§3.3.3 The "Frequent Extension" Lemma

We now prove another theorem on the existence of imbedding extensions, showing that under certain conditions ID is "very frequently" well founded.

Let $\alpha > \omega_1$ be regular. Let $\gamma > \alpha$, $\overline{\overline{\gamma}} = \alpha$. Let $P = J_\gamma^A$ be acceptable. Let $\langle \alpha_\xi \mid \xi \leq \alpha \rangle$ be normal s.t. $\alpha_\alpha = \alpha$. Let $P_\xi = J_{\gamma_\xi}^{A_\xi}$ where $P_\alpha = P$ and $\alpha_\xi < \gamma_\xi < \alpha$ for $\xi < \alpha$. Let commuting maps $\sigma_{\xi\zeta}$ $(\xi \leq \zeta \leq \alpha)$ be given s.t.

(a) $\sigma_{\xi\zeta} : P_\xi \longrightarrow_{\Sigma_0} P_\zeta$

(b) $\alpha_\xi = \text{crit}(\sigma_{\xi\zeta})$ $(\xi < \zeta)$

(c) $P_\lambda = \bigcup_{\xi < \lambda} \text{rng}(\sigma_{\xi\lambda})$ for $\text{Lim}(\lambda)$

Let $S \subset \alpha$ be a stationary set of points of cofinality $> \omega$. For $\xi \in S$ let $\alpha_\xi < \nu_\xi \leq \gamma_\xi$ and set: $\nu_\xi^* = \sup \sigma_{\xi\alpha}{}''\nu_\xi$, $\sigma_\xi = \sigma_{\xi\alpha} \restriction \nu_\xi$.

Set: $Q_\xi = J^{A_\xi}_{\nu_\xi}$, $Q^*_\xi = J^{A}_{\nu^*_\xi}$.

Then $\sigma_\xi : Q_\xi \longrightarrow_{\Sigma_0} Q^*_\xi$ cofinally.

Suppose moreover that:

(d) Q_ξ has a largest cardinal β_ξ , and that for each $\xi \in S$ we have chosen a mouse M_ξ which "end extends" Q_ξ — i.e.

(e) $Q_\xi = J^{E}_{\nu_\xi}$ where $M_\xi = \langle J^E_\gamma , F \rangle$

(f) $\nu_\xi = On \cap M_\xi$ or ν_ξ is regular in M_ξ.

We also assume:

(g) M_ξ is bound to Q_ξ (i.e. if $\omega\rho^{n+1}_{M_\xi} < \nu_\xi \leq \omega\rho^{n}_{M_\xi}$, then M_ξ is the closure of some $\nu_\xi \cup \{x\}$ under good $\Sigma^{(n)}_1$ functions),

For each $\xi \in S$ we form $\mathbb{D}_\xi$ in the attempt to produce a canonical extension of $\sigma_\xi : Q_\xi \rightarrow Q^*_\xi$ wrt M_ξ.

Lemma 1 There is a cub $C \subset \alpha$ s.t. $\mathbb{D}_\xi$ is well founded for $\xi \in C \cap S$.

The proof is rather lengthy and will require some sublemmas. Suppose not. Let γ be minimal for counterexamples. Let $S' \subset S$ be stationary s.t. $\mathbb{D}_\xi$ is not well founded for $\xi \in S'$. Let ν_ξ be chosen minimally for the existence of M_ξ with this property and let M_ξ be $<^*$-minimal (in the sense of §2.4) for this property at ν_ξ.

Sublemma 1.1 Let $\xi \in S'$. Let $Z \subset M_\xi$ be countable. There is a sequence $g_i \in \Gamma_\xi = \Gamma_{Q_\xi, M_\xi}$ $(i < \omega)$ s.t., letting $X = \bigcup_i \mathrm{rng}(g_i)$:

(a) $Z \subset X$

(b) $\nu_\xi \subset X$ + $\nu_\xi \in X$ if $\nu_\xi \in M_\xi$

(c) There is $\pi : N \xrightarrow{\sim} \langle X, \in, E \cap X, F \cap X \rangle$ s.t. N is a mouse and $\pi : N \longrightarrow_{\Sigma^*} M_\xi$ (where $M_\xi = \langle J^E_\gamma, F \rangle$

pf. of Sublemma 1.1

First select $\langle a_i, f_i \rangle$ $(i<\omega)$ which witness the failure of well foundedness in $\mathbb{D}_\zeta$ — i.e.

$\langle a_{i+1}, f_{i+1} \rangle \in \langle a_i, f_i \rangle$ in $\mathbb{D}_\zeta$.

Pick $p_i \in M_\zeta$ s.t. either $f_i = p_i \in M_\zeta$ or f_i is a good $\Sigma_1^{(n)}(M_\zeta)$ function in the parameter p_i (by a functionally absolute definition), where $\nu_\zeta \leq \omega\rho^{n+}_{M_\zeta}$,

Assume (w.l.o.g) that

$\alpha_\zeta, \beta_\zeta, p_{M_\zeta}, p_i \in Z$ $(i<\omega)$ and

$\nu_\zeta \in Z$ if $\nu_\zeta \in M_\zeta$.

Assume (w.l.o.g.): $\alpha_\xi, \nu_\xi, P_{M_\xi}, P_i \in Z$ $(i<\omega)$

We proceed by cases as follows:

<u>Case 1</u> $\nu_\xi \leq \omega\rho^n_{M_\xi}$ for all n.

Let $\langle F_i \mid i<\omega\rangle$ enumerate the fcns which, for some n, are $\Sigma_1^{(n)}(M_\xi)$ to H^n (lightface), with ω many repetitions of each fcn. Let $\langle z_i\rangle$ enumerate Z.

Define g_i by ind. on i as follows:

$g_{3i} = \{\langle z_i, 0\rangle\}$

g_{3i+1} is defined by:

$g_{3i+1}(\langle \xi, \zeta\rangle) \simeq F_{(i)_0}(g_{(i)_1}(\xi), g_{(i)_2}(\zeta))$

where $i = \langle (i)_0, (i)_1, (i)_2\rangle$.

(Thus $\mathrm{dom}(g_{3i+1})$ is a bounded subset of ν_ξ)

$g_{3i+2} = \mathrm{id} \upharpoonright \tau_i$

where τ_i = the least $\tau_i > \sup_{h<i} \tau_i$ s.t.

s.t. $\nu_\xi \cap \mathrm{rng}(g_h) \cup \mathrm{dom}(g_h) \subset \tau_h$ for $h \le i$.

. . . . . .

Note that $\mathrm{dom}(g_i)$ is a bounded subset of ν_i for $i < \omega$.

Let $X = \bigcup_i \mathrm{rng}(g_i)$. Then

(1) $X \cap \nu_\xi = \bigcup_i \mathrm{dom}(g_i) = \bigcup_i \tau_i \subseteq \nu_\xi$

Set $\bar{\nu} = X \cap \nu_\xi$.

(2) $F''X^n \subset X$ for all good $\Sigma^*(M_\xi)$ – function.

Set: $\tilde{X} = \langle X, \in, E^{M_\xi} \cap X, E^{M_\xi}_\gamma \cap X \rangle$.

Then:

(3) $\tilde{X} \prec_{\Sigma^*} M_\xi$, where x^n is taken as ranging over $X \cap H^n_{M_\xi}$ in $\tilde{X}$.

