§2.2 Normal Iteration

If $\langle M_i \rangle$ is an iteration w. indices $\langle \nu_i, \alpha_i \rangle$, we write $\kappa_i \simeq$ that κ s.t. E_{ν_i} is normal on κ_i in $M_i|\nu_i$ ...

<u>Def</u> $\langle M_i \rangle$ is a <u>normal iteration</u> iff $\nu_i > \nu_j$ whenever $j < i$ and $\kappa_i > \nu_j$ if $j < i$ and $E^{M_i}_{\nu_i} \neq \emptyset$

It follows trivially that if $E^{M_i}_{\nu_i} \neq \emptyset$, then $M_{i+1}|\nu_i = M_k|\nu_i$ for $k > i$ and $M_{i+1}|\nu_i = \langle J^E_{\nu_i}, \emptyset \rangle$ where $M_i|\nu_i = \langle J^{E|\nu_i}_{\nu_i}, E_{\nu_i} \rangle$.
If $E^{M_i}_{\nu_i} = \emptyset$, then $M_i|\nu_i = M_k|\nu_i$ for $k \geq i$.

We call two mice M, N <u>comparable</u> if one is an initial segment of the other — i.e. either $M = N$ or $\bigvee \nu \in N \; M = N|\nu$ or $\bigvee \nu \in M \; N = M|\nu$.

Our aim is to show that any two mice have comparable iterates and that the iteration is simple on at least one side. With this end in mind we define the <u>coiteration</u> of two mice, which is designed to terminate in a pair of comparable iterates.

<u>Def</u> Let N^0, N^1 be pms. The <u>coiteration</u> $\langle N_i^k \rangle$ $(k = 0, 1)$ with the common indices $\langle \nu_i \rangle$ is defined by:

$\nu_i \simeq$ the least ν s.t. $E^0_{\nu_i} \neq E^1_{\nu_i}$

(where $N_i^k = \langle J_\beta^{E^k}, E_\beta^k \rangle$),

the iterations $\langle N_i^k \rangle$ being standard.

<u>Lemma 1.1</u> Let $\langle N_i^k \rangle$ $(k = 0, 1)$ be the coiteration of N^0, N^1. Then each of the iteration $\langle N_i^k \rangle$ is normal.

prf.

Suppose not. There is a least i s.t. $\nu_i \leq \nu_j$ for a $j < i$ or else $E^k_{\nu_i} \neq \emptyset$ and $\kappa_i^k \leq \nu_j$. But then $\langle N_l^k \mid l \leq i \rangle$ is normal. Hence for $j < i$ we have $N_{j+1}^k | \nu_j = N_i^k | \nu_j$; hence $N_i^0 | \nu_j = N_i^1 | \nu_j$, violating the def. of ν_i. QED

<u>Lemma 1.2</u> Let $\bar{\bar{N^0}}, \bar{\bar{N^1}} < \theta$, where θ is regular. Then the coiteration terminates below θ.

prf.

Suppose not. Let $X \prec H_{\theta^+}$ s.t. $N^0, N^1 \in X$, $\bar{\bar{X}} < \theta$ and $X \cap \theta = \bar{\theta}$ is transitive. Let $\sigma : \bar{H} \xleftrightarrow{\sim} X$, where $\bar{H}$ is transitive. Clearly $\langle N_i^k \mid i \leq \theta \rangle \in X$ $(k = 0, 1)$ and $\sigma(\langle N_i^k \mid i < \bar{\theta} \rangle) = \langle N_i^k \mid i < \theta \rangle$.

Since $N^k_{\bar\theta}, \langle \pi^k_{i\bar\theta} \mid i < \bar\theta \rangle = \lim_{i \leq j < \bar\theta} (N^k_i, \pi^k_{ij})$,

we conclude that $\sigma(\pi^k_{i\bar\theta}) = \pi^k_{i\theta}$ for

$i < \bar\theta$. Hence $\pi^k_{\bar\theta\theta} \circ \pi^h_{i\bar\theta} = \pi^k_{i\theta} = \sigma(\pi^k_{i\bar\theta}) =$

$= \sigma \circ \pi^h_{i\bar\theta}$ for $i < \bar\theta$. Hence:

(1) $\sigma \upharpoonright N^k_{\bar\theta} = \pi^h_{\bar\theta\theta}$ $(k = 0, 1)$.

Hence $\bar\theta = \mathrm{crit}(\pi^k_{\bar\theta\theta})$. By normality it follows that:

(2) $E^h_{\nu_{\bar\theta}}$ is a measure on $\bar\theta$ in $N^k_{\bar\theta} \| \nu_{\bar\theta}$

and:

(3) $X \in E^k_{\nu_{\bar\theta}} \longleftrightarrow \bar\theta \in \pi_{\bar\theta\theta}(X)$

for $X \in \mathbb{P}(\bar\theta) \cap N^k_{\bar\theta} \| \nu_{\bar\theta}$.

