§ 3.3.4 Weak Covering

We now prove Mitchell's covering lemma, announced at the outset of § 3.3. We assume $\neg O^{\kappa}$. Let W be an α_0-full weasel and let $\beta > \alpha_0^{\omega}$ be a singular cardinal. We claim: $\beta^+ = \beta^{+W}$.

Suppose not. Let $\tau =$ the least regular $\tau > \alpha_0^{\omega}$ s.t. $\mathrm{cf}(\beta^{+W}) < \tau$. Generically collapse β^{+W} to τ, adding $f : \tau \xrightarrow{\text{onto}} W|\beta^+$ (where $\beta^+ =_{\mathrm{Df}} \beta^{+W}$ from now on).

For $\alpha \leq \tau$ set: $X_\alpha = f''\alpha$. Set:

$C = \{\alpha < \tau \mid \alpha = \tau \cap X_\alpha \wedge X_\alpha \prec W|\beta^+ \text{ cofinally}\}$.

Then C is cub in τ. For $\alpha \in C \cup \{\tau\}$ set: $\sigma_\alpha : \overline{W}_\alpha \stackrel{\sim}{\longleftrightarrow} \langle X_\alpha, E \cap X_\alpha, E_{\beta^+} \cap X_\alpha \rangle$ (where $W = J_\infty^E$), $\overline{W}_\alpha$ being transitive. Set $\sigma_{\alpha\alpha'} = \sigma_{\alpha'}^{-1} \sigma_\alpha$ $(\alpha \leq \alpha')$.

Then $\overline{W}_\tau = W|\beta^+$ and

$$\langle \sigma_\alpha \rangle, \overline{W}_\tau = \lim_{\alpha \le \alpha' < \tau} (\sigma_{\alpha\alpha'}, \overline{W}_\alpha),$$

where all maps are elementary hence Σ^*-preserving. Hence each $\overline{W}_\alpha$ is a mouse and $\overline{W}_\alpha \models ZF^-$.

<u>Lemma 1</u> There is a cub $C' \subset C$ s.t $\mathbb{P}(\alpha) \cap W \not\subset \overline{W}_\alpha$ for $\alpha \in C'$ s.t. $cf(\alpha) = \alpha_0$

proof

Suppose not.

Let $S \subset C$ be stationary s.t $\mathbb{P}(\alpha) \cap W \subset \overline{W}_\alpha$ & $cf(\alpha) = \alpha_0$ for $\alpha \in S$.

(1) $\mathbb{P}(\alpha) \cap W = \mathbb{P}(\alpha) \cap \overline{W}_\alpha$ for $\alpha \in S$.

prf.

Coiterate $\overline{W}_\alpha, W$ to $\overline{W}', W'$. The coiteration terminates below ∞ + $\overline{W}'$ is a simple iterate of $\overline{W}_\alpha$ by the universality of W. Moreover the coiteration is above α. Hence $\mathbb{P}(\alpha) \cap \overline{W}_\alpha = \mathbb{P}(\alpha) \cap \overline{W}' \subset \mathbb{P}(\alpha) \cap W' \subset W$.

QED(1)

But then, for $\alpha \in S$:

(2) $(H_{\alpha^+})^W = (H_{\alpha^+})^{\bar{W}}$

Fix $\alpha \in S$ and define an ultra-filter $U = U_\alpha$ on $\mathbb{P}(\alpha) \cap W$ by:

$U_\alpha = \{X \mid \alpha \in \sigma_\alpha(X)\}$. Trivially:

(3) U_α is normal wrt. functions $f \in (H_{\alpha^+})^W$.

Moreover:

(4) $\langle W|\alpha^+, U_\alpha\rangle$ is amenable

(where $\alpha^+ =_{\mathrm{df}} \alpha^{+W}$).

proof.

Let $X = \langle X_\nu \mid \nu < \alpha\rangle \in W$, where $X_\nu \subset \alpha$ for $\nu < \alpha$. Then

$\{\nu \mid X_\nu \in U_\alpha\} = \{\nu \mid \alpha \in \sigma_\alpha(X)_\nu\} \in W$.