Let $\pi : N \widetilde{\longleftrightarrow} \tilde{X}$ where $N = \langle J^{\bar{E}}_{\bar{\gamma}}, \bar{E}_{\bar{\gamma}} \rangle$ is transitive. Then $\bar{\nu} = \beta_\xi^{+N}$ and

(4) $\pi \upharpoonright \bar{\nu} = \mathrm{id}$ and $\pi(\bar{\nu}) = \nu_\xi$ if $\nu_\xi \in M_\xi$

Set: $H_n = \pi^{-1}{}''(X \cap H^n_{M_\xi})$,

Then $H_n = |J^{\bar{E}}_{\rho^n}|$, where $\omega\rho^n = \mathrm{On} \cap H_n$. Clearly:

(5) $\pi : N \to_{\Sigma^*} M_\xi$, if x^n is taken as ranging over H_n in N.

Thus to prove that π is really Σ^*-preserving we must show:

We use induction on n →

<u>Claim</u> $\rho^n = \rho^n_N$ $(1 \leq n < \omega)$

$(\leq)$ Let $\bar{A}$ be $\underset{=}{\Sigma}_1^{(n-1)}(N)$. $y^n \simeq A \cap x^n$ is a $\underset{=}{\Sigma}_0^{(n)}(N)$ fcn. Since $\sigma : N \to_{\Sigma^*} M_\xi$ under the above interpretation, we conclude that $x \cap A \in H_n$ for $x \in H_n$

$(\geq)$ Let $p = p_{M_\xi}$, $\bar{p} = \pi^{-1}(p)$. Let $\bar{A}$ be $\Sigma_1^{(n-1)}(N)$ in $\bar{p}$ by the same def. as A in p over M_ξ. Then $\bar{A} \subset H_n$. <u>Claim</u> $\bar{A} \notin N$.

Suppose not. Let $\bar{A} = x$. Then $\bar{A} = x \cap H_n$ is a $\Pi_1^{(n)}(N)$ statement in $x, \bar{p}$. Hence the same statement holds in M_ξ in $\pi(x), p$. Hence $A = \pi(x) \cap H^n_{M_\xi} \in M_\xi$. Contr! QED (Claim)

Thus $\pi : N \to_{\Sigma^*} M_\xi$ & hence N is a mouse.

Set: $\bar{Q} = J_{\bar{\nu}}^{A_\xi}$, $\bar{\nu}^* = \sup \sigma_\xi{}''\bar{\nu}$, $\bar{Q}^* = J_{\bar{\nu}^*}^{A}$;
$\bar{\sigma}_\xi = \sigma_\xi \restriction \bar{Q}$. Then

(5) $\bar{\sigma} : \bar{Q} \to_{\Sigma_0} \bar{Q}^*$ cofinally and
$\bar{\nu}$ is regular in N, where
$\bar{Q} = J_{\bar{\nu}}^{E^N}$.

Hence we can form $\bar{\mathbb{D}} = \mathbb{D}_{\bar{\sigma}, \bar{Q}, N}$.
Let $\bar{f}_i$ be defined in N from
$\bar{p}_i = \pi^{-1}(p_i)$ as f_i was defined
in M_ξ from p_i. Then

(6) $\pi(\bar{f}_i) = f_i$
(using obvious notation).

(7) $\{\langle u, v\rangle \mid \bar{f}_{i+1}(u) \in \bar{f}_i(u)\} = \{\langle u, v\rangle \mid f_{i+1}(u) \in f_i(v)\}$.

But then $\bar{\mathbb{D}}$ is not well founded.
By the minimal choice of ν_ξ
it follows that:

(8) $\bar{\nu} = \nu_\xi$.

The remaining verification are
trivial. QED (Case 1)

<u>Case 2</u> $\omega\rho^1_{M_\xi} < \nu_\xi$

Let $\bigvee y\, \varphi_i(y, x, z)$ $(i<\omega)$ enumerate the Σ_0 fmlas w. two free vbls.

Define g_i, $\tau_i < \nu_\xi$, $\eta_i < On \cap M_\xi$ by induction on i:

$g_{3i} = \{\langle z_i, 0\rangle\}$

$g_{3i+1}(\langle \xi, \zeta\rangle) \simeq$ the $<_{M_\xi}$ - least $y \in S^{E^{M_\xi}}_{\eta_i}$ s.t.

$$\varphi_{(i)_0}(y, g_{(i)_1}(\xi), g_{(i)_2}(\zeta))$$

where η_i = least $\eta > \sup_{h<i} \eta_h$ s.t. $z_i \in S^E_\eta$ and $E \cap S^E_{\eta_h} \in S^E_\eta$, $F \cap S^E_{\eta_h} \in S^E_\eta$ $(h<i)$, where $M_\xi = \langle J^E_\nu, F\rangle$.

$g_{3i+2} = \mathrm{id} \restriction \tau_i$; τ_i = least $\tau > \sup_{h<i} \tau_i$ s.t. $\mathrm{dom}(g_h) \subset \tau$ and $Q \cap \mathrm{rng}(g_h) \subset J^E_\tau$ for $h<i$.

Set $\omega\eta = \sup_i \eta_i$. Then $\overline{M} = \langle J^E_\eta, F \cap J^E_\eta \rangle$ is a J-model + , in fact, a premouse. Hence $\overline{M} = M_\xi \| \eta$ by the initial segment property. Hence $\overline{M}$ is a mouse. But $f_i = p_i \in \overline{M}$ for $i < \omega$. Hence $\mathbb{D}_{\sigma_\xi, \overline{M}}$ is not well founded,

Hence $\overline{M} = M_\xi$ and $\eta = \gamma$ by the $<_*$ - minimality of M_ξ.

Hence, letting $X = \bigcup_i \text{rng}(g_i)$:

(1) $X \prec_{\Sigma_0} M_\xi$ cofinally

Let $\pi : N \leftrightarrow X$, N trans.

Set: $\bar{\nu} = \sup_i \tau_i = \sup_i (\nu_\xi \cap \text{rng}(g_i)) = \bigcup_i \text{dom}(g_i)$. Then

(2) $\pi \restriction \bar{\nu} = \text{id}$, $\pi(\bar{\nu}) = \nu_\xi$, $\nu > \omega\rho^1_{M_\xi}$

Since $H^n_{M_\xi} = J_{\rho^n_{M_\xi}} \in J_\nu \subseteq N$, we clearly have:

(3) $\pi : N \to_{\Sigma^*} M_\xi$, interpreting x^n as ranging over $H^n_{M_\xi}$.

Just as in the previous case we then get:

(4) $\rho^n_{M_\zeta} = \rho^n_N \quad (n < \omega)$.

Hence $\pi : N \to_{\Sigma^*} M_\zeta$ and N is a mouse. But, letting $\pi(\bar{f}_i) = f_i \quad (i < \omega)$, we have $\{\langle u, v\rangle \mid \bar{f}_{i+1}(u) \in \bar{f}_i(v)\} =$
$= \{\langle u, v\rangle \mid f_{i+1}(u) \in f_i(v)\} \dagger$
hence $\bar{\mathbb{D}}$ is not well founded where $\bar{\mathbb{D}}$ is based on $\bar{\sigma} : J^E_{\bar{\nu}} \longrightarrow J^E_{\bar{\nu}^*}$ ($\bar{\nu}^* = \sup \sigma_\zeta{}''\bar{\nu}$, $\bar{\sigma} = \sigma_\zeta \upharpoonright J^E_{\bar{\nu}}$). By the minimal choice of ν_ζ we conclude:

(5) $\nu_\zeta = \bar{\nu}$. QED (Case 2)

<u>Case 3</u> The above cases fail.

