§2.3 Core Mice

Def Set $\rho^\omega = \inf\{\rho^n \mid n < \omega\}$

Let N be a mouse. N is pure iff there is a sound mouse $\bar{N}$ s.t.

(a) N is a simple iterate of $\bar{N}$ above $\omega\rho^\omega_{\bar{N}}$

(b) If $M, \bar{N}$ have a common simple iterate above $\omega\rho^\omega$, then M is a simple iterate of $\bar{N}$ above $\omega\rho^\omega_{\bar{N}}$.

It is clear that if N is pure and M, N have a common simple iterate above $\omega\rho^\omega_N$, then M is pure.

Note that $\bar{N}$ is the unique sound mouse satisfying (a), since if $\bar{N}'$ were another one, $\bar{N}'$ would be a simple iterate of $\bar{N}$ above $\omega\rho^\omega_{\bar{N}}$. But as in the argument of §2.2 Lemma 1.3, a proper iterate of this sort would not be sound. Hence we may define:

Def Let N be pure. The core of N (core(N)) is the unique sound mouse $\bar{N}$ s.t. N is a simple iterate of $\bar{N}$ above $\omega\rho^\omega_{\bar{N}}$.

In this section we show that every mouse is pure. We also show that in the coiteration of two mice, at least one side must be simple. We also define a $p_N \in P_N^*$ which we call the <u>standard parameter</u> and show that it is preserved under simple iterations (i.e. $\pi_{NM}(p_N) = p_M$). These facts are so interwoven that we must, in effect, prove them simultaneously by a single induction on mice. The relation over which we induct is defined by:

<u>Def</u> MRN iff M is a non simple iterate of N.

R is well founded by §2.1 Lemma 2. We shall prove by R-induction that N is <u>superpure</u> in the following sense:

<u>Def</u> N is <u>superpure</u> iff M is pure whenever N is a simple iterate of M.

We then define:

<u>Def</u> N is <u>hereditarily pure</u> (HP) iff iff every MRN is superpure.

From now on, assume N to be H.P. In the course of the induction we shall establish the above facts.

Lemma 1 Coiterate N with a mouse M. Then at least one side of the coiteration is simple.

prf.

Suppose not. Let $\langle M_{i \leq \theta} \rangle$, $\langle N_{i \leq \theta} \rangle$ be the coiteration. There are maximal j_M, j_N s.t. $\omega\alpha^M_{i_M} \in M_{i_M}$, $\omega\alpha^N_{i_N} \in N_{i_N}$.

Set $\bar{M} = M_{i_M} | \alpha^M_{i_M}$, $\bar{N} = N_{i_N} | \alpha^N_{i_N}$. Then $E^{\bar{M}}_{\nu_{i_{\bar{M}}}}$ is a measure on $\bar{\kappa}$ where $\omega\rho^\omega_{\bar{M}} \leq \bar{\kappa}$.

Thus M_θ is a simple iterate of $\bar{M}$ above $\omega\rho^\omega_{\bar{M}}$. Similarly for $\bar{N}$. Hence:

(1) $\bar{M} = \mathrm{core}(M_\theta) = \mathrm{core}(N_\theta) = \bar{N}$.

Hence $i_N \neq i_M$, since otherwise $E^{N_j}_{\nu_j} = E^{M_j}_{\nu_j}$ $(j = j_N = i_M)$. Let e.g. $i_N < i_M$.

Then $E^{N_{i_M}}_{\nu_{i_N}} = \emptyset$ and $E^{M_{i_M}}_{\nu_{i_N}} = E^{\bar{N}}_{\nu_{i_N}} \neq \emptyset$.

Contradiction!, since $\nu_{i_N} < \nu_{i_M}$.

QED (Lemma 1)

Using this we can show:

Lemma 2 If M, N have a common simple iterate, then M is HP.

proof of Lemma 2

Let $Q R M$. Claim Q is superpure.

