§2 Mice

§2.1

<u>Def</u> $M = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ is a <u>premouse</u> (pm) iff

(a) $E = \{\langle x, \nu \rangle \mid \nu \leq \alpha \wedge x \in E_\nu\}$ s.t. $(M|\nu) =_{Df} \langle J_\nu^E, E_\nu \rangle$ is acceptable for $\nu \leq \alpha$ and sound for $\nu < \alpha$.

(b) If $E_\nu \neq \emptyset$, there is $\kappa < \nu$ s.t.

(i) κ = the largest cardinal in $(M|\nu)$

(ii) E_ν is a normal measure on κ in $(M|\nu)$

*/ (iii) If $\pi : (M|\nu) \xrightarrow[E_\nu]{} N$, then $E^N|\nu = E^M|\nu$ and $E^N_\nu = \emptyset$.

**/ (c) If E_ν, κ are as in (b), $\kappa < \tau < \nu$, and $\langle J_\tau^E, E_\nu \cap J_\tau^E \rangle$ satisfies (a), (b), then $E_\tau = E_\nu \cap J_\tau^E$.

. . . . . .

<u>Note</u> If M is a pm and $\omega\alpha \leq On \cap M$, then $M|\alpha$ is a pm.

<u>Note</u> If E_ν is a measure on κ in M and $\pi : M \xrightarrow[E_\nu]{*} N$, then N is a pm with $E^N|\nu = E^M|\nu$, $E^N_\nu = \emptyset$, $\nu = \kappa^{+N}$.

*/ We think of (iii) as holding even when N is not well founded. The property of being a pm is then a Q-condition.

Def Let M be a pm. By an iteration of M with indices $\langle\langle\nu_i,\alpha_i\rangle \mid i+1<\theta\rangle$ we mean a sequence $\langle M_i \mid i<\theta\rangle$ of transitive iterates with iteration maps $\langle \pi_{ij} \mid i\leq j<\theta\rangle$ s.t.

(i) $M_0 = M$

(ii) The π_{ij} commute.

(iii) $\omega\nu_i \leq \omega\alpha_i \leq On \cap M_i$

(iv) If $E_{\nu_i} = \emptyset$, then $M_{i+1} = M_i \| \alpha_i$,

$$\pi_{i,i+1} = id \upharpoonright (M_i \| \alpha_i).$$

(v) If $E_{\nu_i} \neq \emptyset$, then

$$\pi_{i,i+1} : (M_i \| \alpha_i) \xrightarrow[E_{\nu_i}]{*} M_{i+1}$$

(vi) $\{i \mid \omega\alpha_i \in M_i\}$ is finite.

(vii) If λ is a limit ordinal, then

$$M_\lambda, \langle \pi_{i\lambda} \mid i<\lambda\rangle = \lim_{i\leq j<\lambda} (M_i, \pi_{ij}).$$

Note The meaning of (vii) is straight forward, since $\mathrm{dom}(\pi_{i\lambda}) = M_i$ for sufficiently large $i<\lambda$, by (vi).

<u>Note</u> It follows that M_i is a pm and that $\pi_{ij} : (M_i | \beta) \longrightarrow_{\Sigma^*} M_j$ for some β $(i \leq j < \theta)$.

<u>Def</u> An iteration is <u>standard</u> iff

(a) $\omega\alpha_i = On \cap M_i$ if $E_{\nu_i} = \emptyset$

(b) $\alpha_i =$ the max α s.t. E_{ν_i} is a measure in $M_i | \alpha$ for $E_{\nu_i} \neq \emptyset$.

<u>Note</u> There is a max. α s.t. E_{ν_i} is a measure in $M_i | \alpha$, since if it fails for $\omega\alpha = On \cap M_i$, then there is a least $\alpha \geq \nu_i$ s.t. $\mathbb{P}(\nu_i) \cap J^E_{\alpha+1} \not\subset J^E_\alpha$.

If an it. is standard, we refer to $\langle \nu_i \mid i < \theta \rangle$ as its sequence of indices.

<u>Def</u> An iteration is <u>simple</u> iff $\omega\alpha_i = On \cap M_i$ for all i. (In this case we call the M_i <u>simple iterates</u> of M).