Then $\omega\rho^{n+1}_{M_\xi} < \nu_\xi \leq \omega\rho^{n}_{M_\xi}$ for an $n > 0$.

Let $r \in M_\xi$ witness that M_ξ is bound to ν_ξ.

We combine the methods of the previous cases. Let $\langle F_i \mid i < \omega \rangle$ enumerate the good $\Sigma_1^{(h)}(M_\xi)$ fcns of two variables for $h < n$, with ω many repetitions of each.

Let $\langle \bigvee y\, \varphi_i(y, x, z) \mid i < \omega \rangle$ enumerate the $\Sigma_0^{(n)}$ fmlas, again with ω repetitions, let the Σ_1 Skolem function h be uniformly defined by:

$$y = h(i, x) \text{ iff } \bigvee z\, \Psi(i, z, y, x)$$

where Ψ is Σ_0. We define g_i, $\tau_i < \nu_\xi$, $\eta_i < \omega\rho^n_{M_\xi}$ by ind. on i as follows:

$g_{4i} = \{\langle z_i, 0 \rangle\}$

$g_{4i+1}(\langle \xi, \zeta \rangle) \simeq F_{(i)_0}(g_{(i)_1}(\xi), g_{(i)_2}(\zeta))$

$g_{4i+2}(\langle \xi, \zeta \rangle) \simeq$ the $<_{M_\xi}$-least $y \in J^E_{\eta_{(i)_0}}$

s.t. $M_\xi \models \varphi_i(y, g_{(i)_1}(\xi), g_{(i)_2}(\zeta))$,

where:

η_i = the least $\eta > \sup_{h<i} \eta_h$ s.t.

(a) $A^{m\,r}_{M_\xi} \cap J^E_{\eta_h} \in J^E_\eta$ for $h < i$

(b) $\bigvee j \bigvee \rho < \beta_\xi \bigvee z \in J^E_\eta$ $M_\xi \models \Psi(j, z, \tau_h, \langle \rho, r$

for all $h < i$

(c) $\bigvee j \bigvee \rho < \beta_\xi \bigvee z \in J^E_\eta$ $M_\xi \models \Psi(j, z, s_h, \langle \rho,$

for $h < i$, where $s_h \in J^E_\eta$ is

such that $g_h = s_h$ or g_h is

a good $\Sigma_1^{(k)}$ function in

the parameter r, s_h (by

a func. absolute def.).

This is possible by our condition on r.

$g_{4i+3} = \mathrm{id} \restriction \tau_i$, where τ_i = the least

$\tau > \sup_{h<i} \tau_h$ s.t. $\mathrm{dom}(g_h) \subset \tau$ and

$Q_\xi \cap \mathrm{rng}(g_h) \subset J^E_\tau$ for $h < i$.

Set $\omega\rho^h = On \cap H_h$ $(h \leq n)$

(4) $\rho^h = \rho^h_{\bar{N}}$ $(h \leq n)$

proof.

For $h < n$ exactly as in Case 1.

$\rho^n \leq \rho^n_{\bar{N}}$, since, letting $\bar{A} =$

$= \bigcup_i \pi^{-1}(A^{n,r}_{M_\xi} \cap J^E_{\gamma_i})$, we have

$\langle H_n, \bar{A} \rangle$ is cofinal and

$\pi \upharpoonright H_n : \langle H_n, \bar{A} \rangle \to_{\Sigma_0} M^{n,r}_\xi$,

it follows easily that

$\bar{A} = A^{n\bar{r}}_{\bar{N}} \cap H_n$, where $\pi(\bar{r}) = r$.

Finally, we note that

$\bar{\nu} \subset h_{\langle H_n, \bar{A} \rangle}(\beta_\xi)$,

where $\bar{\nu} = \beta_\xi^{+\bar{N}}$. Hence

$\rho^n_{\bar{N}} \leq \rho^n$. QED(4)

The proof of (4) also shows:

(5) $\rho^{n+1}_{\bar{N}} < \bar{\nu} \leq \rho^n_{\bar{N}}$ and

$\bar{N}$ is $\bar{\nu}$-clear (in the sense of §3.3.2).

Then $\langle H_n, \bar{A} \rangle$ is amenable and

$$\pi \restriction H_n : \langle H_n, \bar{A} \rangle \to_{\Sigma_0} M_{\bar{\xi}}^{n,\alpha}.$$

It follows easily that $\bar{A} = A_{\bar{N}}^{n\bar{\alpha}} \cap H_n$, where $\pi(\bar{\alpha}) = \alpha$. This proves $\rho^n \leq \rho_{\bar{N}}^n$. But by construction, $\bar{\nu} \cap h_{\langle H_n, \bar{A} \rangle}(\beta_\xi)$ is cofinal in $\bar{\nu}$. Hence $\bar{A} \notin \bar{N}$, since $\rho_{\bar{N}}^n \geq \rho^n \geq \bar{\nu}$. and $\bar{\nu} = \beta_\xi^{+\bar{N}}$. Hence $\rho_{\bar{N}}^n = \rho^n$.

QED (4).

The proof of (4) shows:

(5) $\rho_{\bar{N}}^{n+1} < \bar{\nu} \leq \rho_{\bar{N}}^n$ and $\bar{N}$ is $\bar{\nu}$-clean (in the sense of §3.3.2).

But if $\pi(\bar{g}_i) = g_i$ $(i < \omega)$, then by our construction, for each $\bar{g}_i$ there is $s_i \in h_{\bar{N}^{n,\bar{\alpha}}}(\beta_\xi)$ s.t. either $\bar{g}_i = s_i$ or $\bar{g}_i$ is a good $\Sigma_1^{(n)}(\bar{N})$ function in $s_i, \bar{\alpha}$. Hence:

(6) $\bar{N}$ is bound to $\bar{\nu}$ as witnessed by $\bar{\alpha}$.

Now let $\bar{Q} = J^{E}_{\bar{\eta}}$, $Q = J^{E}_{\eta}$, where $\bar{\eta} = \rho^{n}_{\bar{N}}$, $\eta = \sup \pi''\bar{\eta}$. Then

$$\pi \restriction \bar{Q} : \bar{Q} \longrightarrow_{\Sigma_0} Q \text{ cofinally.}$$

By the interpolation lemma there is $\pi_0 : \bar{N} \to N$ canonically extending $\pi \restriction \bar{Q}$ w.r.t. $\bar{N}$. Since $\bar{N}$ is bound to $\omega\rho^{n}_{\bar{N}}$ as witnessed by $\bar{\pi}$ (by (6)), we have:

(7) $\pi_0 : \bar{N} \longrightarrow_{\Sigma_0^{(n)}} N$ cofinally.

Moreover there is a unique π_1 s.t.

(8) $\pi_1 : N \longrightarrow_{\Sigma_0^{(n)}} M_3$, $\pi_1 \circ \pi_0 = \pi$.

By (8), $\pi : N \longrightarrow_{\Sigma_1^{(n-1)}} M_3$, hence by §2.2 Lemma 2:

(9) N is iterable above $\omega\rho^{n}_{N}$.

