§ 1.4 Extendability

Def N is $*$-extendable by E iff
iff there exists $\pi : N \xrightarrow[E]{*} N'$.

In this section we develop criteria of extendability.

Def $\sigma : \langle N, E \rangle \longrightarrow^{*} \langle N', E' \rangle$ iff
iff the following hold;

(a) $\sigma : N \longrightarrow_{\Sigma^*} N'$;

(b) E is a weak extender on N s.t. $\tilde{E} = \{\langle a, x\rangle \mid x \in E_a\}$ is rud over N in a parameter p;

(c) E' is a weak extender on N' and $\tilde{E}'$ is rud over N in $\sigma(p)$ by the same definition.

Lemma 1 Let $\sigma : \langle \bar{N}, \bar{E} \rangle \longrightarrow^{*} \langle N, E \rangle$ and $\pi : N \xrightarrow[E]{*} N'$. Then there is $\bar{\pi} : \bar{N} \xrightarrow[\bar{E}]{*} \bar{N}'$.
Moreover there is a unique $\sigma' : \bar{N}' \longrightarrow_{\Sigma^*} N'$ s.t. $\sigma'\bar{\pi} = \pi\sigma$ and $\sigma' \upharpoonright \bar{\nu} = \sigma \upharpoonright \bar{\nu}$, where $\bar{E}$ is at $\bar{\kappa}, \bar{\nu}$. σ' is defined by: $\sigma'(\bar{\pi}(f)(a)) = \pi\sigma(f)(\sigma(a))$ for $f \in \Gamma(\bar{\kappa}, \bar{N})$, $a \in [\bar{\nu}]^{<\omega}$.

proof of Lemma 1.

Let $\vartheta = \langle \bar{D}, \bar{e}, \bar{I}, \bar{\vec{B}} \rangle$ be defined from $\bar{N}, \bar{E}$ in the usual way. Then

$\langle a, f \rangle \bar{I} \langle b, g \rangle$ iff $\{u \mid f^{ac}(u) = g^{bc}(u)\} \in \bar{E}_c$

iff $\{u \mid \sigma(f)^{\sigma(a)\sigma(c)}(u) = \sigma(g)^{\sigma(b)\sigma(c)}(u)\}$

$\in E_{\sigma(c)}$

iff $\pi\sigma(f)(\sigma(a)) = \pi\sigma(g)(\sigma(b))$.

Similarly:

$\langle a, f \rangle \bar{e} \langle b, g \rangle$ iff $\pi\sigma(f)(\sigma(a)) \in \pi\sigma(g)(\sigma(b))$

Thus $\bar{e}$ is well founded and $\bar{\pi} : \bar{N} \xrightarrow[\bar{E}]{} {}^* \bar{N}'$ exists. Clearly there is a structure preserving map $\sigma' : \bar{N}' \longrightarrow N'$ defined by: $\sigma'(\bar{\pi}(f)(a)) = \pi\sigma(f)(\sigma(a))$.

<u>Claim</u> $\sigma' : \bar{N}' \xrightarrow[\Sigma^*]{} N'$.

proof.

(1) Let $\bar{n} < \omega\rho_N^{n+1}$. Then $\sigma' : \bar{N}' \xrightarrow[\Sigma_1^{(n)}]{} N'$.

pf. Let φ be $\Sigma_1^{(n)}$.

$\bar{N}' \models \varphi[\bar{\pi}(\vec{f})(a)] \longleftrightarrow \underbrace{\{u \mid \bar{N} \models \varphi[\vec{f}(u)]\} \in \bar{E}_a}_{\Delta_1^{(n+1)}}$

$\longleftrightarrow \{u \mid N \models \varphi[\sigma(\vec{f})(u)]\} \in E_{\sigma(a)}$

$\longleftrightarrow N' \models \varphi[\pi\sigma(\vec{f})(\sigma(a))]$. QED(1)

(2) Let $\omega\rho_{\bar{N}}^{n+1} \leq \bar{\kappa} < \omega\rho_{\bar{N}}^{n}$. Then

$$\sigma' : \bar{N}' \longrightarrow_{\Sigma_1^{(n)}} N'.$$

proof.
Let $\bar{A}$ be $\Sigma_1^{(n)}(\bar{N}')$ & let A be $\Sigma_1^{(n)}(N')$ by the same definition.
Let $x \in \bar{N}'$. We claim:

<u>Claim</u> $\bar{A}(x) \longleftrightarrow A(\sigma'(x))$.