QED (4)

<u>Claim</u> U_α is ω-complete for stationarily many $\alpha \in S$.

prf. of Claim.

<u>Case 1</u> $\mathrm{cf}(\beta^+) > \omega$.

It holds for all $\alpha \in S$: Let $X_i \in U_\alpha$ $(i < \omega)$. Then $X_i \in \overline{W}_\alpha | \delta$ for a $\delta < \beta^+$ and we can form the diagonal intersection of $U_\alpha \cap (\overline{W} | \delta)$ to get $Y \in U_\alpha$ almost contained in each X_i. Hence $Y \cap \bigcap_i X_i \neq \emptyset$, since $\mathrm{cf}(\alpha) > \omega$. QED (Case 1)

<u>Case 2</u> $\mathrm{cf}(\beta^+) = \omega$.

Then $\tau = (\alpha_0^\omega)^+$. Assume w.l.o.g. that there are $X_i^\alpha \in U_\alpha$ $(i < \omega)$ for $\alpha \in S$ s.t. $\bigcap_i X_i^\alpha = \emptyset$. For each limit pt. $\alpha \in S$ of C, $\mathrm{cf}(\alpha) > \omega$ + hence there is $\alpha^* < \alpha$ s.t. $X_i^\alpha \in \mathrm{rng}(\sigma_{\alpha^*, \alpha})$ for $i < \omega$. But then $\alpha^* = \gamma$ is constant on a stationary set.

Since $\overline{(\overline{W}_\gamma)}^\omega \leq \alpha^\omega < \tau$, there is a fixed $\langle X_i \mid i<\omega\rangle$ s.t. $\sigma_{\gamma\alpha}(X_i) = X_i^\alpha$ on a stationary set of α. Hence there are $\alpha < \alpha'$ s.t. $\sigma_{\alpha\alpha'}(X_i^\alpha) = X_i^{\alpha'}$ $(i<\omega)$ – i.e. $\alpha \in \bigcap_i X_i^{\alpha'}$. Contr!

QED (Claim).

Now pick $\alpha \in S$ s.t. U_α is ω-complete. Set $\tilde{W} = \langle W|\alpha^+, U_\alpha\rangle$. Then $\tilde{W}$ is iterable, since $W|\alpha^+$ is iterable and U_α is ω-complete. (This is like the proof that a bicephalus is iterable). If $E^W_{\alpha+} \neq \emptyset$, then $\tilde{W}$ is an r-mouse and O^r exists. Contradiction! Hence $E^W_{\alpha+} = \emptyset$ and we identify $\tilde{W}$ with the mouse $\langle J^{E^W}_{\alpha+}, U_\alpha\rangle$. But then $U_\alpha = E^W_{\alpha+} \neq \emptyset$ by α_0-fullness. Contr! QED (Lemma 1)

Let $S \subset C$ be stationary s.t.
$\mathcal{P}(\alpha) \cap W \not\subset \overline{W}_\alpha$ for $\alpha \in S$.
Then for $\alpha \in S$ there is a mouse
$M = M_\alpha = W|\delta$ s.t. (letting $\overline{W} = \overline{W}_\alpha$)
(a) $J_\alpha^{E^{\overline{W}}} = J_\alpha^{E^M}$
(b) M is sound
(c) $\rho_M^\omega \leq \alpha$ and $\rho_M^\omega \leq \delta$ if $\tau = \delta^{+W}$
(d) $\mathcal{P}(\alpha) \cap \Sigma^*(M) \not\subset W$.

Set $\hat{\alpha} = \delta$ if $\tau = \delta^{+W}$ and $\hat{\alpha} = \alpha$ if τ is a limit cardinal in W. Then
(e) $\overline{W}|\hat{\alpha} = M|\hat{\alpha}$,
since $E_\alpha^{\overline{W}} = E_\alpha^M = \emptyset$ if τ, hence α, is a limit cardinal in W.
Coiterate $M, \overline{W}$ to $M', \overline{W}'$. The coiteration is above $\hat{\alpha}$.