Straightforward use of the initial segment property gives:

(10) $E^{N}_{\eta} = E^{M_3}_{\eta}$.

Hence N, M_3 are coiterable. Let R be a common iterate of of N, M_3 and a simple iterate of one. If R is a non simple iterate of M_3, then N is a mouse and $N <_* M_3$. But by our construction

(11) $\sigma_3 : Q_3 \to Q_3^*$ has no canonical extension wrt. N,

contradicting the $<_*$-minimality of M_3. If R is a non simple iterate of N, then, letting $r_0 = \pi_0(\bar{r}) = \pi_1^{-1}(r)$, we have:

$$A_N^{n,r} = A_{M_3}^{n,r} \cap J_\eta^E \in N.$$

This is a contradiction, since $r_0 \in R_N^n$ by §3.3.2 Lemma 4. Hence R is a simple iterate of both M_3 and N, and is an iterate of N above $\eta = \omega\rho_N^n$. Hence $\eta = \omega\rho_{M_3}^n$. Hence, since $\pi = \pi_1 \pi_0$:

(12) $\pi : \bar{N} \longrightarrow_{\Sigma_0^{(n)}} M_3$ cofinally

It follows just as in Case 2 that

(13) $\pi : \bar{N} \longrightarrow_{\Sigma^*} M_3$ and

$$H_{\bar{N}}^h = H_{M_3}^h \text{ for } h \geq n.$$

Then $\bar{N}$ is a mouse. Obviously $\sigma_\xi \upharpoonright J^E_{\bar{\nu}}$ cannot be extended w.r.t. $\bar{N}$. Hence, as before, $\bar{\nu} = \nu_\xi$ by minimality.

QED (Sublemma 1.1)

Note that, by this analysis, $cf(\nu_\xi) = \omega$ for $\xi \in S'$. Recall that we had defined: $Q_\xi = J^{A_\xi}_{\nu_\xi}$ where $P_\xi = J^{A_\xi}_{\gamma_\xi}$ and $\alpha_\xi < \nu_\xi \leq \gamma_\xi$; and $\sigma_\xi = \sigma_{\xi\alpha} : Q_\xi \xrightarrow[\Sigma_0]{} Q^*_\xi$ cofinally, where $\sigma_{\xi\alpha} : P_\xi \xrightarrow[\Sigma_0]{} P = P_\alpha$. We now get:

<u>Sublemma 1.2</u>. There is a club $C \subset \alpha$ s.t. for all $\xi \in C \cap S'$ we have: $Q_\xi = P_\xi$ and $Q^*_\xi = P$ (hence $\sigma_\xi = \sigma_{\xi\alpha}$ is cofinal and $\sigma_{\xi\zeta}(\beta_\xi) = \beta_\zeta$)

prf.

We first show that $\nu_\xi = \gamma_\xi$ for $\xi \in C \cap S'$, $C \subset \alpha$ a club set. Suppose not. Then $\nu_\xi < \gamma_\xi$

for $\xi \in S'' \subset S'$, where S'' is stationary.

Then for $\xi \in S''$ we have $\nu_\xi = \sigma_{\bar{\xi}\xi}(\bar{\nu})$, $\bar{\xi} < \xi$. It follows easily that there is a stat. S''' + fixed $\bar{\xi}$, $\bar{\nu}$ s.t.

$\nu_\xi = \sigma_{\bar{\xi}\xi}(\bar{\nu})$ for $\xi \in S'''$. Set:

$\nu^* = \sigma_{\xi\alpha}(\nu_\xi) = \sigma_{\bar{\xi}\alpha}(\bar{\nu})$ for

$\xi \in S''$. Then we can get a counterexample to Lemma 3 with ν^* in place of $\gamma > \nu^*$, contradicting the minimality of γ.

Thus $\nu_\xi = \gamma_\xi$ for $\xi \in C \cap S'$. But ν_ξ is ω-cofinal. It follows easily that $\sigma_{\xi\alpha}$ is cofinal into γ for sufficiently large $\xi < \alpha$.

QED (Sublemma 1.2)

From now on assume w.l.o.g. that $Q_\xi = P_\xi$, $Q^*_\xi = P$ for $\xi \in S'$.

We also set: $Q =_{df} P$.

We also note that we may assume w.l.o.g. that

(*) $E_{\nu_\xi}^{M_\xi} = \emptyset$ for $\xi \in S'$.

To see this, note that otherwise we may replace M_ξ by M' where $k : M_\xi \xrightarrow[E_{\nu_\xi}]{*} M'$. If $\langle f_n \mid n < \omega \rangle$ witnesses the non well foundedness of $\mathbb{D}_{\sigma_\xi}, M_\xi$, then $k \circ f_n \in M'$ $(n < \omega)$ by standard methods (since $\mathrm{dom}(f_n) \leq \kappa_\xi$ in M_ξ) and $\langle k \circ f_n \mid n < \omega \rangle$ witnesses the non well foundedness of $\mathbb{D}_{\sigma_\xi}, M'_\xi$.

From now on assume (*).

For $\xi \in S'$ let $\langle g_i^\xi \mid i<\omega \rangle$ be the sequence of Sublemma 1.1 (in particular, we suppose it to be defined as in the proof of Sublemma 1.1). We recall that the def. of $\langle g_i^\xi \rangle$ made use of a sequence $\langle f_i^\xi \rangle, \langle a_i^\xi \rangle$ which witnessed the non-well foundedness of $\mathbb{D}_\xi$ – i.e.

$$\langle a_{i+1}^\xi, f_{i+1}^\xi \rangle \in \langle a_i^\xi, f_i^\xi \rangle \quad (i<\omega).$$

Let $\xi^* =$ the least $\xi^* > \xi$ s.t. $a_i^\xi \in \mathrm{rng}(\sigma_{\xi^* \alpha})$ for all $i<\omega$.

Then if $\xi, \zeta \in S'$, $\xi^* \leq \zeta$, then $\sigma_{\xi\zeta} : Q_\xi \longrightarrow Q_\zeta$ has no canonical extension wrt. M_ξ. Define a cub $C \subset \alpha$ by:

$C =$ the set of limit pts λ of S' s.t. $\xi^* < \lambda$ for $\xi \in S' \cap \lambda$.

Fix $\lambda \in C \cap S'$. We shall define a countable $W \subset Q_\lambda$ and observe that $W \subset \mathrm{rng}(\sigma_{\xi\lambda})$ for a $\xi \in S' \cap \lambda$, since $\mathrm{cf}(\lambda) > \omega$. We then use this to derive a contradiction.

We shall use the machinery developed in the proof of Sublemma 1.1 and shall also consider the same three cases.

<u>Case 1</u> $\nu_\lambda \leq \omega\rho^n_{M_\lambda}$ for all $n < \omega$.

For each Σ^* fmla $\varphi = \varphi(v_1^{j_1}, \ldots, v_m^{j_m})$ and each $i_1, \ldots, i_m < \omega$, set:

$t_{\varphi, \vec{i}}$ = the set of $\langle x_1, \ldots, x_m \rangle$ s.t. $x_h \in \mathrm{dom}(g_h)$, $g_h(x_h) \in H^h_{M_\lambda}$ $(h = 1, \ldots, m)$ and $M_\lambda \models \varphi[g_{i_1}(x_1), \ldots, g_{i_m}(x_m)]$,

where $\langle g_h \rangle$ are the canonical maps developed in the proof of Sublemma 1.1.