Let $x = [\langle a, f\rangle]^{\bar{N}'} = \pi(f)(a)$ where $a \in [\bar{\nu}]^m$, $f \in \Gamma(\bar{\kappa}, \bar{N})$. Then
$\sigma'(x) = [\langle \sigma(a), \sigma(f)\rangle]^{N'} = \pi(f)(a)$.
Choose $p \in \bar{N}$ s.t. $f = p$ or f is a good $\Sigma_1^{(n-1)}(\bar{N})$ function in p by a func. absolute definition. Then $\sigma(f)$ bears the same relation to $\sigma(p)$ in N.
The prf. of (10) in §1.3 Lemma 2 shows that: There is $\bar{A}^*$ which is $\Sigma_1^{(n)}(\bar{N})$ in p (unif. in the def. of $\bar{A}, f$) s.t. $\bar{A}^*(a) \longleftrightarrow \bar{A}([\langle a, f\rangle])$ for all $a \in [\bar{\nu}]^m$. Similarly, if A^* is $\Sigma_1^{(n)}(N)$ in $\sigma(p)$ by

the same definition, we have:

$$\bar{A}(\bar{\pi}(f)(a)) \longleftrightarrow \bar{A}^*(a)$$
$$\longleftrightarrow A^*(\sigma(a))$$
$$\longleftrightarrow A(\pi(\sigma(f))(\sigma(a)))$$

QED (2)

But $H^{\bar{N}}_{\bar{\kappa}} = H^{\bar{N}'}_{\bar{\kappa}}$ and $\bar{\pi} \upharpoonright H^{\bar{N}}_{\bar{\kappa}} = \mathrm{id}$. We can carry the above argument a step further and show that if $\bar{A}(\vec{y}, \vec{x})$ is $\Sigma_1^{(n)}(\bar{N}')$ and $x = [\langle a, f\rangle]$ and p are as above, there is $\bar{A}^*$ which is $\Sigma_1^{(n)}(\bar{N})$ in p (unif. in the def. of $\bar{A}$, f) s.t.

$$\bar{A}(\vec{y}, \bar{\pi}(f)(a)) \longleftrightarrow \bar{A}(\vec{y}, a)$$

for all $\vec{y} \in H^{\bar{N}}_{\bar{\kappa}}$. Since $\sigma' \upharpoonright H^m_{\bar{N}'} = \sigma \upharpoonright H^m_{\bar{N}}$ for $\omega\rho^m_{\bar{N}} \leq \bar{\kappa}$ and σ is Σ^*-preserving, we may conclude:

(3) $\sigma' : \bar{N}' \longrightarrow_{\Sigma_1^{(n)}} N'$ for $\omega\rho^n_{\bar{N}} \leq \bar{\kappa}$.

This proves that σ' is Σ^*-preserving. The uniqueness of σ' is straightforward.

QED (Lemma 1)

<u>Note</u> In Lemma 1 we can obviously replace the assumption $\pi : N \xrightarrow[E]{*} N'$ by:

$\pi : N \xrightarrow[\Sigma^*]{} N'$ s.t. $\mathrm{crit}(\pi) = \kappa$ and $a \in \pi(X)$ iff $X \in E_a$, whenever $a \in [\nu]^{<\omega}$ and $X \in N \cap \mathbb{P}([\kappa]^{|a|})$.

. . . .

<u>Def</u> Let E be a weak extender at κ, ν on N. E is ω-<u>complete</u> iff for every sequence $\langle a_i, X_i \rangle$ s.t. $X_i \in E_{a_i}$ $(i < \omega)$, there is an order preserving map $\sigma : \bigcup_i a_i \longrightarrow \kappa$ s.t. $\sigma(a_i) \in X_i$ for $i < \omega$.