<u>Lemma 2</u> There is a stationary set of $\alpha \in S$ s.t. $\overline{W}' = \overline{W}$.
proof Suppose not.
Assume w.l.o.g. $\overline{W}' \neq \overline{W}$ for $\alpha \in S$.
Let $M_i, \overline{W}_i$ $(i \leq \theta)$ be the coiteration of $M, \overline{W}$ with indices ν_i, κ_i. There is a least i s.t. $\overline{W}_{i+1} \neq \overline{W}$.

Then $E^{\bar W}_{\nu_i} \neq \emptyset$ and $E^{\bar W_j}_{\nu_i} = \emptyset$ for $i < j$.

(1) $\rho^{\omega}_{M_i} \leq \kappa_i$ and M_i is "sound above κ_i"

(i.e. M_i = the closure of $\kappa_i \cup \{p_{M_i}\}$ under good $\Sigma_1^{(n)}$ func where $\omega\rho^{n+1}_{M_i} \leq \kappa_i < \omega\rho^{n}_{M_i}$).

prf.

$\rho^{\omega}_{M} \leq \hat{\alpha} \leq \kappa_0$ and M is sound. If M_i is a simple iterate of M, the conclusion is immediate. Otherwise M_i is a simple iterate of $M_j|\gamma$ for a $j < i$ s.t. $\rho^{\omega}_{M_j|\gamma} \leq \kappa_j < \kappa_i$ and $M_j|\gamma$ is sound. QED (1)

Set: $\bar M = M_i|\delta$, where δ is maximal s.t. $\omega\delta \leq On \cap M_i$ and $\kappa_i^+ = \nu_i$ in $M_i|\delta$. Then:

(2) $\rho^{\omega}_{\bar M} \leq \kappa_i$ + $\bar M$ is sound above κ_i

Recall that if $E^{M_i}_{\nu_i} \neq \emptyset$, then $\pi_{M_i, M_{i+1}} : \bar M \xrightarrow[E_{\nu_i}]{} M_{i+1}$. Clearly:

(3) $E^{\bar M}_{\nu_i} \neq E^{\bar W}_{\nu_i}$.

Set: $\bar{Q} = J^{E^{\bar{W}}}_{\nu_i} = J^{\bar{M}}_{\nu_i}$; $\tilde{Q} = J^{E^W}_{\tilde{\nu}}$,

where $\tilde{\nu} = \text{lub}\ \sigma_\alpha{}''\nu_i$, $\tilde{\kappa} = \sigma_\alpha(\kappa)$.

By the frequent extension lemma there is a stationary set of $\alpha \in S$ s.t. $\sigma_\alpha \upharpoonright \bar{Q} : \bar{Q} \to \tilde{Q}$ has a canonical extension $\tilde{\sigma} : \bar{M} \to \tilde{M}$ wrt. $\bar{M}$ and $\tilde{M}$ is iterable above $\tilde{\nu}$. Moreover by §3.3.3 Corollary 3:

(4) $\tilde{M}$ is coiterable with W

on a stationary set. We assume wlog that this holds for all $\alpha \in S$.

(5) $\tilde{\sigma} : \bar{M} \xrightarrow[\Sigma_0^{(n)}]{} \tilde{M}$ cofinally

where $\omega\rho^{n+1} < \nu_i \leq \omega\rho^n_{\bar{M}}$.

(6) $\omega\rho^{n+1}_{\tilde{M}} \leq \tilde{\kappa}$

prf.

Set $\bar{p} = p_{\bar{M}}$, $\tilde{p} = \tilde{\sigma}(p)$, Define partial $\bar{h}$ on κ_i by:

$$\bar{h}(\xi) \simeq \begin{cases} h_M((\xi)_0, \langle(\xi)_1, \bar{p}\rangle) & \text{if } n = 0 \\ h_{M^{n,\bar{p}}}((\xi)_0, (\xi)_1) & \text{if not} \end{cases}$$