Let W be the set of all $t_{\varphi,\vec{i}}$ and suppose that $W \subset \text{rng}(\sigma_{\xi,\lambda})$ for a fixed $\xi \in S' \cap \lambda$.

Set: $d_i = \text{dom}(g_i)$

$e_{ij} = \{\langle x,y\rangle \mid g_i(x) \in g_j(y)\}$

$I_{ij} = \{ \quad \text{"} \quad \mid \quad = \quad \}$

$E_i = \{x \mid g_i(x) \in E\}$, $F_i = \{x \mid g_i(x) \in F\}$

where $M_\lambda = \langle J_\beta^E, F\rangle$. Then

$d_i, e_{ij}, I_{ij}, E_i, F_i \in \text{rng}(\sigma_{\xi\lambda})$.

Set: $\tilde{D} = \{\langle i,x\rangle \mid x \in d_i\}$ and define $\tilde{e}, \tilde{I}, \tilde{E}, \tilde{F}$ on $\tilde{D}$ by:

$$\langle i,x\rangle \tilde{e} \langle j,y\rangle \text{ iff } \langle x,y\rangle \in e_{ij}$$

(sim. for $\tilde{I}, \tilde{E}, \tilde{F}$). Set:

$$\tilde{\mathbb{D}} = \langle \tilde{D}, \tilde{e}, \tilde{I}, \tilde{E}, \tilde{F}\rangle.$$

Then $k : \tilde{\mathbb{D}} \xrightarrow{\sim} M_\lambda$ where $k(\langle i,x\rangle) = g_i(x)$

Moreover, if we interpret v^n as ranging over $\tilde{H}_n = \{\langle i,x\rangle \mid g_i(x) \in H_n^{M_\lambda}\}$ in $\tilde{\mathbb{D}}$, we have $k : \tilde{\mathbb{D}} \prec_{\Sigma^*} M_\lambda$.

Now let $\sigma_{\zeta\lambda}(\bar{d}_i, \bar{e}_{ij}, \bar{E}_i, \bar{F}_j) =$
$= d_i, e_{ij}, E_i, F_j$ and define:

$$\bar{\mathbb{D}} = \langle \bar{D}, \bar{e}, \bar{I}, \bar{E}, \bar{F} \rangle$$

from $\bar{d}_i, \bar{e}_{ij}, \bar{E}_i, \bar{F}_j$ as $\mathbb{D}$ was defined from d_i, e_{ij}, E_i, F_j. Define a pseudo satisfaction relation on $\bar{\mathbb{D}}$ by:

$$\Vdash \varphi[\langle i_1, x_1\rangle, \ldots, \langle i_n, x_n\rangle] \text{ iff}$$

$$\text{iff } \langle x_1, \ldots, x_n\rangle \in \bar{t}_{\varphi, \vec{i}},$$

where $\sigma_{\zeta\lambda}(\bar{t}_{\varphi,\vec{i}}) = t_{\varphi,\vec{i}}$.

Using $\sigma_{\zeta\lambda} : Q_\zeta \to_{\Sigma_0} Q_\lambda$ cofinally, it is easily seen that fr. im.

$$\Vdash (\bigvee v_j \varphi)[\vec{z}] \longleftrightarrow$$

$$\longleftrightarrow \bigvee x \in \bar{H}_j \Vdash \varphi[x, \vec{z}],$$

where $\bar{H}_j = \sigma_{\zeta\lambda}^{-1}{}''\tilde{H}_j$. It follows easily that $\sigma_{\xi\lambda} : \bar{\mathbb{D}} \longrightarrow_{\Sigma^*} \tilde{\mathbb{D}}$, letting v^n range over $\bar{H}_n$, $\tilde{H}_n$ resp. in $\bar{\mathbb{D}}$, $\tilde{\mathbb{D}}$. [Note: We could in fact show that $\sigma_{\xi\lambda}$ is $\Sigma_2^{(n)}$ preserving in this sense for $n < \omega$]. But then $\bar{\mathbb{D}}$ is well founded and satisfies extensionality. Hence there is $\bar{k} : \bar{\mathbb{D}} \stackrel{\sim}{\longrightarrow} \bar{N}$, where $\bar{N}$ is transitive. Set: $\pi = k\sigma_{\xi\lambda}\bar{k}^{-1}$. Then $\pi : \bar{N} \longrightarrow_{\Sigma^*} M_\xi$, letting v^n range over $H_n = \bar{k}``\bar{H}_n = \pi^{-1}``H^n_M$ in $\bar{N}$. Since $g_{3i+2} = \mathrm{id} \restriction \tau_i \in \mathrm{rng}(\sigma_{\xi\lambda})$, and $\nu_\xi = g_{3i}(0) \in \mathrm{rng}(\pi)$ for some i, we have: $\nu_\xi + 1 \subset \bar{N}$ and $\pi(\nu_\xi) = \nu_\lambda$, $\pi \restriction \nu_\xi = \sigma_{\xi\lambda} \restriction \nu_\xi$. Hence $Q_\xi = J^{\bar{E}}_{\nu_\xi}$, where $\bar{N} = \langle J^{\bar{E}}_{\bar{\beta}}, \bar{F} \rangle$ and $\pi \restriction Q_\xi = \sigma_{\xi\lambda}$. Now let $H_n = J^{\bar{E}}_{\omega\rho^n}$. Exactly as in the proof of

Sublemma 1.1 Case 1, we get: $\rho^n = \rho^n_{\bar{N}}$ $(n < \omega)$. Hence

$\pi : \bar{N} \longrightarrow_{\Sigma^*} M_\lambda$ and $\bar{N}$ is a

mouse. Form $\mathbb{D}_{\sigma_\xi, \bar{N}}$ in the usual way. Clearly there are $\bar{f}_n \in$ $\in \Gamma_{Q_\xi, \bar{N}}$ s.t. $\pi(\bar{f}_n) = f_n$ &

hence $\langle a_{n+1}, a_n \rangle \in \sigma_\xi(\{\langle x, y\rangle \mid \bar{f}_{n+1}(x) \in \bar{f}_n(y)\})$ for $n < \omega$. Hence $\mathbb{D}_{\sigma_\xi, \bar{N}}$ is not well founded & σ_ξ has no extension wrt. $\bar{N}$.

Now coiterate $\bar{N}, M_\xi$ to $\bar{N}', M'$. The first point moved is $\geq \nu_\xi$. Hence, by a trivial argument, $\mathbb{D}_{\sigma_\xi, \bar{N}'}$ and $\mathbb{D}_{\sigma_\xi, M'}$ are well founded. But then $\bar{N}'$ is not a proper initial segment of M' and M' is a simple iterate of M_ξ, since otherwise $\bar{N} <_* M_\xi$, contradicting the $<_*$-minimality of M_ξ. But there is an iterate N' of M_λ and a map $\pi' : \bar{N}' \underset{\Sigma^*}{\longrightarrow} N'$ s.t. $\pi' \pi_{\bar{N}\bar{N}'} = \pi_{M_\xi N'} \bar{\pi}$. Hence $\pi' \pi_{M_\xi M'} : M_\xi \underset{\Sigma^*}{\longrightarrow} M''$ for some initial segment M'' of N'. Hence $\mathbb{D}_{\sigma_{\xi\lambda}, M_\xi}$ is well founded, contradicting the def. of C. Contr! QED (Case 1)

<u>Case 2</u> $\omega\rho^1_{M_\lambda} < \nu_\lambda$.