Lemma 2.1 Let $\sigma : \langle \bar{N}, \bar{E} \rangle \to^{*} \langle N, E \rangle$ where $\bar{N}$ is countable and E is ω-complete. Then $\pi : \bar{N} \to^{*}_{\bar{E}} \bar{N}'$ exists. Moreover there is a $\sigma' : \bar{N}' \to_{\Sigma^*} N$ s.t. $\sigma' \bar{\pi} = \pi$.

proof.

Let E be at κ, ν and $\bar{E}$ at $\bar{\kappa}, \bar{\nu}$. By ω-completeness there is $\delta : \bar{\nu} \to \kappa$ which is order preserving s.t. $\delta(a) \in \sigma(X)$ whenever $X \in \bar{E}_a$.

Form $\bar{\mathfrak{D}} = \langle \bar{D}, \bar{e}, \bar{I}, \dot{\bar{B}} \rangle$ as usual. It is easily seen that:

$$\langle a, f \rangle \underset{\bar{e}}{\bar{I}} \langle b, g \rangle \longleftrightarrow \sigma(f)(\delta(a)) \underset{\in}{=} \sigma(g)(\delta(b))$$

Thus $\bar{e}$ is well founded. Hence $\bar{\pi} : \bar{N} \to^{*}_{\bar{E}} \bar{N}'$ exists. Moreover there is a structure preserving $\sigma' : \bar{N}' \to N$ defined by:

$\sigma'(\bar{\pi}(f)(a)) = \pi\sigma(f)(\delta(a))$.

A virtual repetition of the proof of Lemma 1 then shows that σ' is Σ^* preserving.

QED (Lemma 2.1)

<u>Cor 2.2</u> Let E be a weak extender on N s.t. $\tilde{E}$ is <u>snd</u> over N. If E is ω-complete, then N is $*$-extendable by E.

prf.
Suppose not. Let $N \in H_\theta$ for a regular θ and let $N \in X \prec H_\theta$ where X is countable. Let $F : \bar{H} \leftrightarrow X$ where $\bar{H}$ is transitive. Set: $\bar{N} = F^{-1}(N)$, $\bar{E} = F^{-1}(E)$. Then $\bar{N}$ is not extendable by $\bar{E}$, by absoluteness. But $\sigma : \langle \bar{N}, \bar{E} \rangle \xrightarrow{*} \langle N, E \rangle$ where $\sigma = F \upharpoonright \bar{N}$. Hence $\bar{N}$ is extendable. Contradiction! QED

Finally we note a variant of Lemma 1 which will be used in the sequel. We first alter the def. of $\longrightarrow^*$ as follows:

Def $\sigma : \langle N, E\rangle \longrightarrow^{(n)} \langle N', E'\rangle$ iff

(a) $\sigma : N \longrightarrow_{\Sigma_1^{(n)}} N'$ and

(b), (c) as in the def. of $\longrightarrow^*$.

The proof of Lemma 1 then trivially gives:

Lemma 1' Let $\sigma : \langle \bar N, \bar E\rangle \longrightarrow^{(n)} \langle N, E\rangle$ where $\bar E$ is an extender on $\bar\kappa \geq \omega\rho^n_{\bar N}$. Let $\pi : N \longrightarrow^*_E N'$. Then there is $\bar\pi : \bar N \longrightarrow^*_{\bar E} \bar N'$. Moreover there is a unique $\sigma' : \bar N' \longrightarrow_{\Sigma_1^{(n)}} N'$ s.t. $\sigma'\bar\pi = \pi\sigma$ and $\sigma' \upharpoonright \bar\nu = \sigma \upharpoonright \bar\nu$ (where E is at $\bar\kappa, \bar\nu$). σ' is defined by:

$$\sigma'(\pi(f)(a)) = \pi\sigma(f)(\sigma(a))$$

for $f \in \Gamma(\bar\kappa, \bar N)$, $a \in [\bar\nu]^{<\omega}$.