Iterate M, N simply to a common M'. Coiterate Q, M' to Q', M''. Then:

(a) Q' is an initial segment of M''

(b) If the iteration from M' to M'' is simple, then Q' is a proper segment. Otherwise M'' would be both a simple and non simple iterate of M'. But then $Q' R N$ by (a), (b) & hence Q' is superpure. But Q' is a simple iterate of Q, since otherwise M'' would be an initial segment of Q' and both sides of the iteration would be non simple, violating Lemma 1.

Hence Q is superpure. QED (Lemma 2)

Def Let M, N be mice. M is a derived iterate of N with derived iteration map π iff the following hold:

(a) M, N have a common simple iterate Q with $\mathrm{rng}(\pi_{NQ}) \subset \mathrm{rng}(\pi_{MQ})$

(b) $\pi = \pi_{MQ}^{-1}\pi_{NQ}$.

The derived iteration map is in fact unique, as is shown by the following fact.

<u>Fact</u> If M, N have a common simple iterate Q, then:

(a) If Q' is a common iterate of M, N and a simple iterate of M or N, then Q' is a simple iterate of both (hence M, N coiterate to a common Q').

(b) If $\pi = \pi_{MQ}^{-1} \pi_{NQ}$ is a derived iteration map, then $\pi = \pi_{MQ'}^{-1} \pi_{NQ'}$

To see (a), coiterate Q, Q' to $\tilde{Q}, \tilde{Q}'$. Q, Q' are both simple iterates of a $P = M$ or N. If e.g. $\tilde{Q}$ were a simple iterate of Q and $\tilde{Q}'$ a non simple iterate of Q', then $\tilde{Q}$, as an initial segment of $\tilde{Q}'$ would be a simple and non simple iterate of P. Contr!

To see (b) note that

$$\pi_{N\tilde{Q}} = \pi_{Q\tilde{Q}} \pi_{NQ} = \pi_{Q'\tilde{Q}} \pi_{NQ'}$$

and similarly for M.

Lemma 3 Let M be a derived iterate of N with map π. Then M is an iterate of N by a normal iteration and $\pi = \pi_{NM}$.

prf.

It is easily seen that N, M coiterate to a common Q and $\pi = \pi_{MQ}^{-1} \pi_{NQ}$. Both sides of the iteration are simple and normal. Hence it suffices to prove:

Claim $Q = M$.

Suppose not. Let $\langle N_{i \leq \theta} \rangle$, $\langle M_{i \leq \theta} \rangle$ be the two sides of the coiteration and let $i =$ = the least i s.t. either:

(a) $M_i \neq M$ or:

(b) M is not a derived iterate of N_i

Then $i > 0$. Suppose $i = \lambda$, lim(λ). Then $M_\lambda = M$ since $M_j = M$ for $j < \lambda$. But for $j < \lambda$ M is a derived iterate of N_j; hence

$rng(\pi_{N_i Q}) \subset rng(\pi_{MQ})$ for $i < \lambda$.
Hence $rng(\pi_{N_\lambda Q}) = \bigcup_{i<\lambda} rng(\pi_{N_i Q}) \subset$
$\subset rng(\pi_{MQ})$. Hence M is a derived iterate of N_λ. Contr!
Thus $i = j+1$. Assume w.l.o.g. that $j = 0$. Let $\nu = \nu_0 =$ the least ν s.t. $E^N_\nu \neq E^M_\nu$.

<u>Case 1</u> $E^N_\nu = \emptyset$.
Then $\pi_{NQ}(\kappa) = \kappa$ where E^M_ν is normal on κ, but $\kappa = crit(\pi_{MQ})$.
Hence $rng(\pi_{NQ}) \not\subset rng(\pi_{MQ})$.
Contr!

