§3.5 The Sequence Lemma

From now on assume $\neg 0^{\P}$.

The definition of K can break down in two ways. Either, at some limit λ, the structure $\langle J^E_\lambda, F\rangle$ fails to be strong for any F or else the uniqueness condition for picking $F \neq \emptyset$ fails. In this section we show that the first type of failure cannot occur.

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We first introduce some new concepts.

<u>Def</u> Let N be strong. Let $\alpha > \omega$ be regular. The <u>canonical</u> α-<u>full</u> <u>hierarchy</u> $W_\nu = W_\nu[N]$ <u>over</u> N is defined like the canonical ω-complete hierarchy over N (cf §3.4.1) except that at

limit λ, having formed $J^E_{\tilde{\lambda}} = \bigcup_{\xi < \lambda} J^{E^{W_\xi}}_{\sigma_\xi}$, we set:

$W_\lambda = \langle J^E_{\tilde{\lambda}}, \emptyset \rangle$ <u>unless</u> there is an ω-complete F s.t. $\langle J^E_{\tilde{\lambda}}, F\rangle$ is a mouse and F is a measure on κ s.t. $cf(\kappa) = \alpha$. In the latter case set: $W_\lambda = \langle J^E_{\tilde{\lambda}}, F\rangle$.

It follows as before that $W_\nu = W_\nu[N]$ exists for all ν and we set: $W_\infty = W[N] = \bigcup_\nu W_\nu$.

Then W is a weasel and it follows as before that whenever $\nu = \kappa^{+W}$, $cf(\kappa) = \alpha$, and F is ω-complete s.t. $\langle J^{E^W}_\nu, F\rangle$ is a premouse, then $F = E^W_\nu$.

From this it follows as before that W is universal, The interesting new feature of W is that whenever $\nu = \kappa^{+W}$, $\nu \geq On \cap N$ and $cf(\kappa) \neq \alpha$, then $E_\nu^W = \emptyset$. Recall that if τ is regular in a weasel W and $E_\nu^W = \emptyset$ for $\nu = \tau^{+W}$, then any simple iteration of W will take τ cofinally to its image. For $W = W[N]$ it follows that if β is a cardinal in W and $cf(\beta) \neq \alpha$, $cf(\beta) \geq On \cap N$, then any simple iteration will take β cofinally to its image.

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It was Mitchell who introduced this notion & realized its utility.

Def Let W be a weasel. $X \subset On$ is *massive in* W iff X is unbounded in On and for sufficiently large regular τ we have: If β is a limit point of X and $cf(\beta) = \tau$, then:

(a) $\beta \in X$

(b) If β is a cardinal in W and $\nu = \beta^{+W}$, then $E_\nu^W = \emptyset$.

(c) If $\beta > \tau$ is a cardinal in V, then $\beta^{+W} = \beta^+$ and β^+ is a limit point of X. (Hence $|X \cap \beta^+| = \beta^+$).

It is obvious by the covering lemma that:

Lemma 1.1 If W is the canonical α-full weasel over a strong N, then On is massive in W.

Moreover:

Lemma 1.2 If X_ν is massive in W ($\nu < \Theta < \infty$), then $\bigcap_\nu X_\nu$ is massive in W.

Lemma 1.3 Let X be massive in W. Let W' be a simple iterate of W. Then $X' = \{\alpha \in X \mid \pi_{WW'}(\alpha) = \alpha\}$ is massive in W and W'.

Lemma 1.4 Let $\sigma : \overline{W} \rightarrow_{\Sigma_1} W$ where $X \subset rng(\sigma)$ is massive in W. Then $X' = \{\alpha \in X \mid \sigma(\alpha) = \alpha\}$ is massive in $\overline{W}$, W.

The proofs are straightforward.

