§ 1.3 Σ^* Ultrapowers

Suppose that $\bar{N}$ is a J-model and E is an extender at κ, ν, where $\kappa \in \bar{N}$ and E is "weakly amenable" wrt. $\bar{N}$ [i.e. If $\langle X_i \mid i < \kappa \rangle \in \bar{N}$, $X_i \in [\kappa]^{|a|}$ for $i < \kappa$, $a \in [\nu]^{<\omega}$, then $\{i \mid X_i \in E_a\} \in \bar{N}$].

We write: $\pi : \bar{N} \longrightarrow_E N$ to mean that N is the transitivized ultrapower of $\bar{N}$ by E (using functions $f \in \bar{N}$) and π is the canonical injection. $\pi : \bar{N} \longrightarrow_E N$ can be alternatively characterized by the following conditions:

(a) $\pi : \bar{N} \longrightarrow_{\Sigma_0} N$ cofinally where N is transitive,

(b) $N =$ the Σ_0 closure of $\nu \cup \mathrm{rng}(\pi)$ in N.

(c) $\mathrm{crit}(\pi) = \kappa < \nu \leq \pi(\kappa)$,

(d) $E = \langle E_a \mid a \in [\nu]^{<\omega} \rangle$ where

$$E_a = \{X \in \mathbb{P}([\kappa]^{|a|}) \cap \bar{N} \mid a \in \pi(X)\}$$

From time to time we may drop the requirement that N be transitive (though we still require its well founded core to be so). As this case we call E a <u>weak</u> <u>extender</u> on $\bar{N}$. E is a weak extender if it satisfies a few simple structural conditions which ensure that the ultraproduct construction works. In particular, if $\langle \bar{N}, \tilde{E} \rangle$ is amenable, where $\tilde{E} = \{\langle x, a \rangle \mid x \in E_a\}$, then being a weak extender is a Π_1 condition on $\langle \bar{N}, \tilde{E} \rangle$.

. . . . . .

If $\pi : \bar{N} \to_E N$, then π is Σ_1 preserving but not, in general, Σ^* preserving. We shall therefore introduce a new notion of Σ^*-ultrapower and write $\pi : \bar{N} \to_E^* N$ to mean that N is the Σ^*-ultrapower of $\bar{N}$ by E and π is the canonical injection. The notion $\pi : \bar{N} \to_E^* N$ can actually be characterized by a short set of conditions analogous to (a) - (e) above. To wit:

(a) $\pi : \bar{N} \xrightarrow[\Sigma^*]{} N$ where N is transitive.

(b) Let $\rho = \inf\{\rho^n_{\bar{N}} \mid \kappa < \omega\rho^n_{\bar{N}}\}$. Let $\bar{H} = H^{\bar{N}}_{\omega\rho}$, $H = \bigcup_{x \in \bar{H}} \pi(x)$. Then

$$\pi \upharpoonright \bar{H} : \bar{H} \xrightarrow[E]{} H.$$

(c) N = the closure of $H \cup \mathrm{rng}(\pi)$ under good $\Sigma^{(n)}_1(N)$ functions for n s.t. $\kappa < \omega\rho^{n+1}_{\bar{N}}$.

(If there is no such n, this means: $|N| = H$).

As in the ordinary case, however, this def. is of little use without an ultraproduct construction which enables us to prove the existence of $\pi : \bar{N} \xrightarrow[E]{*} N$. We shall therefore describe that construction in detail and take that as our official definition of $\pi : \bar{N} \xrightarrow[E]{*} N$. N will be the ultraproduct of $\bar{N}$ by E using an appropriate set of Σ^* fcns.

In place of weak amenability we make the stronger assumption:

(*) $\tilde{E}$ is $\underline{\text{rud}}$ in $\bar{N}$. ($\tilde{E} = \{\langle x, a\rangle \mid x \in E_a\}$

(Here $\underline{\text{rud}}$ means rud in some parameter). (__Note__ For most of what we do it would suffice to assume that E_a is $\underline{\text{rud}}$ in $\bar{N}$ for $a \in [\nu]^{<\omega}$)

__Def__ Let $\kappa \in \bar{N}$.

$\Gamma = \Gamma(\kappa, \bar{N})$ = the set of f s.t. $f : [\kappa]^i \to \bar{N}$ for some $i < \omega$ and either $f \in \bar{N}$ or f is a good $\underline{\Sigma}_1^{(n)}(\bar{N})$ fcn for an n s.t. $\omega\rho_{\bar{N}}^{n+1} > \kappa$.