<u>Case 2</u> $E^M_\nu = \emptyset$.
Then $M_1 = M$ and $\pi_{01} : N \xrightarrow[E_\nu]{} N_1$.
If $f \in \Gamma(N, \kappa)$, we clearly have $\pi_{N_1 Q}(\pi_{01}(f)(\kappa)) =$
$$= \pi_{NQ}(f)(\kappa) = (\pi_{MQ}\pi(f))(\kappa) =$$
$$= (\pi_{MQ}\pi(f))(\pi_{MQ}(\kappa)) =$$
$$= \pi_{MQ}(\pi(f)(\kappa)).$$ Hence

$\mathrm{rng}(\pi_{N_1Q}) \subset \mathrm{rng}(\pi_{MQ})$ and M is a derived iterate of N_1. Contradiction!

Case 3 The above cases fail.

Let $X \in \mathbb{P}(\kappa) \cap N = \mathbb{P}(\kappa) \cap M$. Let $\pi_{NQ}(X) = \pi_{MQ}(Y)$. Then $Y \subset \pi(\kappa) \geq \kappa$ and

$$\pi_{MQ}(X) = \pi_{MQ}(Y) \cap \pi_{MQ}(\kappa) = \pi_{NQ}(X) \cap \pi_{MQ}(\kappa)$$

where $\pi_{MQ}(\kappa) > \kappa$. Hence:

$$X \in E_\nu^N \longleftrightarrow \kappa \in \pi_{NQ}(X)$$
$$\longleftrightarrow \kappa \in \pi_{MQ}(X) \longleftrightarrow X \in E_\nu^M.$$

Thus $E_\nu^N = E_\nu^M$. Contradiction!

QED (Lemma 3)

Lemma 3.1 Let $p \in P_N^n$. Let $\bar{N}, \bar{p}$ be s.t. $\bar{N}^{n\bar{p}} = N^{np}$ and $\bar{p} \in R_{\bar{N}}^n$.
Then $\bar{N}, N$ have a common simple iterate above $\omega\rho^n$.

proof.

Let $\sigma : \bar{N} \to_{\Sigma_1^{(n)}} N$ s.t. $\sigma \upharpoonright \bar{N}^{n\bar{p}} = id$; $\sigma(\bar{p}) = p$, $\bar{N}$ is iterable above $\omega\rho^n$ by the existence of σ.

Claim $E_\nu^{\bar{N}} = E_\nu^N$ for $\nu \leq \omega\rho^n$

pf. For $\nu < \omega\rho^n$ this is trivial. Now let $\nu = \omega\rho^n$. If $E_\nu^{\bar{N}} = E_\nu^N = \emptyset$ there is nothing to prove. Otherwise κ is the largest cardinal in $\bar{N}^{n\bar{p}} = N^{np}$, where $E_\nu^{\bar{N}}$ or E_ν^N has critical pt. κ. If $\omega\rho^n = On \cap N$ there is nothing to prove. If not, then $\nu = \kappa^{+N}$. But then $\nu \in \bar{N}$, since otherwise $N^{np} = \bar{N}^{n\bar{p}} \in N$. Hence $\nu = \kappa^{+\bar{N}}$, since $\sigma(\kappa) = \kappa$.

Hence $\sigma(\nu) = \nu$ and $\sigma(E^{\bar{N}}_\nu) = E^N_\nu$.
But $\sigma \upharpoonright \mathbb{R}(\kappa)_{\bar{N}} = id$. QED (Claim)

But then $\bar{N}, N$ are coiterable. Coiterate them to $\bar{N}', N'$. Then neither is a proper initial segment of the other, since otherwise $A \in \bar{N}$ or $A \in N$, where $A = A^{np}_N$ QED (Lemma 3.1)

<u>Lemma 3.2</u> Let $p \in P^n_N$. Then $\rho^{n+m}_N = \rho^m_{N^{np}}$ $(m \geq 0)$. (In other words, p can be lengthened to a $q \in P^{n+m}_N$ for any $m \geq 0$).

