle \eta, \zeta \rangle \in F(I)\}$

Hence there is $\gamma < \beta$ s.t.

$\langle \alpha, \gamma \rangle \in F(I)$; i.e. $\langle \alpha, f \rangle \cong \langle id, \gamma \rangle$.

QED (2)

By (1), $\cong$ is a congruence relation for $\mathbb{D}$. Let $p : \mathbb{D} \to \mathbb{D}/\cong$ be the natural projection. Let $\sigma : (\mathbb{D}/\cong) \xrightarrow{\sim} N$, where $wfc(N)$ is transitive. Set:

$$[t] = \sigma p(t) \quad \text{for } t \in \mathbb{D}$$

$$\pi(x) = [\langle 0, \mathrm{const}_x \rangle] \quad \text{for } x \in M.$$

(3) $\pi : M \to_{\Sigma_0} N$ cofinally.

prf.

Σ_0 preservation follows by Łoś Thm.

Cofinality follows by:

$$[\nu, f] \subset \pi(\bigcup \mathrm{rng}(f)) \text{ in } N.$$

QED(3)

(4) $[\alpha, \mathrm{id}] = \alpha$ for $\alpha < \lambda$

prf.

Set: $\tilde{\alpha} = \langle \alpha, \mathrm{id} \rangle$. Then $\tilde{\alpha} \in wfc(\mathbb{D})$ by (2) + hence $[\tilde{\alpha}] \in wfc(N)$, where $wfc(N)$ is transitive. Hence

$$[\tilde{\alpha}] = \{[t] \mid \mathbb{D} \models t \in \tilde{\alpha}\} \text{ for } \alpha < \lambda.$$

By ind. on α, using (2), we get:

$[\tilde{\alpha}] = \alpha$. QED(4)

(5) $[\alpha, f] = \pi(f)(\alpha)$

prf.

Let $I = I_{ff} = \{\langle \xi, \zeta \rangle \mid f(\xi) = f(\zeta)\}$.

$[\alpha, f] = \pi(f)(\alpha) \longleftrightarrow$

$\longleftrightarrow ID \models (\langle \alpha, f \rangle = \langle 0, \mathrm{cnst}_f \rangle (\langle \alpha, id \rangle))$

$\longleftrightarrow \langle 0, \alpha, \alpha \rangle \in F(\{\langle \xi, \zeta, \eta \rangle \mid f(\zeta) = f(\eta)\})$

$\longleftrightarrow$ " $\in \{\langle \xi, \zeta, \eta \rangle \mid \langle \xi, \eta \rangle \in F(I)\}$

$\longleftrightarrow \langle \alpha, \alpha \rangle \in F(I) \longleftrightarrow \langle \alpha, f \rangle \cong \langle \alpha, f \rangle$.

$\longleftrightarrow [\alpha, f] = [\alpha, f]$. QED (5)

Hence:

(6) $N = \{\pi(f)(\alpha) \mid \alpha < \lambda, f : \kappa \to M, f \in M\}$

(7) $\pi \upharpoonright \kappa = id$

prf.

$\langle 0, \nu \rangle \in F(\{\langle \xi, \zeta \rangle \mid \mathrm{cnst}_\nu(\xi) = id(\zeta)\})$

for $\nu < \kappa$, since $F(A) \cap \kappa = A$,

Hence $\pi(\nu) = [0, \mathrm{cnst}_\nu] = [\nu, id] = \nu$.

QED (7)

(8) $\alpha < \pi(\kappa)$ in N for $\alpha < \lambda$

prf.

$[\alpha, id] < [0, \mathrm{cnst}_\kappa]$, since

$\langle \alpha, 0 \rangle \in F(\{\langle \xi, \tau \rangle \mid id(\xi) < \kappa\}) = F(\kappa) = \lambda$

QED (8)

Thus:

(9) $\kappa = \mathrm{crit}(\pi)$

It remains only to show:

(10) $F(A) = \lambda \cap \pi(A)$ for $A \in \mathbb{P}(\kappa) \cap M$.

prf.

Let $\alpha < \lambda$. Then $\alpha \in \pi(A) \longleftrightarrow$

$\longleftrightarrow [\alpha, \mathrm{id}] \in [0, \mathrm{const}_A] \longleftrightarrow$

$\longleftrightarrow \langle \alpha, 0 \rangle \in F(\{\langle \xi, \xi \rangle \mid \xi \in A\}) \longleftrightarrow \alpha \in F(A$

QED (Lemma 3)

<u>Lemma 4</u> Let $\pi : M \xrightarrow[F]{} N$ where $\kappa = \mathrm{crit}(\pi)$ is the largest cardinal in N. Assume $F(\kappa) = \pi(\kappa)$. Then $\langle N, F \rangle$ is amenable.

prf.