We assume E to be an extender at κ, ν where $\kappa \in \bar{N}$ and will define N to be the ultrapower of $\bar{N}$ by E using the fcns Γ ($N = \mathrm{Ult}(\bar{N}, E; \Gamma)$).

Explicitly, we first define a domain $D = \{\langle a, f\rangle \mid a \in [\nu]^{<\omega}, f \in \Gamma, \mathrm{dom}(f) = [\kappa]^{|a|}\}$

We define a pseudo $\in$ - relation e on D by:

$\langle a,f\rangle e \langle b,g\rangle$ iff

$$\{u \mid f^{a,c}(u) \in g^{b,c}(u)\} \in E_c$$

where $c \supset a \cup b$ and $f^{a,c}$ is defined by:

$$f^{a,c}(\{\xi_1,\dots,\xi_n\}) = f(\{\xi_{j_1},\dots,\xi_{j_m}\})$$

$(\xi_1 < \dots < \xi_n)$ where $c = \{\gamma_1,\dots,\gamma_n\}$ $(\gamma_1 < \dots < \gamma_n)$ and $a = \{\gamma_{j_1},\dots,\gamma_{j_m}\}$ $(j_1 < \dots < j_m)$.

We similarly define:

$\langle a,f\rangle I \langle b,g\rangle$ iff

$$\{u \mid f^{ac}(u) = g^{ac}(u)\} \in E_c$$

$\mathring{B}(\langle a,f\rangle)$ iff $\{u \mid \bar{B}(f(u))\} \in E_b$

where $\bar{N} = J_{\bar{\beta}}^{\bar{B}}$.

We now show that the structure $\mathfrak{A} = \langle D, I, e, \mathring{B}\rangle$ satisfies Łoś theorem for Σ_0 formulae:

<u>Lemma 1.1</u> Let $\langle a, f_1 \rangle, \ldots, \langle a, f_n \rangle \in D$. Let φ be Σ_0. Then:

$$\mathfrak{A} \models \varphi[\overrightarrow{\langle a, f \rangle}] \quad \longleftrightarrow \quad \{u \mid \bar{N} \models \varphi[f(u)]\} \in E_a$$

We prove this by induction on φ, making use of the following lemma as necessary:

<u>Lemma 1.2</u> Let $R(y^m, x^{j_1}, \ldots, x^{j_r})$ be $\Sigma_1^{(m)}(\bar{N})$ where $\omega\rho_N^m > \kappa$ and $\vec{j} \leq m$. Let $n \geq m$ s.t. $\omega\rho_N^{n+1} > \kappa$ and let $f_1, \ldots, f_r \in \Gamma$ be good $\underset{=}{\Sigma}_1^{(m)}(\bar{N})$ maps where f_i is to H^{j_i} and $\mathrm{dom}(f_i) = [\kappa]^p$ for a fixed $p < \omega$. Then there is a good $\underset{=}{\Sigma}_1^{(n)}(\bar{N})$ map $g \in \Gamma$ s.t.

$$\bigvee y^m\, R(y^m, \vec{f}(u)) \longleftrightarrow R(g(u), \vec{f}(u))$$

for all $u \in [\kappa]^p$.

(Note: $m = j_1 = \ldots = j_r = 0$ in the case used in the proof of Lemma 1.1).

proof of Lemma 1.2.

By Lemma 5.3 of §1.2 there is a $\Sigma_1^{(m)}$ fcn F to H^m s.t.

$$\bigvee y^m R(y^m, \vec{x}) \longleftrightarrow R(F(\vec{x}), \vec{x}).$$

Set: $G(u) \simeq F(\vec{f}(u))$ $(u \in [\alpha]^p)$.
Then G is $\underline{\Sigma}_1^{(m)}$ and good. Clearly $\mathrm{dom}(G) \in \bar{N}$, since $\alpha < \omega\rho_{\bar{N}}^{n+1}$.
But then $g \in \Gamma$ has the desired property where $g(u) = G(u)$ if defined, $g(u) = 0$ if not. * QED (Lemma 1.2)