proof.
Let $\bar{N}, \bar{p}$ be as above + let Q be a common iterate above $\omega\rho^n$.
Then $Q^{n\pi_{\bar{N}Q}(\bar{p})} = \bar{N}^{n\bar{p}} = N^{np} = Q^{n\pi_{NQ}(p)}$
Since $\bar{p} \in R^n_{\bar{N}}$ we can lengthen it to $\bar{q} \in P^{n+m}_{\bar{N}}$. Hence $\bar{N}^{n+m,\bar{q}} =$
$Q^{n+m,\pi_{\bar{N}Q}(\bar{q})} = Q^{n+m,\pi_{NQ}(q)} = N^{n+m,q}$
where q is the same lengthening of p. QED (Lemma 3.2)

<u>Def</u> p_N = the $<_*$ - least $p \in P_N^*$, where $<_*$ is the well ordering of $[On]^{<\omega}$ defined by:

$$a <_* b \longleftrightarrow \bigvee \nu \, (a \setminus (\nu+1) = b \setminus (\nu+1) \wedge \nu \in b \setminus a).$$

If we set: $p_N^i = p_N \cap [\omega\rho_N^{i+1}, \omega\rho_N^i)$ $(i < \omega)$, then it follows from Lemma 3.2 that:

$$p^i = \text{the } <_* \text{- least } p \in P_{N^i, pr^i}.$$

(In the following we adopt the convention that $p^i =: p \cap [\omega\rho^{i+1}, \omega\rho^i)$ where $p \subset On_N$ is finite, and $pr^i =: \langle p^0, \ldots, p^{i-1} \rangle$.)

We call p_N the <u>standard parameter</u> of N. The main lemma in this section states the <u>preservation property</u> of standard parameters: $\pi_{NM}(p_N) = p_M$.

As preparation for this we define the <u>Friedman witnesses</u>: */

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*/ This is something of a misnomer, since these witnesses, though discovered independently by Friedman, turn out to have been defined earlier by Mitchell.

<u>Def</u> Let $\nu \in p^i = p^i_N$ $(i<\omega)$.
The <u>Friedman</u> <u>witness</u> <u>for</u> ν $(W = W^\nu_N)$
is defined as follows: Let
$$X = h_{N^{i,p\restriction i}}(\nu \cup (p^i \setminus (\nu+1)))$$
and $\bar{W} \simeq (N^{i,p\restriction i} | X)$ where $\bar{W}$ is
transitive. $W, \bar{p}$ are the unique
objects s.t. $W^{i p} = \bar{W}$ and $\bar{p} \in R^i_W$.

(Note that $\bar{p} = p_W \restriction i$, and $\omega\rho^{i+1}_W \leq \nu$).

The following lemma marks our first
use of (c) in the def. of premouse.

<u>Lemma 4.1</u> Let $\nu \in p = p_N$. Then $W^\nu_N \in N$.

proof
Let $\bar{\sigma} : \bar{W} \stackrel{\sim}{\longleftrightarrow} (N^{i,p\restriction i} | X)$, $\sigma \supset \bar{\sigma}$,
with $\sigma : W \longrightarrow_{\Sigma^{(i)}_1} N$, $\sigma(\bar{p}) = p = p_N \restriction i$
(where $\bar{W}, W, \bar{p}$ are as in the above def.).
The existence of σ tells us that
(1) W is iterable above $\omega\rho^{i+1}_W \leq \nu$.

(2) W, N are coiterable.

proof of (2)

Clearly $E_\xi^N = E_\xi^W$ for $\xi < \nu$. Coiterability could only fail if $E_\nu^N \neq E_\nu^W \neq \emptyset$.

Assume $E_\nu^W \neq \emptyset$. If $\nu \in W$, then $E_\nu^W = E_{\sigma(\nu)}^N \cap J_\nu^{E^N}$ where $E_{\sigma(\nu)}^N$ is a measure on the same κ as E_ν^W. Hence, since W is a pm, the def. of pm tells us that $E_\nu^N = E_{\sigma(\nu)}^N \cap J_\nu^{E^N} = E_\nu^W$.