Lemma 1.5 Let X be massive in W. Then W is universal.

proof.
Suppose not. Let N be a premouse which coiterates with W but whose coiteration does not terminate. Let N_i, W_i be

the coiterates with indices κ_i, ν_i.
Let $C \subset On$ be cub s.t.
$\pi_{N_i N_j}(\kappa_i) = \kappa_j$ and $\pi_{W_0 W_i}(\kappa_i) = \kappa_i$
for $i, j \in C$, $i \leq j$. We may also assume that $i = \kappa_i$ for $i \in C$.
Let $\beta \in C \cap X$ be a singular limit cardinal of sufficiently large cofinality s.t. $\beta > \bar{N}$. Then $\beta > \bar{\bar{N}}_i$ for $i < \beta$ & hence $\beta = \bar{\bar{N}}_\beta$.
But $\nu_\beta = \beta^{+N_\beta} = \beta^{+W_\beta} = \beta^{+W} = \beta^+$.
Contr! QED (Lemma 1.5)

<u>Lemma 1.6</u> Let W be universal.
For any α there is a simple iterate W' of W above α s.t. some X is massive in W'.

proof.
Let W^* = the canonical ω_1-full weasel over $W|\alpha$. Coiterate W, W^* simply to W' and set:
$X = \{\alpha \mid \pi_{W^* W'}(\alpha) = \alpha\}$.
QED (Lemma 1.6

<u>Lemma 1.7</u> Let W be weakly universal. There is a simple iterate W' of W s.t. some X is massive in W'.

prf. as above.

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Using these lemmas we now can prove the <u>sequence lemma</u>:

<u>Lemma 2</u> Let J_λ^E be s.t. $\langle J_\nu^E, E_\nu \rangle$ is strong for $\nu < \lambda$, where $\lim(\lambda)$. Then there is F s.t. $\langle J_\lambda^E, F \rangle$ is strong.

We begin the proof with a rather technical definitión.

<u>Def</u> A <u>T-system</u> is a sequence $\langle W_i \mid i < \theta \rangle$ of weasels s.t. W_j is a simple iterate of W_i for $j \geq i$.

We now assign to each sequence $\vec{W} = \langle W_i \mid i < \gamma \rangle$ $(\gamma < \infty)$ of universal weasels a T-sequence $\vec{W}' = T(\vec{W})$.

Def Let $\vec{W} = \langle W_i \mid i < \gamma \rangle$ be a sequence of universal weasels ($\gamma < \infty$). By ind. on γ we define $\vec{W}' = T(\vec{W})$:

Case 1 $\gamma = 0$. $\vec{W}' = \emptyset$

Case 2 $\gamma = \tau + 1$. Let $\vec{W}^\tau = T(\vec{W} \restriction \tau)$. Coiterate W_τ, W_j^τ to W_j^* for $j < \tau$. Define $\vec{W}'$ by:

$$\vec{W}' \restriction \tau = T(\vec{W}^*) \ , \quad W'_\tau = W_\tau$$

Case 3 $\operatorname{Lim}(\gamma)$. For $\tau < \gamma$ set: $\vec{W}^\tau = T(\vec{W} \restriction \tau)$. Suppose that

(*) W_i^τ is a simple iterate of W_i^ν for $i < \nu \leq \tau < \gamma$.

$\vec{W}'$ is then defined by:

$$W'_i = \lim_{i < \tau < \gamma} W_i^\tau \ .$$

If (*) fails, then $\vec{W}'$ is undefined.

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We verify by ind. on γ that $\vec{W}'$ is defined and that:

<u>Lemma 2.1</u> Set: $\vec{W}^\tau = T(\vec{W} \upharpoonright \tau)$ $(\tau \leq \gamma)$.

(a) W_i^τ is a simple iterate of W_i $(i < \tau \leq \gamma$

(b) W_i^τ " " " " " W_j^τ $(i \leq j < \tau \leq$

(c) W_i^τ " " " " " W_i^ν $(i < \nu \leq \tau \leq \gamma$

(d) If $W_i | \eta = W_j | \eta$ for $i \leq j < \gamma$, then W_j^τ is an iterate of W_j above η for $i \leq j < \tau \leq \gamma$.