Then $\bar{h}$ is $\Sigma_1^{(n)}(\bar{M})$ in $\bar{p}$. Let $\tilde{h}$ have the same $\Sigma_1^{(n)}(\tilde{M})$ def in $\tilde{p}, \tilde{\kappa}$. Then

$$\tilde{h}''\tilde{\kappa} \supset \tilde{\sigma}''\bar{h}''\kappa_i = \tilde{\sigma}''\bar{M}^{n,\bar{p}},$$

by the κ_i-soundness of $\bar{M}$. Hence $\tilde{\nu} \cap \tilde{h}''\tilde{\kappa}$ is cofinal in $\tilde{\nu} = \kappa_i^{+\bar{M}}$. But $\tilde{h}''\tilde{\kappa} \prec_{\Sigma_1} \tilde{M}^{n,\tilde{p}}$ + hence $\tilde{\nu} \subset \tilde{h}''\tilde{\kappa}$. QED (6)

(7) $\tilde{M}$ is the closure of $\tilde{\kappa} \cup \{\tilde{p}\}$ under good $\Sigma_1^{(n)}(\tilde{M})$ functions.

prf.
$\tilde{M}$ is the closure of $\tilde{\nu} \cup \{\tilde{p}\}$ under such fcns by construction. It remains to observe that $\tilde{\nu} \subset \tilde{h}''\tilde{\kappa}$ where $\tilde{h}$ is the good $\Sigma_1^{(n)}$ fcn in $\tilde{p}$ constructed above.

QED (7)

By universality, the coiteration of $\tilde{M}, W$ terminates. Coiterate $\tilde{M}, W$ to M', W'. The coiteration is above $\tilde{\kappa}$, since $\tilde{\nu}$ is the earliest possible point of difference. Then M' is a simple iterate of $\tilde{M}$ and

$$\mathcal{P}(\alpha) \cap \underline{\Sigma}^*(\tilde{M}) = \mathcal{P}(\alpha) \cap \underline{\Sigma}^*(M') \subset$$

$$\subset \mathcal{P}(\alpha) \cap \underline{\Sigma}^*(W') \subset W.$$

But by (7) it is easily seen that there is a $\underline{\Sigma}^*(\tilde{M})$ subset of $\tilde{\kappa}$ which codes $\tilde{M}$ (i.e. $\tilde{M} \in L[a]$). Hence:

(8) $\tilde{M} \in W$

By the coiteration of $\tilde{M}, W$ we also know:

(9) $\tilde{M}$ is a mouse.

By (6), (8) and the fact that $\tilde{\kappa}$ is a cardinal in W:

(10) $\rho^{\omega}_{\tilde{M}} = \rho^{n+1}_{\tilde{M}} = \tilde{\kappa}$,

Recall that by §3.3.2 we have:

(11) $\tilde{\sigma} : \overline{M} \longrightarrow_{\Sigma_2^{(h)}} \tilde{M}$ for $h < n$.

An examination of the proof of §2.3 Lemma 4.2 shows that the assumption (5), (11) + (10) are enough to prove:

(12) $\tilde{P} \setminus \kappa = P_{\tilde{M}}$

Hence by (7):

(13) $\tilde{M}$ is sound.

We also note:

(14) $\sigma_0 \upharpoonright \overline{Q} : \overline{W} | \nu_i \longrightarrow_{\Sigma_0} W | \tilde{\nu}$ cofinally

proof

We know $\sigma_\alpha : \overline{Q} \longrightarrow_{\Sigma_0} Q$ cofinally, where $\overline{W} | \nu_i = \langle \overline{Q}, E^{\overline{W}}_{\nu_i} \rangle$ and $W | \tilde{\nu} = \langle Q, E^{W}_{\nu_i} \rangle$. Let F be defined by:

$\sigma_\alpha \upharpoonright \bar{Q} : \bar{W}|\nu_i \xrightarrow[\Sigma_0]{} \langle Q, F \rangle$ cofinally

(i.e. $F = \bigcup_{x \in \bar{Q}} \sigma_\alpha(x \cap E^{\bar{W}}_{\nu_i})$).

Then $\langle Q, F \rangle$ is a premouse and $F = Q \cap E^{W}_{\sigma_\alpha(\nu_i)}$. Hence $F = E^{\bar{W}}_{\nu_i}$ by the initial segment condition.