If $\nu \notin W$, then $E_\nu^W = E_\alpha^N \cap J_\nu^{E^N}$ where $W\alpha = On \cap N$ and we apply the same reasoning. QED (2)

Note that by the prf. of (2), the W side of the coiteration is above ν, and the N side is above $\omega\rho_N^{i+1}$.

Coiterate N, W to N', W'.

(3) $N' \notin W'$.

Suppose not. Then N' is a simple iterate of N. Imitating the W side of the coiteration, we get an iterate N'' of N' above ν and a map $\sigma' : W' \xrightarrow[\Sigma_1^{(i)}]{} N''$ s.t. $\sigma' \pi_{WW'} = \pi_{NN''} \sigma$.

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\sigma"'] & W' \arrow[d, "\sigma'"] \\ N \arrow[r] \arrow[ur, "N'" near end, phantom] & N'' \end{tikzcd}$$

Hence $\sigma'\pi_{NN'} : N \xrightarrow[\Sigma^*]{} \sigma'(N')$ where $\sigma'(N')$ is a proper initial segment of N'', hence a non simple iterate of N. Contr! by §2.1 Lemma 3. QED(3)

(4) $N' \neq W'$ if N' is a simple iterate of N.

prf.

The iteration from W to W' is simple, since otherwise N'' is a non simple iterate of N and and we get the same contradiction as in (3). But then $\rho^{i+1}_N = \rho^{i+1}_{N'} = \rho^{i+1}_{W'} = \rho^{i+1}_W$, since both iterations are above $\omega\rho^{i+1}$. Let $A \subset \omega\rho^{i+1}$ be $\Sigma_1(\bar{W})$ s.t. $A \notin N$.

Set: $\bar{p}^* = \bar{\sigma}^{-1}(p^*)$ where $p^* = p^i_N \setminus (\nu+1)$

Since $h_{\bar{W}}(\nu \cup \bar{p}^*) = \bar{W}$, there must be $\vec{\xi} < \nu$ s.t. A is $\Sigma_1(\bar{W})$ in $\{\vec{\xi}\} \cup \bar{p}^*$. Hence A is $\Sigma_1(N^{i,p^i})$ in $\{\vec{\xi}\} \cup p^*$, contradicting the def of P_N.

QED(4)

Hence:

(5) Either $W' \in N'$ or $W' = N'$ and the iteration from N to N' is non simple.

We now prove $W \in N$ by cases:

<u>Case 1</u> There is $\beta \in N$ s.t. $\bar{\bar{\beta}}^N > \nu$.

Set: $D = \{\langle \xi, \zeta \rangle \in \nu^2 \mid h_{\bar{W}}(\xi, \bar{p}^*) \in h_{\bar{W}}(\zeta, \bar{p}^*)\}$.

Clearly, ν is p.r. closed & hence W can be constructed from D inside any admissible set. It suffices to prove:

<u>Claim</u> $D \in N$,

since then $D \in J_\beta^{E^N}$ where $J_\beta^{E^N}$ is admissib

prf. of Claim:

If $W' \in D'$, then $D \in N'$ since D is $\underline{\Sigma}^*(W')$. Hence $D \in N_1$ since N' is an iterate of N' above ν ($\langle N_\xi \mid \xi \leq \theta \rangle$ being the normal iteration. But then $D \in N$, since N_1 is either an it. of N above ν or the Σ^* ultraproduct of an $N|\alpha$ by E_ν^N (hence $\mathcal{P}(\nu) \cap N_1 \subset N$).

Now let $W' = N'$. Then N' is a non simple iterate of N and there is a maximal $\xi < \theta$ s.t. $\pi : (N_\xi | \alpha) \xrightarrow[E_{\bar{\nu}}]{} N_{\xi+1}$ with $\omega\alpha \in N_\xi$.