Let $x \in N$. <u>Claim</u> $x \cap F \in N$.

Let $x \subset \pi(X)$. Then $x \cap F =$ $= x \cap (\pi(X) \cap F)$ and it suffices to show: <u>Claim</u> $\pi(X) \cap F \in N$.

But F is a function and $\langle a, F(a) \rangle \in$ $\in \pi(X) \rightarrow F(a) \in \bigcup{}^n \pi(X) =$ $= \pi(\bigcup{}^n X)$. Hence $F \cap \pi(X) \subset$ $\subset F \restriction \bigcup{}^n X$ and it suffices to show: Claim $F \upharpoonright X \in N$ for $X \in M$.

Assume w.l.o.g. $F \upharpoonright X \neq \emptyset$ (i.e. $\mathbb{P}(\kappa) \cap X \neq \emptyset$). Let $f \in M$ s.t. $f : \kappa \xrightarrow{\text{onto}} \mathbb{P}(\kappa) \cap X$. Then $\pi(f) : \pi(\kappa) \xrightarrow{\text{onto}} \mathbb{P}(\pi(\kappa)) \cap \pi(X)$. Moreover, $f = \langle \pi(f)(\alpha) \cap \kappa \mid \alpha < \kappa \rangle \in N$. But $F(f(\alpha)) = \pi(f(\alpha)) = \pi(f)(\alpha)$ for $\alpha < \kappa$. Hence:

$$F \upharpoonright X = \{ \langle f(\alpha), \pi(f)(\alpha) \rangle \mid \alpha < \kappa \} \in N.$$

QED (Lemma 4)

Def Let F be an extender on U of length λ, $\kappa = \mathrm{crit}(F)$. F is weakly amenable iff whenever $X \in U^2$, $\delta < \lambda$, then $\{ \xi \mid X''\{\xi\} \in F_\delta \} \in U$.

Def Let $\pi : M \to_F N$ weakly. F is Σ_1-amenable wrt. M iff iff F_δ is $\Sigma_1(M)$ for $\delta < \lambda$.

<u>Note</u> Either of the properties: weakly amenable and Σ_1 amenable can hold without the other. Both can fail.

If $\pi : M \to_F N$ weakly, it follows easily that:

(a) $\mathbb{P}(\kappa) \cap M \subset N$, since $X = \kappa \cap \pi(X)$ for $X \in \mathbb{P}(\kappa) \cap M$.

(b) $\mathbb{P}(\kappa) \cap N \subset M$ iff F is weakly amenable.

. . . . . .

If F is an extender on U of length λ, then whenever $\pi : M \to_F N$ weakly, the ordinals $< \pi(\kappa)$ in N will all have the form $[\alpha, f]$, where $\alpha < \lambda$ and $f \in U^2$, $f : \kappa \to \kappa$. Moreover, every such $[\alpha, f]$ is an ordinal $< \pi(\kappa)$ in N. Hence $\lambda = \pi(\kappa)$ iff $[\alpha, f] \in \lambda$ for all such f.

We define:

<u>Def</u> Let F be an extender on $\mathcal{U}$ of length λ. F is <u>whole</u> iff whenever $f \in \mathcal{U}^2$, $f: \kappa \to \kappa$, $\alpha < \lambda$ then there is $\beta < \lambda$ s.t. $\langle \alpha, \beta \rangle \in F(\{\langle \xi, \eta \rangle \mid f(\xi) = \eta\})$.

By the above remarks it is obvious that:

<u>Lemma 5</u> Let $\pi : M \to_F N$ weakly, where F is at κ, λ (i.e. $\kappa = \mathrm{crit}(F)$, $F(\kappa) = \lambda$). F is whole iff $\lambda = \pi(\kappa)$

[Hence $\pi(\kappa) \in \mathrm{wfc}(N)$ if F is whole.

We also note:

<u>Fact</u> Let $\mathcal{U}, \mathcal{U}'$ be suitable for κ, where $\mathcal{U}' \subset \mathcal{U}$. Let F be an extender on $\mathcal{U}$. Then $F \upharpoonright \mathcal{U}'$ is an extender on $\mathcal{U}'$.

<u>Def</u>. F is an <u>extender</u> <u>on</u> M <u>at</u> κ, λ iff F is an extender on $\mathbb{P}(\kappa) \cap M$ of length λ, where M is suitable for κ.

<u>Def</u> $N = \mathrm{Ult}(M, F)$ iff $\bigvee \pi \, (\pi : M \to_F N)$.