W = the set of $t_{\varphi, \vec{i}}$ s.t. $\varphi = \varphi(v_1, \ldots, v_n)$ is a Σ_0-fmla. Define $\tilde{\mathbb{D}}$, k as before. Then $d_i, e_{ij}, I_{ij}, E_i, F_i \in$ $\in \mathrm{rng}(\sigma_{\xi\lambda})$ + we can define $\bar{\mathbb{D}}$ as before. As before we get: $\sigma_{\xi\lambda} : \bar{\mathbb{D}} \underset{\Sigma_0}{\longrightarrow} \tilde{\mathbb{D}}$. But $\sigma_{\xi\lambda}$ is cofinal wrt. $\tilde{e}$, since there are arb. large $\eta < \nu_\lambda$ s.t. $g_{3i+1}(\xi) = J_\eta^E$ $(M_\lambda = \langle J_\beta^E, F\rangle)$ for appropriate ξ, i. Hence $\sigma_{\xi\lambda} : \mathbb{D} \longrightarrow_{\Sigma_1} \tilde{\mathbb{D}}$. As before, there is $\bar{k} : \bar{\mathbb{D}} \xrightarrow{\sim} \bar{N}$, $\bar{N}$ trans., & hence $\pi : \bar{N} \longrightarrow_{\Sigma_0} M_\lambda$ cofinally, where $\pi = k\sigma_{\xi\lambda}\bar{k}^{-1}$. As before, $Q \subseteq \bar{N}$, $\pi \restriction Q = \sigma_{\xi\lambda}$, and $\pi(\nu_\xi) = \nu_\lambda$. As before, $\mathbb{D}_{\sigma_\xi}, \bar{N}$ is not well founded. We can then form the $(*, 0)$-coiteration of $M_\xi, \bar{N}$, terminating in $M', \bar{N}'$. Exactly as before, M' is a segment of N' & a simple iterate of M_ξ, since otherwise $N' <_* M'$. But this yields a contradiction exactly as before. QED (Case 2)

<u>Case 3</u> $\omega\rho^{n+1}_{M_\lambda} < \nu_\lambda \leq \omega\rho^{n}_{\lambda}$ $(n > 0)$

Again let W be the set consisting of $t_{\varphi,\vec{i}}$ for $\Sigma_0^{(n)}$ formulae $\varphi = \varphi(\vec{v})$ and $\vec{i} < \omega$. Define $\tilde{\mathbb{D}}$, $\bar{\mathbb{D}}$ as before. Then $\sigma_{\zeta\lambda} \restriction \bar{\mathbb{D}} : \bar{\mathbb{D}} \longrightarrow_{\Sigma_0^{(n)}} \tilde{\mathbb{D}}$, interpreting v^h by the appropriate $\bar{H}^h, \tilde{H}^h$ in $\bar{\mathbb{D}}, \tilde{\mathbb{D}}$ resp. Again we get $\bar{k} : \bar{\mathbb{D}} \tilde{\longrightarrow} \bar{N}$, $\bar{N}$ transitive, and define $\pi = k\sigma_{\zeta\lambda}\bar{k}^{-1}$. Then $\pi : \bar{N} \longrightarrow_{\Sigma_0^{(n)}} M_\lambda$ cofinally, with v^h . . ranging over $H_h = \pi^{-1}{}''H^h_{M_\lambda}$ in $\bar{N}$. Again we have: $Q \subseteq \bar{N}$, $\pi \restriction Q = \sigma_{\zeta\lambda}$, $\pi(\nu_\zeta) = \nu_\lambda$. Letting $H_h = J^{\bar{E}}_{\rho^h}$ $(\bar{N} = \langle J^{\bar{E}}_{\beta}, \bar{F}\rangle)$, we get $\rho^n = \rho^n_{\bar{N}}$ by the methods of Sublemma 1.1 Case 3. We also get: $\bar{N}$ is ν_ζ-clear and bound to ν_ζ.

Hence $\omega\rho_{\bar{N}}^{n+1} \leq \beta_3 < \nu_3$ and $\bar{N}$ is iterable above β_3 by §2.2 Lemma 2. We coiterate $\bar{N}, M_3$ + derive a contradiction as before.

QED (Lemma 1)

We now improve Lemma 1 to

<u>Lemma 2</u> Let $\langle Q_3 \rangle, \langle M_3 \rangle$ etc be as in Lemma 1. There is a cub $C \subset \alpha$ s.t. for all $3 \in C \cap S$ we have:

(a) The canonical extension $\tilde{\sigma}_3 : M_3 \to M_3^*$ of σ_3 wrt. M_3 exists.

(b) M_3^* is iterable above ν_3^*.

The proof is virtually the same, once we have developed a small amount of additional machinery. Hence we first develop this machinery and then briefly sketch the modification to be made.

Suppose we are given a premouse M and an iteration $\langle M_i \mid i < \theta \rangle$ with indices $\langle \nu_i, \alpha_i \rangle$ and maps π_{ij}. Now let $\sigma : \bar{M} \longrightarrow_{\Sigma^*} M$. There is a canonical way of trying to "mirror" the iteration of M by an iteration $\langle \bar{M}_i \rangle$ of $\bar{M}$ with indices $\langle \bar{\nu}_i, \bar{\alpha}_i \rangle$ and maps $\bar{\pi}_{ij}$, simultaneously producing imbeddings $\sigma_i : \bar{M}_i \longrightarrow_{\Sigma^*} M_i$ s.t. $\sigma_i \bar{\pi}_{hi} = \pi_{hi} \sigma_h$.

The definitions are as follows:

$\sigma_0 = \sigma$. Now let $\bar{M}_i$, σ_i be defined. We attempt to define $\bar{\nu}_i$, $\bar{\alpha}_i$ by:

$$\bar{\nu}_i \simeq \begin{cases} 0 \text{ if } E_{\nu_i}^{M_i} = \emptyset \\ \sigma_i^{-1}(\kappa_i) + \bar{M}_i \text{ if not} \end{cases}$$

$$\omega\bar{\alpha}_i \simeq \begin{cases} On \wedge \bar{M}_i \text{ if } \omega\alpha_i = On \wedge M_i \\ \sigma_i^{-1}(\omega\alpha_i) \text{ if not.} \end{cases}$$

This defines $\bar{M}_{i+1}$, $\bar{\pi}_{i,i+1}$ and we define σ_{i+1} by:

$\sigma_{i+1} = \sigma_i \upharpoonright \bar{M}_{i+1}$ if $E_{\nu_i}^{M_i} = \emptyset$;

otherwise:

$\sigma_{i+1}(\bar{\pi}_{i,i+1}(f)(\bar{\kappa}_i)) = \pi_{i,i+1}\sigma_i(f)(\kappa_i)$

for $f \in \Gamma(\bar{\kappa}_i, \bar{M}_i \| \bar{\alpha}_i)$.

For limit λ define σ_λ by:

$\sigma_\lambda \bar{\pi}_{i\lambda} = \pi_{i\lambda}\sigma_i$.

. . . . .

It is clear that this definition can break down only at successor stages. We now formulate a condition which is sufficient to prevent breakdown.