QED (12)

Now let δ = the least δ s.t. $\rho^{\omega}_{W|\delta} \leq \tilde{\kappa}$. Set $N = W|\delta$.

Then $W|\tilde{\nu} = N|\tilde{\nu}$ and $\rho^{\omega}_{N} = \tilde{\kappa}$, since $\tilde{\kappa}$ is a cardinal in W. Since N is sound and $\tilde{\nu} = \tilde{\kappa}^{+N}$, there is $a \in \underset{=}{\Sigma}^{*}(N)$, $a \subset \tilde{\kappa}$ coding a well ordering of type $\tilde{\nu}$. Hence $a \notin \tilde{M}$. By exactly the same argument there is $\tilde{a} \in \Sigma^{*}(\tilde{M})$, $\tilde{a} \subset \tilde{\kappa}$ s.t. $\tilde{a} \notin N$. Since $N, \tilde{M}$ coiterate above $\tilde{\kappa}$, it follows easily that they coiterate to the same mouse & hence by soundness:

(15) $N = \tilde{M}$.

But then $E^{W}_{\tilde{\nu}} = E^{\bar{M}}_{\tilde{\nu}}$ and hence by (14): $E^{\bar{W}}_{\nu_i} = E^{\bar{M}}_{\nu_i}$. Contradiction!

QED (Lemma 2)

. . . . . .

Assume w.l.o.g. that $\bar{W}' = \bar{W}$ at all $\alpha \in S$. Then $\bar{W} = \langle J^{\bar{E}}_{\bar{\gamma}}, \bar{E}_{\gamma} \rangle = M' | \gamma$. Let $\bar{\beta} = \sigma_{\alpha}^{-1}(\beta) =$ the largest cardinal in $\bar{W}$. Let $\nu_i, \varkappa_i$ be the indices of the coiteration. Clearly $\varkappa_i \leq \bar{\beta}$ for all i. Let $M'' = M_{i_0}$ where $i_0 =$ the least i s.t. $M_i = M$ or $\varkappa_i = \bar{\beta}$. Exactly as in the proof of Lemma 2:

(1) $\omega\rho^{\omega}_{M''} \leq \bar{\beta}$ and M'' is sound above;

Let $\bar{M} = M'' | \delta$, where δ is least s.t. $\rho^{\omega}_{M''|\delta} \leq \bar{\beta}$. Note that $\bar{Q} = J^{\bar{E}}_{\bar{\gamma}}$ is an initial segment of M'', hence of $\bar{M}$ (i.e. $\bar{Q} = J^{E^{\bar{M}}}_{\bar{\gamma}}$). Exactly as in Lemma 2 we get:

(2) $\omega\rho^{\omega}_{\bar{M}} \leq \bar{\beta}$ and $\bar{M}$ is sound above $\bar{\beta}$.

Let $\tilde{Q} = J_{\tilde{\gamma}}^{E^W}$, where $\tilde{\gamma} = \sup \sigma_\alpha{}'' \bar{\gamma}$.

By the frequent extension lemma there is a stationary set of $\alpha \in S$ s.t. $\sigma_\alpha : \bar{Q} \to \tilde{Q}$ has a canonical extension $\tilde{\sigma} : \bar{M} \to \tilde{M}$ s.t. $\bar{M}$ + $\tilde{M}$ is iterable above $\tilde{\gamma}$. Moreover, as before we may assume:

(3) $\tilde{M}$ is coiterable with W.

Exactly as before we get:

(4) $\tilde{M} \in W$

(5) $\tilde{M}$ is the closure of $\beta \cup \{\tilde{p}\}$ under good $\Sigma_1^{(n)}(\tilde{M})$ functions ($\omega\rho_{\bar{M}}^n \geq \bar{\gamma}$) where $\tilde{p} = \tilde{\sigma}(p_{\bar{M}})$

But then $\overline{\overline{\tilde{\gamma}}} \leq \beta$ in W.

Contradiction! since $\tilde{\gamma} = \beta^{+W}$.

QED