Def A pm M is a mouse iff it is iterable in the sense that every iteration $\langle M_i \mid i < \theta \rangle$ can be continued — i.e.

(a) If $\theta = k+1$, $\omega\nu \leq \omega\alpha \leq On \cap M_k$, and E_ν is a measure in $M_k \mid \alpha$, then $\pi : (M_k \mid \alpha) \xrightarrow[E_\nu]{*} M'$ exists.

(b) If $\lim(\theta)$, then $\langle M_i \rangle$, $\langle \pi_{ij} \rangle$ has a well founded limit.

<u>Lemma 1.1</u> Let M be a mouse and let $\sigma : \overline{M} \xrightarrow[\Sigma^*]{} M$. Then $\overline{M}$ is a mouse.

proof.

$\overline{M}$ is clearly a pm. We must show iterability. Let $\langle \overline{M}_i \mid i < \theta \rangle$ be an iteration of $\overline{M}$ with indices $\langle \overline{\nu}_i, \overline{\alpha}_i \rangle$ and iteration maps $\overline{\pi}_{ij}$. We show that $\langle \overline{M}_i \rangle$ can be continued. We first define an iteration $\langle M_i \mid i < \theta \rangle$ of M with indices $\langle \nu_i, \alpha_i \rangle$, maps π_{ij}, and imbeddings $\sigma_i : \overline{M}_i \xrightarrow[\Sigma^*]{} M_i$ s.t. $\sigma_j \overline{\pi}_{ij} = \pi_{ij} \sigma_i$. Set $\sigma_0 = \sigma$. Let σ_i be defined. Set $\nu_i = \sigma_i(\overline{\nu}_i)$ if $\omega\overline{\nu}_i \in \overline{M}_i$, $\omega\nu_i = On \cap M_i$ if not. Similarly for α_i. If E_{ν_i} is a measure in M_i; we set:

$\pi_{i,i+1} : (M_i \| \alpha_i) \xrightarrow[E_i]{*} M_{i+1}$ and define

$\sigma_{i+1}(\overline{\pi}_{i,i+1}(f)(\overline{u})) = \pi_{i,i+1}\sigma_i(f)(u)$

for $f \in \Gamma(\overline{u}, \overline{M}_i \| \overline{\alpha}_i)$. Otherwise set:

$M_{i+1} = (M_i \| \alpha_i)$, $\sigma_{i+1} = \sigma_i \upharpoonright \overline{M}_{i+1}$.

For limit λ define M_λ, $\langle \pi_{i\lambda} \rangle$ as the direct limit and define σ_λ

σ_λ by : $\sigma_\lambda \bar{\pi}_{i\lambda} = \pi_{i\lambda} \sigma_i \quad (i < \lambda)$.

We now show that $\langle \bar{M}_i \rangle$ can be continued.

<u>Case 1</u> $\theta = k+1$. Let $\omega\bar{\delta} \leq On \wedge \bar{M}_k$, $\bar{N} = (\bar{M}_k | \bar{\delta})$, $E_{\bar{\nu}}^{\bar{N}}$ a measure in $\bar{N}$, let $\omega\delta = \sigma_k(\omega\bar{\delta})$ if $\omega\bar{\delta} \in \bar{M}_k$; $\omega\delta = On \wedge M_k$ if not. Define ν similarly. Set $N = (M_k | \delta)$. Then $\sigma_k \upharpoonright \bar{N} : \bar{N} \to_{\Sigma^*} N$ and E_ν^N is a measure in N. Hence $\pi : N \to_{E_\nu} N'$ exists. Hence so does $\bar{\pi} : \bar{N} \to_{E_{\bar{\nu}}} \bar{N}'$.

<u>Case 2</u> $\lim(\theta)$. Let $N, \langle \pi_i \rangle = \lim_{i \leq j} (M_i, \pi_{ij})$ and let $\bar{N}, \langle \bar{\pi}_i \rangle$ be the direct limit of $\langle \bar{M}_i \rangle, \langle \bar{\pi}_{ij} \rangle$. We can define $\in$-preserving $\sigma : \bar{N} \to N$ by : $\sigma \bar{\pi}_i = \pi_i \sigma_i$. Hence $\bar{N}$ is well founded.