By the choice of $\nu_{\bar\theta}$ we have:

(4) $J^{E^0}_{\nu_{\bar\theta}} = J^{E^1}_{\nu_{\bar\theta}}$ where $N^k_{\bar\theta} = \langle J^{E^k}_\beta, E^k_\rho \rangle$.

But (1)-(4) give: E^k

(5) $E^0_{\nu_{\bar\theta}} = E^1_{\nu_{\bar\theta}}$,

contradicting the def. of $\nu_{\bar\theta}$.

QED Lemma 1.2

If N^0, N^1 are mice, it follows that they coiterate to a pair of comparable mice. We may, of course, have performed truncations on one or both sides of the iteration. However:

<u>Lemma 1.3</u> Let $\bar{M}, \bar{N}$ be mice and coiterate them to M, N. If the coiteration of $\bar{M}$ to M is not simple, then N is an initial segment of M (i.e. $N = M$ or $\bigvee \nu \in M \ N = M|\nu$).

proof.

Suppose not. Then M is a proper initial segment of N. Hence M is sound. Let $\langle M_{i \leq \theta} \rangle$ be the M-side of the coiteration. There is a maximal $i < \theta$ s.t. $\omega\alpha_i \in M_i$. Thus, letting $Q = M_i|\alpha_i$, we have $\rho^n_Q \leq \kappa_i^{M}$. Then M is a simple iterate of Q. By normality we have: $\rho^n_Q = \rho^n_M$. Moreover, if $p \in P^n_Q$, then $\pi_{QM}(p) \in P^n_M$, but $\pi_{QM}(p) \notin R^n_M$. Hence M is not sound. Contradiction! QED

<u>Corollary 1.4</u> Let $\bar{M}, \bar{N}, M, N$ be as above. If neither side of the coiteration is simple, then $M = N$.

The possibility described in Cor 1.4 must be regarded as a defect of the coiteration process. We shall show later that, in fact, one side of the coiteration is always simple.

. . . . .

<u>Iterability above a point</u>:

<u>Def</u> Let M be a pm. Let $\langle M_i \rangle$ be an iteration of M with <u>indices</u> $\langle \nu_i, h_i \rangle$. We call $\langle M_i \rangle$ an iteration <u>above</u> τ iff $\kappa_i \geq \tau$ whenever defined and $\nu_i > \tau$ for all i.

M is <u>iterable above</u> τ iff every iteration above τ can be continued.

Imitating the proof of §2.1 Lemma 1.1, using §1.4 Lemma 1', we get:

<u>Lemma 2</u> Let M be a mouse and let $\sigma : \bar{M} \rightarrow_{\Sigma_1^{(n)}} M$. Then $\bar{M}$ is iterable above $\omega\rho_{\bar{M}}^{n+1}$. (Moreover, if $\bar{M}'$ is an iterate of $\bar{M}$ above $\omega\rho_{\bar{M}}^{n+1}$ with iteration map $\bar{\pi}$, there exist an iterate M' of M with iteration map π and a map $\sigma' : \bar{M}' \rightarrow_{\Sigma_1^{(n)}} M'$ s.t. $\sigma'\bar{\pi} = \pi\sigma$.)

It follows exactly as before that:

<u>Lemma 3</u> Let M be iterable above τ. Then M has no degenerate iteration above τ.

<u>Lemma 4</u> Let $\bar{M}$ be iterable above τ & let M be an it. of $\bar{M}$ above τ with iteration map π. Let $\sigma : \bar{M} \rightarrow_{\Sigma^*} M$. Then M is a simple iterate of $\bar{M}$ and $\pi(\xi) \leq \sigma(\xi)$ for $\xi \in \bar{M}$.

<u>Def</u> M, N are <u>coiterable</u> <u>above</u> τ iff they are iterable above τ and both sides of the coiteration are above τ.

Clearly, if $\bar{M}, \bar{N}$ are coiterable above a point, then the conclusion of Lemmas 1.3, 1.4 hold.

<u>Lemma 5</u> Let M be an iterate of N above $\bar{\tau}$. Then

$$\mathbb{P}(\bar{\tau}) \cap \underline{\Sigma}^{*}_{N}(N) = \mathbb{P}(\bar{\tau}) \cap \underline{\Sigma}^{*}_{M}(M).$$

prf. Ind. on the length of the iteration, using §1.3 Cor 2.5.2