Def Let M, $\langle M_i \mid i < \theta \rangle$ etc. be as above. We assign to each $x \in M_i$ an element $u_i(x)$ of M s.t. $\pi_{hi}(\bar{x}) = x \rightarrow u_i(x) = u_h(\bar{x})$. We define u_i by induction on i as follows:

$u_0(x) = x$, for $x \in M$.

$u_{i+1}(x) = u_i(\bar{x})$ if $\pi_{i,i+1}(\bar{x}) = x$.

If there is no such $\bar{x}$, let $x = \pi_{i,i+1}(f)(\kappa_i)$, where $f \in \Gamma(\kappa_i, M_i)$. Pick p s.t. either $f = p \in M_i$ or else f is a good $\Sigma_1^{(n)}(M_i \| \alpha_i)$ function in p (by a functionally abs. def.) for n s.t. $\kappa_i < \omega\rho^{n+1}_{M_i \| \alpha_i}$

Set: $u_{i+1}(x) = u_i(p)$.

For limit $\lambda < \theta$ set $u_\lambda(x) = u_i(\bar{x})$ for $i < \lambda$ s.t. $\pi_{i\lambda}(\bar{x}) = x$.

. . . .

This completes the definition.

<u>Sublemma 2.0</u> Let M, $\langle M_i \rangle$, $\bar{M}$, σ etc. be as above. Assume:

(a) $u_i(\kappa_i) \in \text{rng}(\sigma)$ for $E^{M_i}_{\kappa_i} \neq \emptyset$

(b) $u_i(\omega\alpha_i) \in \text{rng}(\sigma)$ for $\omega\alpha_i \in M_i$.

Then $\bar{M}_i$, σ_i are defined for $i < \theta$. Moreover:

$$u_i(x) \in \text{rng}(\sigma) \rightarrow x \in \text{rng}(\sigma_i)$$

for $x \in M_i$, $i < \theta$.

Proof. Induction on i.

The details are left to the reader.

Note that the construction of the "mirror" iteration $\langle \bar{M}_i \rangle$ + the maps σ_i goes thru under somewhat weaker assumptions than $\sigma : \bar{M} \underset{\Sigma^*}{\longrightarrow} M$. We can replace this by:

(*) $\sigma : \bar{M} \to M$ and $\bar{\nu} \in \bar{M}$ s.t.

(a) $\sigma(\bar{\nu}) = \nu$

(b) $\bar{\nu} < \omega\rho^n_{\bar{M}}$ iff $\nu < \omega\rho^n_M$ $(n < \omega)$

(c) $\sigma : \bar{M} \underset{\Sigma_1^{(n)}}{\longrightarrow} M$ whenever $\nu < \omega\rho^n_M$.

$\bar{M}_i$, σ_i are then defined as before, with $\sigma_i : \bar{M}_i \underset{\Sigma_1^{(n)}}{\longrightarrow} M_i$ whenever $\nu < \omega\rho^n_M$. (This is just as in §2.2 Lemma 2). Sublemma 2.0 continues to hold.

Now suppose Lemma 2 to be false + let $\langle Q_\zeta \mid \zeta \in S \rangle$, $\langle M_\zeta \mid \zeta \in S \rangle$ etc. be a counterexample. Let γ be chosen minimally, as before

By Lemma 1 we may suppose w.l.o.g. that $\sigma_\xi : Q_\xi \to Q^*_\xi$ has a canonical extension $\sigma^*_\xi : M_\xi \to M^*_\xi$ wrt M_ξ for $\xi \in S$. Let $S' \subset S$ be stationary s.t. M^*_ξ is not iterable above ν^*_ξ for $\xi \in S'$. We let ν_ξ be chosen minimally for M_ξ with this property and M_ξ be $<_*$-minimal for this property at ν_ξ. As before:

<u>Sublemma 2.1</u> Let $\xi \in S'$. Let $Z \subset M_\xi$ be countable. There is a sequence $q_i \in \Gamma_\xi = \Gamma_{Q_\xi, M_\xi}$ $(i < \omega)$ s.t.

(a) $Z \subset X$

(b) $\nu_\xi + 1 \subset X$

(c) There is $\pi : N \widetilde{\longleftrightarrow} \langle X, \in, E \cap X, F \cap X \rangle$ s.t. N is a mouse and $\pi : N \to_{\Sigma^*} M_\xi$ (where $M_\xi = \langle J^E_\beta, F \rangle$).

The proof is a straightforward modification of that of Sublemma 1.1. In place of the sequence $\langle a_i, f_i \rangle$ which formed a counterexample to well foundedness of $\mathbb{D}_\zeta$, we use a sequence $r_i = \sigma_\zeta^*(f_i)(a_i)$ which witnesses – in a manner still to be specified – the non iterability of M_ζ^*. $\langle r_i \mid i < \omega \rangle$ is the enumeration of a countable set $R \subset M_\zeta^*$ which is defined as follows: Let $\langle \tilde{M}_i \mid i < \theta \rangle$ be iteration of M_ζ^* above ν_ζ^* which cannot be continued. Let $\langle \tilde{\nu}_i, \tilde{\varkappa}_i \rangle$ be the indices of this iteration + $\tilde{\pi}_{ij}$ the maps.

<u>Case 1</u> $\theta = \gamma + 1$

Then there is $\omega\tilde{\alpha} \leq On \cap \tilde{M}$ and $\tilde{\nu} = \tilde{\varkappa}^{+\tilde{M}}$ with $\tilde{E} = E_{\tilde{\nu}}^{\tilde{M}} \neq \emptyset$ s.t. $\tilde{M}$ is not $*$-extendable by $\tilde{E}$.

In particular there are $g_i \in \Gamma(\tilde{\kappa}, \tilde{M})$ s.t. $\{\zeta \mid g_{i+1}(\zeta) \in g_i(\zeta)\} \in \tilde{E}$ for $i < \omega$. Pick $p_i \in \tilde{M}$ s.t. either $p_i = g_i \in \tilde{M}$ or else p_i is in a good $\Sigma_1^{(n)}(\tilde{M})$ map in p_i where $\tilde{\kappa} < \omega\rho_{\tilde{M}}^{n+1}$ (by a func. abs. def.). Set:

$\tilde{\alpha} = \tilde{\alpha}_\gamma$, $\tilde{\nu} = \tilde{\nu}_\gamma$, $\tilde{\kappa} = \tilde{\kappa}_\gamma$.

Let R = the set containing:

(a) $u_i(\tilde{\kappa}_i)$ for $i \leq \gamma$ s.t. $E_{\tilde{\nu}_i}^{\tilde{M}_i} \neq \emptyset$

(b) $u_i(\omega\tilde{\alpha}_i)$ for $i \leq \gamma$ s.t. $\omega\tilde{\alpha}_i \in \tilde{M}_i$

(c) $u_\gamma(p_i)$ $(i < \omega)$

<u>Case 2</u> $\lim(\theta)$.

Then there is a monotone sequence $\langle i_n \mid n < \omega \rangle$ and $x_n \in \tilde{M}_{i_n}$ s.t. $x_{n+1} \in \pi_{i_n, i_{n+1}}(x_n)$ for $n < \omega$.