QED (Lemma 11)

<u>Def</u> M is <u>countably</u> <u>iterable</u> iff every countable iteration of M can be continued.

Obviously the above proof yields:

<u>Cor 1.2</u> If $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M$ and M is countably iterable, then $\bar{M}$ is countably iterable.

But then:

<u>Lemma 1.3</u> If M is countably iterable, it is a mouse.

proof.

Suppose not. Let $\langle M_i \rangle$ be an iteration which cannot be continued. Let $\langle M_i \rangle \in H_\theta$ where θ is regular + let $\sigma : \bar{H} \prec H_\theta$, where $\bar{H}$ is countable and transitive and $\sigma(\langle \bar{M}_i \rangle) = \langle M_i \rangle$. Then $\langle \bar{M}_i \rangle$ is a countable iteration of $\bar{M} = \bar{M}_0$ which, by absoluteness, cannot be continued.

However, $(\sigma \upharpoonright \bar{M}) : \bar{M} \prec M$, contradicting Cor 1.2

QED (Lemma 1.3)

We made life easier by imposing condition (iv) in the def. of iteration. However, we shall have to deal with the possibility of "iterations" in which (iv) fails.

<u>Def</u> $\langle M_i \mid i<\theta \rangle$ is a <u>degenerate iteration</u> w. indices $\langle \nu_i, \alpha_i \rangle$ $(i+1<\theta)$ if it satisfies (i) – (v), (vii) of the above def., but $\{i \mid \omega\alpha_i \in M_i\}$ is infinite.

<u>Lemma 2</u> Let M be a mouse. Then M has no degenerate iteration.

proof.

Suppose not.

Then there exists a <u>bad</u> <u>sequence</u> $\langle M_i \rangle$, $\langle \nu_i \rangle$ in the following sense:

(1) $M_0 = M$; $\omega\nu_i \in M_{i+1}$

(2) M_{i+1} is a simple iterate of M_i', where $M_0' = M_0$; $M_{i+1}' = (M_{i+1} \| \nu_i)$.

By Lemma 1.1 and a Löwenheim–Skolem argument we may assume:

(3) M_i is countable $(i<\omega)$.

We may also choose the sequence in such a way that:

(4) ν_i is the least s.t. there is a bad sequence $\langle N_i\rangle, \langle \tau_i\rangle$ with $N_0 = M_{i+1}|\nu_i$.

Let $\pi_i : M_i' \to M_{i+1}$ be the iteration map $(i<\omega)$.

<u>Claim 1</u> Let $E_\nu^{M_{i+1}} \neq \emptyset$. Then $\nu \leq \nu_i$ $(i \geq 1)$

proof.

Suppose not. Let $\langle M_j \mid j>i\rangle \in V_\theta$. Iterate $\tilde{M} = M_{i+1}|\nu$ to a $\tilde{\kappa} > \theta$ using the measure $\tilde{E} = E_\nu^{M_{i+1}}$.

Let $\sigma : \tilde{M} \to \tilde{M}'$ be the iteration map. On the admissible structure $\tilde{N} = J_{\tilde{\kappa}}^{E'}$ $(\tilde{M}' = J_{\alpha'}^{E'})$ consider the infinitary language with predicate $\in$, constants $\underline{x}$ $(x \in \tilde{N})$, constant $\dot{H}$, and the following axioms:

ZF^-; $\bigwedge z\,(z \in \underline{x} \longleftrightarrow \bigvee\!\!\!\bigvee_{y \in x} z = \underline{y})$;

$\dot{H}$ is transitive; $rn(\dot{H}) = \theta$;

there is a bad sequence $\langle Q_i \rangle \in \dot{H}$ s.t. $Q_0 = \underline{(M_{i+1} | \nu_i)}$, (where $M_{i+1}|\nu_i = N|\nu_i$