By Lemma 1.1, I is an equality relation for $\mathfrak{r}^c$ and $\mathfrak{r}^c$ satisfies extensionality. Thus, if $\mathfrak{r}^c$ is well founded, there is a structure preserving map $[\;] : \vartheta \twoheadrightarrow N$ onto a transitive N s.t.

$$[x] = [y] \quad \text{iff} \quad x I y$$
$$[x] \in [y] \quad \text{iff} \quad x e y.$$

If we then define $\pi : \bar{N} \to N$ by: $\pi(x) = [\langle \phi, \mathrm{const}_x \rangle]$,

* To see that g is $\underline{\Sigma}_1^{(n)}$, let $F'(\vec{x}, z) \simeq F(\vec{x})$ if $z \in d$ and $F'(\vec{x}, z) = 0$ if $z \notin d$, where $d = \mathrm{dom}(G)$. Then $g(u) = F'(\vec{f}(u), u)$.

we have: $\pi : \bar{N} \longrightarrow_{\Sigma_0} N$. We denote this state of affairs by writing:

$$\pi : \bar{N} \xrightarrow[E]{*} N.$$

. . . . .

Lemma 1.3 Let $\pi : \bar{N} \xrightarrow[E]{*} N$. Let $\bar{H} = H^n_{\bar{N}}$ where $\rho^n_{\bar{N}} = \min\{\rho^m_{\bar{N}} \mid \rho^m_{\bar{N}} > \kappa\}$. Set $H = \bigcup \pi''\bar{H}$. Then

$$\pi \upharpoonright \bar{H} : \bar{H} \longrightarrow_E H.$$

proof.

By the def. of Γ, whenever $f \in \Gamma$ and $\mathrm{rng}(f) \subset u \in \bar{H}$, then $f \in \bar{H}$.

QED (1.3)

Hence, $\kappa = \mathrm{crit}(\pi)$, and $[\langle a, f \rangle] = \pi(f)(a)$ for $f \in \bar{H}$.

<u>Lemma 2</u> Let $\pi : \bar{N} \xrightarrow{*}_{E} N$. Then

(a) $\pi : \bar{N} \longrightarrow_{\Sigma^*} N$

(b) $\pi'' P^*_{\bar{N}} \subset P^*_N$.

proof.

For $\omega\rho^m_{\bar{N}} > \kappa$ set:

$$\Gamma_m = \begin{cases} \{f \in \Gamma \mid \mathrm{rng}(f) \subset H^m_N\} & \text{if } \omega\rho^{m+1}_{\bar{N}} > \kappa \\ \{f \in \Gamma \mid \mathrm{rng}(f) \in H^m_N\} & \text{if} \\ \quad \text{if } \omega\rho^{m+1} \leq \kappa < \omega\rho^m & \text{in } \bar{N} \end{cases}$$

<u>Remark</u> $\Gamma_m = \{f \in \Gamma \mid f \in H^m_N\}$ if

if $\omega\rho^{m+1} \leq \kappa < \omega\rho^m$ in $\bar{N}$.

<u>Remark</u> By the remarks following the proof of §1.2 Lemma 5.1, Γ_m may (but need not) be regarded as a set of fcns <u>to</u> H^n – i.e. defined by $\varphi(u^n, v)$.

Set: $H_m^{\cdot\cdot} = \{[\langle a,f\rangle] \mid \langle a,f\rangle \in D \wedge f \in \Gamma_m\}$ for $\omega\rho_{\bar N}^m > u$. For $\omega\rho_{\bar N}^m \leq u$ set:

$$H_m = H_{\bar N}^n .$$

It is obvious that:

(1) H_m is transitive for $m < \omega$.

In all that follows we shall interpret Σ^* formulae in N by letting x^m range over H_m. Σ^* satisfaction in N is to be understood in this sense as is the notion of a Σ^* relation over N and a Σ^*-preserving map to N. We first show that the map π is Σ^*-preserving in this sense. Afterwards we show that this notion of Σ^* satisfaction is, in fact, the correct one.