<u>Def</u> $N = \langle J_\alpha^A, F \rangle$ is <u>coherent</u> iff
iff J_α^A is acceptable, F is a whole extender on $J_{\bar\alpha}^A$ for an $\bar\alpha < \alpha$ s.t. $\kappa = \mathrm{crit}(F)$ is the largest cardinal in $J_{\bar\alpha}^A$, and
and $J_\alpha^A = \mathrm{Ult}(J_{\bar\alpha}^A, F)$.

<u>Lemma 6</u> Let $N = \langle J_\alpha^A, F \rangle$ be coherent

Then:

(a) N is amenable

(b) Let $\bar\alpha$ = the least $\bar\alpha$ s.t. $\mathrm{dom}(F) = \mathbb{P}(\kappa) \cap J_{\bar\alpha}^A$. Then κ is the largest cardinal in $J_{\bar\alpha}^A$

(c) Let $\lambda = F(\kappa)$. Then λ is the largest cardinal in N.

(d) Let $\bar\beta < \bar\alpha$ s.t. κ is the largest cardinal in $J_{\bar\beta}^A$. Let $\omega\beta = \sup F''\omega\bar\beta$. Set $\bar F = F \upharpoonright J_{\bar\beta}^A$. Then $\langle J_\beta^A, \bar F \rangle$ is coherent.

prf.

(a) is immediate by Lemma 4

(b), (c) are trivial. We prove (d)

Set: $\overline{\pi} = \pi \upharpoonright J^A_{\overline{\beta}}$. Then:

(1) $\overline{\pi} : J^A_{\overline{\beta}} \xrightarrow[\Sigma_0]{} J^A_\beta$ cofinally

(2) $\kappa = \mathrm{crit}(\overline{\pi})$, $\lambda = \overline{\pi}(\kappa) = \overline{F}(\kappa)$

(3) $\overline{F} = \overline{\pi} \upharpoonright \mathbb{P}(\kappa) \cap J^A_{\overline{\beta}}$

It remains only to show:

(4) J^A_β = the Σ_0 closure of $\mathrm{rng}(\overline{\pi}) \cup \lambda$.

prf.

Let $x \in J^A_\beta$, $x \in \overline{\pi}(X)$. Let $f \in J^A_{\overline{\beta}}$, $f : \kappa \xrightarrow{\text{onto}} X$. Then $\overline{\pi}(f) : \lambda \xrightarrow{\text{onto}} \overline{\pi}(X)$ and $x = \overline{\pi}(f)(\alpha)$ for an $\alpha < \lambda$.

QED (Lemma 6)

<u>Lemma 7</u> There is a Q-fmla φ s.t. if $N = \langle J_\alpha^A, F \rangle$ is acceptable, then N is coherent iff $N \models \varphi$.

prf.

φ is the statement that there are arbitrarily large ξ s.t.

(a) $F \cap S_\xi^A$ is a fcn and there is exactly one pair of ordinals $\langle \kappa, \lambda \rangle \in F \cap S_\xi^A$.

(b) $J_\kappa^A \models ZF^-$, $J_\lambda^A \models ZF^-$

(c) If $\langle a, b \rangle \in F \cap S_\xi^A$, then $a \subset \kappa$ and $b \subset \lambda$.

(d) Let $\langle \langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \rangle \in S_\xi^A$ s.t. $\langle a_i, b_i \rangle \in F$ for $i = 1, \ldots, n$. Let a be $\Sigma_1(\langle J_\kappa^A, \vec{a} \rangle)$ in the parameter α + let b be $\Sigma_1(\langle J_\lambda^A, \vec{b} \rangle)$ in the same parameter. Then $\langle a, b \rangle \in F$.

by the same definition

(e) There are $\bar{\xi} < \lambda$, $\zeta > \xi$ s.t.

$\mathrm{dom}(F \cap S_\xi^A)$ $\quad$ $\mathrm{dom}(F \cap S_\zeta^A)$

(f) There is $\xi' \geq \xi$ s.t. if $\beta < \xi$, $\beta < \lambda$, $f \in S_\xi^A$, $f : \kappa \to \kappa$ and $\bar{a} \in \mathrm{dom}(F \wedge S_\xi^!$ where $\bar{a} = \{\langle \delta, \eta \rangle < \kappa \mid f(\delta) = \eta\}$, then there is $a \in S_{\xi'}^A$, s.t. $\langle \bar{a}, a \rangle \in F$ and $\langle \beta, \alpha \rangle \in a$.

(g) There are $\bar{\xi}, \bar{f}, f$ s.t.

