Appendix to §3.2 of Measures of Order Zero

Thm Every weakly universal weasel is universal.

proof.

Let W be a counterexample. Let N be a coiterable premouse whose coiteration does not terminate. Let W_i, N_i be the coiteration with indices ν_i, κ_i ($\nu_i = \kappa_i^{+W_i}$). Let $C \subset On$ be cub s.t. $\pi_{N_i N_j}(\kappa_i) = \kappa_j$ for $i, j \in C$, $i \leq j$. For $i \in C$ set $\bar{N}_i = N|\nu_i$. Then

(1) $\langle W_j \mid j \geq i \rangle$, $\langle \bar{N}_j \mid j \geq i \rangle$ is the coiteration of W_i, $\bar{N}_i$ for $i \in C$.

Hence:

(2) $\bar{N}_i$ is not a mouse.

We shall derive a contradiction by showing that some $\bar{N}_i$ is a mouse.

Set: $W' = W_\infty$. Let Q be the canonical ω-complete model. Coiterate W', Q to W'', Q'. Let W'_i, Q_i be the coiteration with indices $\tilde{\nu}_i$, $\tilde{\kappa}_i$ ($\tilde{\nu}_i = \tilde{\kappa}_i + \tilde{\omega}_i$

Set: $\tilde{W}_i = \begin{cases} W'_i & \text{if defined} \\ W'' & \text{if not} \end{cases}$

(similarly for $\tilde{Q}_i$). Let D be the set of κ s.t:

(a) $\kappa = \tilde{\kappa}_\kappa = \sup\{\tilde{\kappa}_i \mid i < \kappa \wedge \tilde{\kappa}_i \text{ defined}$

(b) $\pi_{W'\tilde{W}_\kappa}(\kappa) = \pi_{Q\tilde{Q}_\kappa}(\kappa) = \kappa$

(c) $cf(\kappa) > \omega$.

For $\kappa \in D$, let $\nu = \kappa^{+\tilde{W}_\kappa}$ and define $\tilde{N}_\kappa = \langle J_\nu^{E^{\tilde{W}_\kappa}}, U_\kappa \rangle$ by:

(3) $\pi_{W'\tilde{W}_\kappa} \upharpoonright \bar{N}_\kappa : \bar{N}_\kappa \xrightarrow[\Sigma_0]{} \tilde{N}_\kappa$ cofinally.

We know that $E^{\bar{N}_\kappa}_{\nu_\kappa}$ is ω-complete and, in fact, that

(4) $X \in E_{\nu_n}^{\bar{N}_n}$ iff $\kappa_i \in X$ for sufficiently large $i < n$.

By the proof of §3.1 Lemma 2.3 it follows that U_n is ω-complete and that

(5) $X \in U_n$ iff $\pi_{W' \tilde{W}_n}(\kappa_i) \in X$ for sufficiently large $i < n$.

A straightforward repetition of the proof of §3.1 Lemma 2.7 shows:

(6) $U_n = E_\nu^{Q_n}$ for some $n \in D$.

Fix n. Then:

(7) $\tilde{N}_n$ is a mouse

(8) $\rho_{\tilde{N}_n}^\omega = \nu$.

Using this we prove:

<u>Claim</u> $\pi : \bar{N}_n \to_{\Sigma^*} \tilde{N}_n$,

where $\pi = \pi_{W', \tilde{W}_\kappa} \upharpoonright \bar{N}_\kappa$, which establishes that $\bar{N}_\kappa$ is a mouse, contradicting (2).

(9) If $\rho^n_{\bar{N}_\kappa} \geq \nu_\kappa$, then $\pi : \bar{N}_\kappa \to_{\Sigma^*_{n+1}} \tilde{N}_\kappa$.

prf. (Ind. on n).

$n = 0$ is trivial. Let it hold for n.

Let $p \in \Gamma^{n+1}_{\bar{N}_\kappa}$, $\bar{B} = A^{n+1,p}_{\bar{N}_\kappa}$,

$\tilde{B} = A^{n+1,\pi(p)}_{\tilde{N}_\kappa}$. <u>We claim</u>:

$$\pi : \langle \bar{N}_\kappa, \bar{B} \rangle \to_{\Sigma_1} \langle \tilde{N}_\kappa, \tilde{B} \rangle.$$

Let $\bar{A} = A^{n, p \upharpoonright m}_{\bar{N}_\kappa}$, $\tilde{A} = A^{n, \pi(p \upharpoonright m)}_{\tilde{N}_\kappa}$.

Set: $B = \bigcup_{x \in \bar{N}_\kappa} \pi(\bar{B} \cap x)$. Then

(10) $\pi : \langle \bar{N}_\kappa, \bar{A}, \bar{B} \rangle \to_{\Sigma_1} \langle \tilde{N}_\kappa, \tilde{A}, B \rangle$

and it suffices to show that $B = \tilde{B}$. $\bar{B}$ is uniformly definable from $\bar{A}, p_n$ by a Σ_1 fmla $\bigvee y \varphi$ + we must show that B is defined from $\tilde{A}, \pi(p_n)$

by the same formula - i.e.

$$z \in B \longleftrightarrow \langle \tilde{N}_\kappa, \tilde{A} \rangle \models \bigvee y \varphi[z, \pi(p_m)]$$

for $z \in \tilde{N}_\kappa$.

($\longleftarrow$) is $\Delta_1(\langle \tilde{N}_\kappa, \tilde{A}, B \rangle)$ + hence follows by (10). To see ($\rightarrow$), let $z \in B$ + suppose that $z \in \pi(u)$ for a $u \in \bar{N}_\kappa$. Note that $\langle \bar{N}_\kappa, \bar{A} \rangle$ is admissible, since $\rho^n_{\bar{N}_\kappa} \geq \kappa$ + κ is the largest cardinal in $\bar{N}_\kappa$. Since

$$\bigwedge z \in \bar{B} \cap u \langle \bar{N}_\kappa, \bar{A} \rangle \models \bigvee y \varphi[z, p_m],$$

there is $w \in \bar{N}_\kappa$ s.t.

$$\bigwedge z \in \bar{B} \cap u \bigvee y \in w \ \langle \bar{N}_\kappa, \bar{A} \rangle \models \varphi[y, z, p_m]$$

Hence by (10):

$$\bigwedge z \in B \cap \pi(u) \bigvee y \in \pi(w) \ \langle \tilde{N}_\kappa, \tilde{A} \rangle \models \varphi[y, z, \pi(p_m)]$$

QED (9)

The Claim then follows by:

(11) $\rho^\omega_{\bar{N}_\kappa} \geq \nu_\kappa$,

since otherwise, if n is

least s.t. $\rho^{n+1}_{\bar{N}_\kappa} < \nu_\kappa$, then

$h_{\bar{N}_\kappa}$"κ is cofinal in ν_κ +

it follows by (9) that

$h_{\tilde{N}_\kappa}$"κ is cofinal in ν.

Hence $\rho^{n+1}_{\tilde{N}_\kappa} < \nu$. Contr!

QED