But N' is a simple iterate of $N_{\xi+1}$ above ν. Hence $D \in \underline{\Sigma}^*(N_{\xi+1})$ by §2.2 Lemma 5. But either $\kappa_\xi \geq \nu$ or $\xi = 0$ and $\nu = \kappa^{+N}$. In the former case, $D \in \underline{\Sigma}^*(N_\xi | \alpha) \subset N_\xi$ and it follows as before that $D \in N$. In the latter case we note that there is $\langle f_\xi \mid \xi < \nu \rangle \in N$ s.t. $f_\xi : \kappa \to \nu$, $f_\xi \in N_\xi$ and $\pi(f_\xi | (\kappa)) = \xi$ for $\xi < \nu$. By §1.3 Cor 2.5.1 it follows easily that $D \in N$. QED (Case 1)

Case 2 Case 1 fails and $\nu \in W$.
Then $\sigma(\nu) = \nu$, since otherwise ν is a cardinal in W and $\sigma(\nu) > \nu$ would be a cardinal in N. But then $i = 0$ and $W = N | X = \langle J_\gamma^{E^N}, E_\alpha^N \cap J_\gamma^{E^N} \rangle$ for some γ, where $\omega\alpha = On \cap N$.
But then $W = W'$, $N = N'$ and $W \in N$ by (5). QED (Case 2)

Case 3 $\omega\gamma = On \cap W$
Then $W = \langle J_\gamma^E, E_\alpha^N \cap J_\gamma^E \rangle \in N$.

QED (Lemma 4.1)

Lemma 4.1 makes it clear in what sense $W = W^\nu_N$ is a "witness" to $\nu \in p_N$. If $\nu \notin p_N$, then there would be an a $A \subset \omega\rho^{i+1}_N$ s.t. $A \notin N$ which is $\Sigma_1(N^{i,p_N,i})$ in parameters from $\nu \cup p_N \setminus (\nu+1)$. Hence $A \in \underline{\Sigma}^*(W) \in N$. Contradiction!

Now suppose $\pi : N \xrightarrow[E_3]{*} M$. Let $\tilde{p} = \pi(p_N)$. Then $\tilde{p} \in P^*_M$. If we can show that each $\nu \in \tilde{p}$ is "witnessed" inside M in the same way, we can conclude that $\tilde{p} = p_M$. Following this strategy we prove:

<u>Lemma 4.2</u> Let $\pi : N \xrightarrow[E_3]{*} M$. Then $\pi(p_N) = p_M$.

proof of Lemma 4.2.

Let $p = p_N$, $q = \pi(p)$. Then $q \in P^*_M$. Let $\tau \in q$. In particular, suppose $\tau \in q^i = q \cap [\omega\rho_M^{i+1}, \omega\rho_M^i)$. Then $\tau = \pi(\nu)$ where $\nu \in p^i$. Set:

$\bar{N} = N^{i p \restriction i}$, $\bar{M} = M^{i q \restriction i}$,

$p^* = p^i \setminus (\nu+1)$, $q^* = q^i \setminus (\tau+1)$.

By our above remarks it suffices to show that there are $Q, \bar{q}^* \in M$ s.t.

(1) $\bigwedge \vec{\xi} < \tau \; (Q \models \varphi(\vec{\xi}, \bar{q}^*) \longleftrightarrow \bar{M} \models \varphi(\vec{\xi}, q^*))$

for Σ_1 formulae φ, since there Q witness the minimality of q in P^*_M. Let E_ξ be a measure on κ.

<u>Case 1</u> $\kappa < \omega\rho_N^{i+1}$

Then $\pi \restriction \bar{N} : \bar{N} \xrightarrow[\Sigma_2]{} \bar{M}$. Since

(2) $\bigwedge \vec{\xi} < \nu \; (\bar{W} \models \varphi(\vec{\xi}, \bar{p}^*) \longleftrightarrow \bar{N} \models \varphi(\vec{\xi}, p^*))$,

the claim holds with $Q = \pi(\bar{W})$.