Clearly this system $\mathcal{L} = \mathcal{L}(\tilde{N}, \theta, \nu_i)$ is consistent. Set $N = J^E_\kappa$. Since $N \prec \tilde{N}$ we conclude that there are $\bar{\theta}, \bar{\nu} < \kappa$ s.t. $\mathcal{L}(N, \bar{\theta}, \bar{\nu})$ is consistent. Since $\kappa \in M_{i+1}$

and $\pi_i : M_i' \underset{\Sigma^*}{\longrightarrow} M_{i+1}$,

we conclude that there is $\hat{\kappa} \in M_i'$ with the property that $\hat{N} = J^{E^{M_i'}}_{\hat{\kappa}}$ is admissible and $\mathcal{L}(\hat{N}, \hat{\theta}, \hat{\nu})$ is consistent for $\hat{\theta}, \hat{\nu} < \hat{\kappa}$. By Barwise' completeness theorem we conclude that there is a bad sequence $\langle Q_i \rangle$ with $Q_0 = M_i' | \hat{\nu} = M_i | \hat{\nu}$. But $\hat{\nu} < \nu_{i+1}$, contradicting (4). QED (Claim 1)

Set $\eta_i = \{\nu \mid E_\nu^{M_{i+1}} \neq \emptyset\}$. We have shown:

(6) $\nu_i \geq \eta_i$ for $i \geq 1$. Set:

u_i = the set of $\eta \geq \eta_i$ s.t. some $E_\nu^{M_{i+1}}$ is a measure in $M_{i+1} | \eta$ but not in $M_{i+1} | \eta + 1$.

For $\eta \in u_i$ let κ_η be the least κ which is a critical point of such an $E_\nu^{M_{i+1}}$.

Then $\eta < \eta' \to \kappa_\eta > \kappa_{\eta'}$. Hence:

(7) u_i is finite.

By (6), $u_i \cap \omega\nu_i$ has the same definition in $M'_{i+1} = M_{i+1} | \nu_i$ as u_i in M_{i+1}. Thus:

(8) $u_{i+1} = \pi_{i+1}{}''(u_i \cap \omega\nu_i)$ $(i \geq 1)$

Hence $\overline{\overline{u}}_{i+1} \leq \overline{\overline{u}}_i$ for $i \geq 1$. It follows that there is $i_0 > 1$ s.t. $\overline{\overline{u}}_i = n$ is constant for $i+1 \geq i_0$.

Hence

(10) $u_i \subset \omega\nu_i$ for $i+1 \geq i_0$.

But this means that, for $i \geq i_0$, every $E_\nu^{M_i}$ which is a measure in M_i' remains a measure in M_i. We now construct a sequence $\tilde{M}_i$ $(i \geq i_0)$ of simple iterates of M_{i_0} with iteration maps $\tilde{\pi}_i : \tilde{M}_i \longrightarrow \tilde{M}_{i+1}$. Simultaneously we define $\tilde{\nu}_{i-1} \in \tilde{M}_i$ and $\sigma_i : M_i' \underset{\Sigma^*}{\longrightarrow} (\tilde{M}_i | \tilde{\nu}_{i-1})$. Set: $\tilde{M}_{i_0} = M_{i_0}$, $\tilde{\nu}_{i_0-1} = \nu_{i_0-1}$, $\sigma_{i_0} = \mathrm{id}$. Since every $E_\xi^{M_{i_0}}$ which is a measure in M_{i_0}' remains so in M_{i_0}, we can imitate the construction in Lemma 1.1 to get an iterate $\tilde{M}_{i_0+1}$ of $\tilde{M}_{i_0}$ with iteration map $\tilde{\pi}_{i_0} : \tilde{M}_{i_0} \rightarrow \tilde{M}_{i_0+1}$ and $\sigma : M_{i_0+1} \underset{\Sigma^*}{\longrightarrow} (\tilde{M}_{i_0+1} | \tilde{\pi}_{i_0}(\nu_{i_0-1}))$ s.t. $\tilde{\pi}_{i_0} \sigma_{i_0} = \sigma \pi_{i_0}$. Set $\tilde{\nu}_{i_0} = \sigma(\nu_{i_0})$; $\sigma_{i_0+1} = \sigma \upharpoonright M_{i_0+1}'$. Then $\tilde{\nu}_{i_0} \supset \sigma(U_{i_0+1}) = U_{\tilde{M}_{i_0+1}}$ in the obvious sense; hence every measure in $\tilde{M}_{i_0+1} | \tilde{\nu}_{i_0}$ is a measure in $\tilde{M}_{i_0+1}$. This gives an iteration $\tilde{\pi}_{i_0+1} : \tilde{M}_{i_0+1} \longrightarrow \tilde{M}_{i_0+2}$ and