The verifications are straightforward. In order to see (b) in the case $\lim(\tau)$, observe that if we coiterate W_i^τ, W_j^τ to W, then $\mathrm{rng}(\pi_{W_j^\tau, W}) =$
$$= \bigcup_{\nu < \tau} \mathrm{rng}(\pi_{W_j^\nu, W}) \subset \bigcup_{\nu < \tau} \mathrm{rng}(\pi_{W_i^\nu, W})$$
$$= \mathrm{rng}(\pi_{W_i^\tau, W}).$$

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Now let J_λ^E be as in the sequence lemma and set:
$$W_\nu = \text{the canonical } \omega_1\text{-full weasel over } \langle J_\nu^E, E_\nu \rangle$$
for $\nu < \lambda$.

Set: $\vec{W}' = T(\vec{W})$.

Then On is massive in W_i $(i<\lambda)$ + hence $X_i' = \{\alpha \in X_i \mid \pi_{W_i W_i'}(\alpha) = \alpha\}$ is massive in W_i'. But then

$$X_i^* = \{\alpha \in X_i' \mid \pi_{W_j' W_i'}(\alpha) = \alpha \text{ for } j \geq i\}$$

is massive in W_i'.

Set: $Z_i =_{Df} \bigcap_{j \geq i} \operatorname{rng}(\pi_{W_j' W_i'})$.

Then $Z_i \prec W_i'$ and $X_i^* \subset Z_i$. Moreover

$$\pi_{W_j' W_i'}{}'' Z_j = Z_i \quad (i \leq j < \lambda).$$

Set $\sigma_i : \overline{W} \stackrel{\sim}{\longleftrightarrow} Z_i$ $(i<\lambda)$ where $\overline{W}$ is transitive.

Then $\overline{X}_i = \{\alpha \in X_i^* \mid \sigma(\alpha) = \alpha\}$ is massive in $\overline{W}$. Hence $\overline{W}$ is universal. Note that $W_i | i = W_j | i$ for $i \leq j < \lambda$. Hence W_i' is an iterate of

W_i above i for $i < \lambda$. Hence $W'_i = W_i | i = W_j | i = W'_j | i$ for $i \le j < \lambda$ and W'_j is an iterate of W'_i above i for $i \le j < \lambda$. Hence $W_i | i \subset Z_i$. Hence $\sigma \upharpoonright (\bigcup_{i<\lambda} W_i | i) = \mathrm{id}$ and $\bigcup_{i<\lambda} W_i | i = \bar{W} | \lambda$. But $W_i | i = \langle J_i^E, E_i \rangle$. Hence $\bar{W} | \lambda = \langle J_\lambda^E, E_\lambda^{\bar{W}} \rangle$ is strong.

QED (Lemma 2).

We close this section by formulating an extension of Lemma 1.3 which will be useful later. We shall have occasion to consider structures of the form $\langle W, U \rangle$ where W is a weasel and for some κ, ν we have: $\nu = \kappa^{+W}$ and U is a normal measure on κ in the amenable structure $\langle J_\nu^{E^W}, U \rangle$. We shall consider simple iteration $\langle \langle W_i, U_i \rangle \mid i \leq \theta \rangle$ $(\theta < \infty)$ of $\langle W, U \rangle$ where for each i, either the index is ν_i and $E^{W_i}_{\nu_i}$ is applied or the index is U_i and U_i is applied: $\pi_{i,i+1} : W_i \xrightarrow[U_i]{} W_{i+1}$. U_i is defined by:

$$(\pi_{0i} \upharpoonright J_\nu^{E^W}) : \langle J_\nu^{E^W}, U \rangle \xrightarrow[\Sigma_0]{} \langle J_{\nu'}^{E^{W_i}}, U_i \rangle$$

cofinally where $\nu' = \pi_{0i}(\nu)$.

It is then clear that Lemma 1.3

also holds for such iterations – i.e.

<u>Lemma 1.3.1</u> If X is massive in W and $\langle W', U' \rangle$ is an iterate of $\langle W, U \rangle$ in the above sense with iteration map π, then $X' = \{\alpha \in X \mid \pi(\alpha) = \alpha\}$ is massive in W, W'.

<u>Note</u> This requires only that the particular iterate exist and not that $\langle W, U \rangle$ be iterable.