(i) $\bar{f} : \kappa \xrightarrow{\text{onto}} S_{\bar{\xi}}^A$, $f : \lambda \xrightarrow{\text{onto}} S_\xi^A$

(ii) If $\bar{a} = \{\langle \alpha, \beta \rangle \mid \bar{f}(\alpha) \in \bar{f}(\beta)\}$ and $a = \{\langle \alpha, \beta \rangle \mid f(\alpha) \in f(\beta)\}$, then $\langle \bar{a}, a \rangle \in F$

(iii) If $\bar{a} = \{\alpha \mid \bar{f}(\alpha) \in A\}$, $a = \{\alpha \mid f(\alpha) \in A\}$ then $\langle \bar{a}, a \rangle \in F$.

(a)–(e) guarantee that F is an extender at some κ, λ on $\mathcal{P}(\kappa) \cap J_{\bar{\alpha}}^A$, where κ is the largest cardinal in $J_{\bar{\alpha}}^A$.

(f) guarantees that if $\pi : J_{\bar{\alpha}}^A \to_F M$ weakly, then $\lambda = \pi(\kappa)$.

(g) then guarantees that $M = J_\alpha^A$.

It is clear that (a)–(g) hold for all coherent N.

QED (Lemma 7)

Let $N = \langle J_\alpha^A, F \rangle$ be coherent + let $\bar{\alpha}$ be least s.t. $\mathrm{dom}(F) = \mathcal{P}(\kappa) \cap J_{\bar{\alpha}}^A$. Clearly we have:

F is weakly amenable iff

iff $\bar{\alpha} = \kappa^{+N}$, since

$\bar{\alpha} = \kappa^{+N}$ iff $\mathcal{P}(\kappa) \cap N \subset J_{\bar{\alpha}}^A$.

Thus if $\bar{F}, \beta, F$ are as in Lemma 6 (d), then $\bar{F}$ is not weakly amenable.

As an example of the use of Σ_1-amenability we prove the following lemma. The argumentation used will be of great importance later in connection with the so called $*$-ultraproducts.

<u>Lemma 8</u> Let F be Σ_1-amenable wrt $\bar{M}$. Let $\pi : \bar{M} \to_F M$.

Let $\kappa = \text{crit}(F)$. Then
$$\mathbb{P}(\kappa) \cap \underline{\Sigma}_1(M) \subset \underline{\Sigma}_1(\bar{M}).$$

proof.
Let A be $\underline{\Sigma}_1(M)$, $A \subset \kappa$. Then
$$A\xi \longleftrightarrow \bigvee z\, R(z, \xi, q),$$
where R is $\Sigma_0(M)$, $q = \pi(f)(\alpha)$, $f \in \bar{M}$, $f : \kappa \to \bar{M}$ and $\alpha <$ length(F).

Hence:
$$A\xi \longleftrightarrow \bigvee u \in \bar{M} \quad \bigvee z \in \pi(u)\, R(z, \xi, \pi(f)(\alpha))$$
$$\longleftrightarrow \quad " \quad \underbrace{P(\pi(u), \xi, \pi(f)(\alpha))}_{\Sigma_0(M).}$$

Let $\bar{P}$ have the same Σ_0 def. over $\bar{M}$. Then:

$$A\xi \leftrightarrow \underbrace{\bigvee u \in \bar{M} \;\; \alpha \in F(\{\zeta \mid \bar{P}(u,\xi,f(\zeta))\})}$$

$\Sigma_1(\bar{M})$ in $\alpha, f, \mathfrak{r}$, where F_α is $\Sigma_1(\bar{M})$ in $\mathfrak{r}$.

QED (Lemma 8).

The same proof shows:

<u>Lemma 9</u> Let $F, \bar{M}, M$ be as above. Let $R(x_1,\dots,x_p, y_1,\dots,y_q)$ be $\Sigma_1(M)$. Let $f_1,\dots,f_q \in \bar{M}$ s.t. $f_i : \kappa \to \bar{M}$, where $\kappa = \operatorname{crit}(F)$. Let $\alpha_1,\dots,\alpha_q < \nu = \operatorname{length}(F)$. Then $\tilde{R} = \{\vec{x} \mid R(\pi(x_1),\dots,\pi(x_p), \pi(f_1)(\alpha_1),\dots,\pi(f_q)(\alpha_q))\}$ is $\Sigma_1(\bar{M})$ in $f_1,\dots,f_q, \alpha_1,\dots,\alpha_q$, and $\mathfrak{r}$, where $F_{\langle\alpha_1,\dots,\alpha_q\rangle}$ is $\Sigma_1(\bar{M})$ in $\mathfrak{r}$ (uniformly in the Σ_1 def of R and the Σ_1 def. of $F_{\vec{\alpha}}$ from $\mathfrak{r}$).