<u>Case 2</u> $\omega\rho_N^{i+1} \leq \kappa < \omega\rho_N^i$

Then $\pi \restriction \bar{N} : \bar{N} \xrightarrow[\Sigma_1]{} \bar{M}$ cofinally.

Case 2.1 There is $X \in \bar{N}$ s.t. $\text{rng}(\bar{\sigma}) \subset X$ (where $\bar{\sigma} : \bar{W} \to \bar{N}$ is defined as before).

Take X as transitive.

(3) $\bigwedge \vec{\xi} < \nu \, (\bar{N} \models \varphi(\vec{\xi}, p^*) \to \bar{W} \models \varphi(\vec{\xi}, \bar{p}^*))$

for Σ_1 fmlae φ. (3) is $\Pi_1(\bar{N})$ in the parameters $\bar{W}, \bar{p}^*, p^*$. Moreover:

(4) $\bigwedge \vec{\xi} < \nu \, (\bar{W} \models \varphi(\vec{\xi}, \bar{p}^*) \to (\bar{N} | X) \models \varphi(\vec{\xi}, p^*))$

which is $\Pi_1(\bar{N})$ in $\bar{W}, (\bar{N}|X), \bar{p}^*, p^*$. It follows easily that the claim holds for $Q = \pi(\bar{W})$.

Case 2.2 Case 2.1 fails

Let $\Gamma = \{u \in W \mid u \text{ is transitive}\}$.

Then $\bar{N} = \bigcup_{u \in \Gamma} \bar{\sigma}(u)$. For $u \in \Gamma$:

(5) $\bigwedge \vec{\xi} < \nu \, ((\bar{W}|u) \models \varphi(\vec{\xi}, \bar{p}^*) \longleftrightarrow (\bar{N}|\bar{\sigma}(u)) \models \varphi(\vec{\xi}, p^*))$

which is $\Pi_1(\bar{N})$ in $\bar{W}|u, \bar{N}|\sigma(u), \bar{p}^*, p^*$.

But $\bar{M} = \bigcup_{u \in \Gamma} \pi\bar{\sigma}(u)$. Hence (1) holds with $Q = \pi(\bar{W}) \,\big|\, \bigcup_{u \in \Gamma} \pi(u)$.

But $Q \in M$, since $Q = \langle J^A_{\tilde{\nu}}, B \cap J^A_{\tilde{\nu}} \rangle$ where $\pi(\bar{W}) = \langle J^A_{\gamma}, B \rangle$ and $\tilde{\nu} = \sup \pi'' \text{On} \cap \bar{W}$.

Case 3 $\omega\rho_N^i \le \kappa$.
Then the claim holds with $Q = \overline{W}$.
QED (Lemma 4.2).

It follows easily from the proof of Lemma 4.2 that:

Cor. 4.3 Let $\pi : N \xrightarrow[E_\xi]{*} M$, $\nu \in P_N$.
Then either $W_M^{\pi(\nu)} = \pi(W_N^\nu)$ or else $(On \cap W_M^{\pi(\nu)}) \in \pi(W_N^\nu)$.

. . . .

We use these lemmas to prove:

Lemma 5 Let M be a simple iterate of N.
Then $\pi_{NM}(P_N) = P_M$.

proof.
Let $\langle N_{i \le \theta} \rangle$ be the iteration from N to M with it. maps π_{ij}.
By induction on i we prove:

Claim $\pi_{hi}(P_{N_h}) = P_{N_i}$ for $h < i$.