a map $\sigma : M_{i_0+2} \xrightarrow[\Sigma^*]{} (\tilde{M}_{i_0+2} | \tilde{\pi}(\nu_{i_0}))$

s.t. $\quad \tilde{\pi}_{i_0+1}\sigma_{i_0+1} = \sigma\pi_{i_0+1}$.

We then set $\tilde{\nu}_{i_0+1} = \sigma(\nu_{i_0+1})$ etc.

Note that under this construction:

(10) $\tilde{\nu}_i < \tilde{\pi}_i(\tilde{\nu}_{i-1})$ for $i \geq i_0$.

Set $\tilde{\pi}_{ii} = \mathrm{id} \restriction \tilde{M}_i$, $\tilde{\pi}_{i\,j+1} = \tilde{\pi}_j \tilde{\pi}_{ij}$

for $i_0 \leq i \leq j < \omega$. By the iterability of M we can form the limit:

$$M^*, \langle \pi_i^* \rangle = \lim_{i \leq j} (\tilde{M}_i, \tilde{\pi}_{ij}).$$

But then $\pi^*_{i+1}(\tilde{\nu}_i) < \pi^*_i(\tilde{\nu}_{i-1})$ for $i \geq i_0$

Contradiction! QED (Lemma 2)

<u>Lemma 3</u> Let M be an iterate of the mouse $\bar{M}$ with iteration map π. Let $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M$. Then the iteration is simple and $\sigma(\xi) \geq \pi(\xi)$ for all $\xi \in \bar{M}$.

prf.

Suppose not. Define a relation R on pairs $\langle N, \nu \rangle$ s.t. N is a mouse and $\nu \in N$ by:

$\langle N, \nu \rangle R \langle \bar{N}, \bar{\nu} \rangle$ iff

iff N is an iterate of $\bar{N}$ with iteration map π and either $\nu < \pi(\bar{\nu})$ or the iteration is nonsimple.

R is easily seen to be well founded using Lemma 2. Now let $\langle \bar{M}, \bar{\xi} \rangle$ be R-minimal for the property: There is an iterate M of $\bar{M}$ with iteration map π and there is $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M$ s.t. either $\sigma(\bar{\xi}) < \pi(\bar{\xi})$ or the iteration is non simple. By the construction of Lemma 1.1 we can then obtain an iterate M' of M with iteration map π' and map $\sigma' : M \xrightarrow[\Sigma^*]{} M'$ s.t.

$$\begin{tikzcd} \bar{M} \arrow[r, "\pi"] \arrow[d, "\sigma"'] & M \arrow[d, "\sigma'"] \\ M \arrow[r, "\pi'"] & M' \end{tikzcd}.$$

If the iteration from $\bar{M}$ to M is non-simple, then so is the "mirror" iteration from M to M'. Otherwise $\sigma(\bar{\xi}) < \xi =_{Df} \pi(\bar{\xi})$ and we get:

$$\sigma'(\xi) = \sigma'\pi(\bar{\xi}) = \pi'\sigma(\bar{\xi}) < \pi'(\xi).$$

This contradicts the minimality of $\langle \bar{M}, \bar{\xi} \rangle$. QED (Lemma 3)

Cor 3.1 M cannot be both a simple and nonsimple iterate of $\bar{M}$.

Cor 3.2 Let M be a simple iterate of the mouse $\bar{M}$. Then there is a unique iterati map.

Def $\pi_{\bar{M}M}$ = the unique simple iteration map from $\bar{M}$ to M.

Def If M_j is a simple iterate of M_i for $i \leq j < \theta$, we write

$$M = \lim_i M_i$$

for: $M, \langle \pi_{M_i, M} \rangle = \lim_{i \leq j} (M_i, \pi_{M_i M_j})$.