Let R = the set containing

(a) $u_i(\tilde{\kappa}_i)$ for $i < \theta$ s.t. $E_{\tilde{\nu}_i}^{\tilde{M}_i} \neq \emptyset$

(b) $u_i(\omega\tilde{\alpha}_i)$ " " s.t. $\omega\tilde{\alpha}_i \in \tilde{M}_i$

(c) $u_{i_n}(x_n)$ $(n < \omega)$.

This gives us $s_i = \sigma_\xi^*(f_i)(a_i)$ $(i<\omega)$. We again pick $p_i \in M_\xi$ s.t. either $p_i = f_i \in M_\xi$ or f_i is a good $\Sigma_1^{(n)}(M_\xi)$ in p_i (by a func. abs. def.) where $\nu_\xi \leq \omega\rho_{M_\xi}^{n+1}$. We make the previous assumptions on Z (in particular $\alpha_\xi, \nu_\xi, p_{M_\xi}, p_i \in Z$ $(i<\omega)$) and construct the fcns g_i $(i<\omega)$ exactly as before, using the same three cases.

We then repeat the proof of Lemma 1.1, replacing non-well foundedness by non-iterability in a rather mechanical way. We exemplify this by sketching the changes to be made in Case 1.

We form $X = \bigcup_i \text{rng}(g_i)$ and $\tilde{X} = \langle X, \in, E \cap X, F \cap X \rangle$

$(M = \langle J^E_\beta, F\rangle)$ as before and form $\pi : N \overset{\sim}{\longleftrightarrow} X$ as before. Exactly as before we prove that $\pi : N \longrightarrow_{\Sigma^*} X$ + hence that N is a mouse. We again set:

$\bar{\nu} = X \cap \nu_3 = \bigcup_i{}' \mathrm{dom}(g_i) = \bigcup_i \bar{\tau}_i$;

As before:

(1) $\pi \upharpoonright \bar{\nu} = \mathrm{id}$; $\pi(\bar{\nu}) = \nu$.

We wish to show: $\bar{\nu} = \nu$.

Set: $\bar{\bar{Q}} = J^{A_3}_{\bar{\nu}}$, $\bar{\nu}^* = \sup \sigma_3{}'' \bar{\nu}$,

$\bar{Q}^* = J^{A}_{\bar{\nu}^*}$; $\bar{\sigma}_3 = \sigma_3 \upharpoonright \bar{Q}$. Then:

(2) $\bar{\sigma} : \bar{Q} \longrightarrow_{\Sigma_0} \bar{Q}^*$ cofinally and $\bar{\nu}$ is regular in N, $\bar{Q} = J^{E^N}_{\bar{\nu}}$

But $\sigma_3^* \pi : N \longrightarrow_{\Sigma^*} M_3^*$ and $\sigma_3^* \pi \upharpoonright \bar{Q} = \bar{\sigma}_3$. Hence by the interpolation lemma $\bar{\sigma}_3$ has a canonical extension $\bar{\sigma}^* : N \longrightarrow N^*$ w.r.t. N. Moreover there is

$\pi^* : N^* \xrightarrow[\Sigma^*]{} M^*$ s.t. $\pi^* \bar{\sigma}^* = \sigma_\zeta^* \pi$.

$$\begin{array}{ccc} N^* & \xrightarrow{\pi^*} & M_\zeta^* \\ \bar{\sigma}^* \uparrow & & \uparrow \sigma_\zeta^* \\ N & \longrightarrow & M_\zeta \end{array}.$$

But by our construction:

$$\bar{r}_i = \bar{\sigma}^*(f_i)(a_i) \in N^* \quad (i<\omega)$$

where $\pi^*(\bar{r}_i) = r_i$. Hence there is an iteration $\langle \tilde{N}_i \mid i<\theta \rangle$ of N^* & $\sigma_i : \tilde{N}_i \xrightarrow[\Sigma^*]{} \tilde{M}_i$ as given by Sublemma 1.0. But, using Sublemma 1.0, our construction ensures that $\langle \tilde{N}_i \rangle$ cannot be continued. Hence N^* is not iterable above $\bar{\nu}^*$. Hence $\bar{\nu} = \nu_\zeta$ by the minimality of ν_ζ. QED (Case 1)

The other cases are similar but require

QED (Sublemma 2.1)

We again have: $cf(\nu_\xi) = \omega$ for $\xi \in S'$.
Arguing exactly as before:

<u>Sublemma 2.2</u> There is a cub $C \subset \alpha$ s.t. for all $\xi \in C \cap S'$ we have $Q_\xi = P_\xi$ and $Q^*_\xi = P$ (hence $\sigma_\xi = \sigma_{\xi\alpha}$ is cofinal).

From now on assume w.l.o.g.

$Q_\xi = P_\xi$, $Q^*_\xi = Q =_{\text{Df}} P$ for $\xi \in S'$.

Without loss of generality we also assume:

(**) $E^{M_\xi}_{\nu_\xi} = \emptyset$ for $\xi \in S'$.

To see this, note that we may replace M_ξ by M' where $k : M_\xi \longrightarrow^*_{E_{\nu_\xi}} M'$. Let $\sigma'^* : M' \longrightarrow M'^*$ be the canonical extension of σ_ξ w.r.t. M'^*. Define $k^* : M^*_\xi \longrightarrow M'^*$ by: $k^*(\tilde{\sigma}_\xi(f)(a)) = \sigma'^*(k \circ f)(a)$.

Then $k^* \tilde{\sigma}_\xi = \sigma'^* k$. It is straightforward (e.g. by the remark after § 2.3 Cor 2.2) to prove that $k^*: M_\xi^* \underset{E_{\nu_\xi^*}}{\longrightarrow} M'^*$.

But then the non iterability of M_ξ^* above ν_ξ^* implies the non iterability of M'^* above $k^*(\nu_\xi^*)$.

For $\xi \in S'$ let $\langle f_i^\xi \mid i<\omega \rangle$, $\langle a_i^\xi \mid i<\omega \rangle$ be the sequence witnessing the non iterability of M_ξ^* ($\pi_i^\xi = \tilde{\sigma}_\xi(f_i^\xi)(a_i^\xi$

Let ξ^* = the least $\xi^* > \xi$ s.t. $a_i^\xi \in \operatorname{rng}(\sigma_{\xi^*, \alpha})$ for all $i<\omega$.

If $\xi, \zeta \in S'$, $\xi^* \leq \zeta$, then the canonical extension of $\sigma_{\xi\zeta}: Q_\xi \to Q_\zeta$ wrt M_ξ is not iterable above ν_ζ, as can be seen using Sublemma 2.0.

Define a club $C \subset \alpha$ by:

C = the set of limit pts λ of S' s.t. $\xi^* < \lambda$ for $\xi \in S' \cap \lambda$.

Fix $\lambda \in C \cap S'$. We define a countable $W \subset Q_\lambda$ exactly as in the proof of Lemma 1 and pick $\xi \in S' \cap \lambda$ s.t. $W \subset \mathrm{rng}(\sigma_{\xi\lambda})$. We derive a contradiction exactly as before.

QED (Lemma 2)

This proof shows more than we claimed. Recalling our justification of the assumption (**), we have:

<u>Corollary 3</u> There is a cub $C \subset \alpha$ s.t. for $\xi \in C \cap S$ we have (a), (b) and:

(c) M^*_ξ is iterable above $\beta = \beta_\alpha$.

Henceforth we refer to the conjunction of Lemmas 1, 2, 3 as the "frequent extension lemma".