Only the limit case is problematic. Let the claim hold for $i < \lambda$. By Cor 4.3 there is an $i_0 < \lambda$ s.t. $\pi_{ij}(W^{\nu}_{N_i}) = W^{\pi_{ij}(\nu)}_{N_j}$ for $\nu \in P_{N_i}$ when $i_0 \leq i \leq j < \lambda$. Set $p = \pi_{i\lambda}(p_{N_i})$ $(i < \lambda)$. Set

$$W^{\nu} = \pi_{i\lambda}\left(W^{\pi_{i\lambda}^{-1}(\nu)}_{N_i}\right) \quad (i_0 \leq i < \lambda)$$

for $\nu \in p$. It follows easily that the W^{ν} witness the $<_*$-minimality of p in $P^*_{N_\lambda}$.

QED (Lemma 5)

We are now ready to prove:

<u>Lemma 6</u> N is pure. Moreover if $\rho_N^\omega = \rho_N^n$ and $\bar{N}, \bar{p}$ are defined by: $\bar{N}^{n\bar{p}} = N^{n\,p\restriction n}$, $\bar{p} \in R_{\bar{N}}^n$, where $p = p_N$, then $\bar{N} = \mathrm{core}(N)$.

proof.

Let $\sigma : \bar{N} \xrightarrow[\Sigma_1^{(n)}]{} N$ s.t. $\sigma \restriction \bar{N}^{n\bar{p}} = \mathrm{id}$ and $\sigma(\bar{p}) = p\restriction n = \langle p^0, \ldots, p^{n-1}\rangle$. Then $\sigma : \bar{N} \xrightarrow[\Sigma^*]{} N$. By Lemma 3.1 $\bar{N}$ and N have a common simple iterate Q above $\omega\rho_N^n = \omega\rho_{\bar{N}}^n$, Hence $\bar{N}$ is HP. Clearly $p_{\bar{N}} = \underset{i<n}{\smile} \bar{p}^i$ and $\pi_{\bar{N}Q}(p_{\bar{N}}) = \pi_{NQ}(p_N) = p_Q$. Since $p_{\bar{N}} \in R_{\bar{N}}^*$ and $\pi_{\bar{N}Q} \restriction \omega\rho_{\bar{N}}^n = \pi_{NQ} \restriction \omega\rho_N^n$, we conclude: $\sigma = \pi_{NQ}^{-1}\pi_{\bar{N}Q}$. Hence by Lemma 3, N is an iterate of $\bar{N}$ and $\sigma = \pi_{\bar{N}N}$. Now suppose that $M, \bar{N}$ have a common simple iterate above $\omega\rho_{\bar{N}}^n$.

The same analysis then shows that M is a simple iterate of $\bar{N}$.
It remains only to show:

<u>Claim</u> $\bar{N}$ is sound.

prf.
It suffices to show for $i < n$ that $P_{\bar{N}}^{i,\bar{p}ri} = R_{\bar{N}}^{i,\bar{p}ri}$. Suppose not, let i be the least counterexample.
Let q be $<_*$-minimal in $P_N \setminus R_N$, where $N = \bar{N}^{i,\bar{p}Ri}$.
Then $p <_* q$, where $p = \bar{p}^i$.
Let N', q' be s.t. $N'^{q'} = N^q$ and $q' \in R_{N'}$. Let $\sigma : N' \longrightarrow_{\Sigma_1} N$ be def. by $\sigma \upharpoonright N'^{q'} = id$, $\sigma(q') = q$. Then

(1) $N \models \bigvee p <_* q \;\; q = h(i, \langle \vec{\xi}, p \rangle)$

for some $\vec{\xi} < \omega\rho_N$, since $p \in R_N$. Hence there is

$p' \in N'$ s.t. $p' <_* q'$ and $q' = h_{N'}(i, \langle \vec{\xi}, p' \rangle)$. Then, ~~nothing~~ $\sigma(p') \in P_N$ and $\sigma(p') <_* q$. But $\sigma(p') \notin \Gamma_N$, since $\sigma \neq id$. Contradiction! by the minimality of q.

QED (Lemma 6)

But if N is a simple iterate of M, then M is HP & hence pure by Lemma 6. Thus N is superpure. This completes the induction, showing that all mice are pure. Hence the previous lemmas hold for all mice.