§3.3 Mitchell's Covering Lemma

§3.3.1 The Theory of O^1

Def An s-premouse (spm) is a J-model $M = \langle J_\beta^E, E_\beta, E_{\beta+1} \rangle$ s.t.

(a) $\langle J_\beta^E, E_\beta \rangle$ is a pm; $E_\beta \neq \emptyset$.

(b) $E_{\beta+1}$ is a normal measure on κ in M (where $\kappa = \text{crit}(E_\beta)$).

(d) If $\pi : M \xrightarrow[E_{\beta+1}]{} M'$, then

$$M' | \beta = \langle J_\beta^E, E_\beta \rangle.$$

A *simple iteration* $\langle M_i \mid i < \theta \rangle$ of an spm with indices $\nu_i \leq \beta_i + 1$ ($i+1 < \theta$, where $M_i = \langle J_{\beta_i}^E, E_{\beta_i}, E_{\beta_i+1} \rangle$ is defined in the obvious way. Note that a proper truncation of an spm is a pm. Hence the notion of an arbitrary iteration with indices $\langle \nu_i, \alpha_i \rangle$ has an obvious definition. The notion of *standard* and *normal* iteration are also defined as before.

Def An s-mouse is an spm which is iterable in the sense that any iteration may be continued.

Call N a generalized mouse (gm) iff N is a mouse or an s-mouse. For gm's the following are easily verifiable:

(a) §2.1 Lemmas 1.1 – 1.3 go through as before.

(b) §2.1 Lemma 2 (The nonexistence of a degenerate iteration) goes thru since any proper truncation of an s-mouse is a mouse.

(c) §2.1 Lemma 3 (The "Dodd-Jensen Lemma") goes thru as before.

(d) The definition of coiteration $\langle N_i^h \mid i < \theta \rangle$ $(h = 0, 1)$ of N^0, N^1 with indices ν_i goes thru as before, using the convention: $E_{\beta+1}^N = \emptyset$ if $N = \langle J_\beta^E, E_\beta \rangle$.

(e) §2.2 Lemmas 1.1 – 1.4 go thru as before.

Note that no s-mouse can be a proper segment of any g-mouse. Hence the coiteration of two s-mice either truncates on both sides or on neither.

(f) §2.2 Lemmas 2 – 5 go thru as before.

(g) By §2.3 any s-mouse is HP (hereditarily pure). But then the argument of §2.3 goes thru for s-mice.

In particular, one side of any co-iteration must be simple. By the above remark, two s-mice must coiterate simply to a common s-mouse. Now let M be an s-mouse and set:

$$\overline{M} \simeq h_M(\emptyset),\ \overline{M} \text{ transitive.}$$

Then $\omega = \omega\rho^1_{\overline{M}}$; $\emptyset \in R^1_{\overline{M}}$. By §2.1 Lemma 2 it follows that $\overline{M}$ is an s-mouse. But since any two r-mice coiterate to a common r-mouse, it follows easily that $\overline{M}$ is the common core of all r-mice.

<u>Def</u> O^r = the core of all r-mice

By our remarks, $\omega\rho^n_{O^r} = \omega$ $(n < \omega)$

and $P_{O^r} = \emptyset$.

We let $\neg O^r$ be the statement that O^r does not exist.

. . . .

This section is devoted to an important consequence of $\neg O^r$. We prove Mitchell's "weak covering lemma" which says that for certain universal weasels the successors of sufficiently large singular cardinals are absolute. This holds in particular for the canonical ω-complete weasel. In order to state the theorem, we define:

<u>Def</u> Let $\alpha > \omega$ be regular. A weasel W is called α-<u>full</u> iff whenever λ is a cardinal in W s.t. $cf(\lambda) = \alpha$ and $\nu = \lambda^{+W}$, and if F is ω-complete s.t. $\langle J_\nu^{E^W}, F \rangle$ is a premouse, then $F = E_\nu^W$.

The proof which showed the canonical ω-complete weasel to be universal also shows every α-full weasel to be universal. Mitchells covering lemma reads:

<u>Theorem</u> Assume $\neg O^\P$. Let W be an α-full weasel. Let $\beta > \alpha^\omega$ be a singular cardinal. Then $\beta^+ = \beta^{+W}$.