We first generalize Łoś Thm by showing:

(2) Let $\langle a, f_1 \rangle, \dots, \langle a, f_n \rangle \in D$ and let φ be a $\Sigma_0^{(n)}$ fmla for an n s.t. $\omega\rho_{\bar{N}}^n > \kappa$ or a $\Sigma_1^{(n)}$ fmla for an n s.t. $\omega\rho_{\bar{N}}^{n+1} > \kappa$. Then:

$$N \models \varphi[[\overrightarrow{\langle a, f \rangle}]] \longleftrightarrow \{u \mid \bar{N} \models \varphi[\vec{f}(u)]\} \in E_a$$

proof. By ind. on n and for given n by ind. on φ, using Lemma 1.2 for the case $\omega\rho^{n+1} > \kappa$. QED (2)

Note that if $\omega\rho^{n+1} \leq \kappa < \omega\rho^n$ in $\bar{N}$, then $\pi \upharpoonright H_{\bar{N}}^n$ is cofinal into H_n. Hence we conclude:

(3) $\pi : \bar{N} \longrightarrow_{\Sigma_1^{(n)}} N$ for $\omega\rho_{\bar{N}}^n > \kappa$.

But since $H_n = H_{\bar{N}}^n$, $\pi \upharpoonright H_n = \mathrm{id}$ for $\omega\rho^n \leq \kappa$, we conclude:

(4) $\pi : \bar{N} \longrightarrow_{\Sigma^*} N$.

We now improve this result by showing:

(5) $\pi : \bar{N} \longrightarrow_{Q^*} N$.

(Recall that $Q^* = \bigcup_n Q^{(n)}$).

proof. of (5).

We prove $\pi : \bar{N} \longrightarrow_{Q^{(n)}} N$ by cases as follows.

Case 1 $\omega\rho^n_{\bar{N}} \leq \kappa$.

Triv. since $H_n = H^n_{\bar{N}}$, $\sigma \upharpoonright H_n = \mathrm{id}$.

Case 2 $\omega\rho^{n+1}_{\bar{N}} \leq \kappa < \omega\rho^n_{\bar{N}}$.

$\pi \upharpoonright H^n_{\bar{N}}$ is a cofinal Σ_0 preserving map.

Case 3 $\kappa < \omega\rho^{n+1}_{\bar{N}}$.

Claim $\pi : \bar{N} \longrightarrow_{\Sigma_2^{(n)}} N$

Let $N \models \bigvee x^n \underbrace{\varphi(x^n, \pi(\vec{z}))}_{\Pi_1^{(n)}}$

Let $\langle a, g\rangle \in D$ s.t. $N \models \varphi[[\langle a,g\rangle], \pi(\vec{z})]$

By Łoś Thm $\{u \mid \bar{N} \models \varphi[g(u), \vec{z}]\} \in E_a$.

Hence $\bar{N} \models \bigvee x^n \varphi(x^n, \vec{z})$. QED(5).

By (5) we have:

(6) N is acceptable.

(7) $H_n = |J^B_{\rho^n}|$ where $\omega\rho^n = On \cap H_n$ and $N = J^B_\beta$.

Hence it remains to show:

<u>Claim</u> $\rho^n = \rho^n_N$ $(0 < n < \omega)$.

But this reduces to:

<u>Claim A</u> If $A \subset \omega\rho^n$ is $\Sigma_1^{(n-1)}(N)$, then $\langle H_n, A \rangle$ is amenable.

<u>Claim B</u> There is an $A \subset \omega\rho^n$ which is $\Sigma_1^{(n-1)}$ s.t. $A \notin N$.

(8) Claim A holds for $\omega\rho^n_{\bar{N}} > \kappa$.

pf.
Let A be $\Sigma_1^{(n-1)}$ in $[\langle a, f \rangle]$. Let $z = [\langle a, g \rangle] \in H_n$, where $g \in \Gamma_n$.

Let $Ax \longleftrightarrow A'(x, [a, f])$, where A' is $\Sigma_1^{(n-1)}(N)$. Let $\bar{A}'$ be $\Sigma_1^{(n-1)}(\bar{N})$ by the same def. Define $h : [\kappa]^{|a|} \to H^n_{\bar{N}}$ by: $h(u) = \{z \in g(u) \mid \bar{A}(z, f(u))\}$. Then $h \in \Gamma_n$. Hence, letting $w = [\langle a, h \rangle] \in H_n$, we have:

$w = z \cap A$ by Łoś Thm. QED(8)

(9) $\mathbb{P}(\kappa) \cap \bar{N} = \mathbb{P}(\kappa) \cap N$.

pf.
Let $[\langle a, f \rangle] \subset \kappa$. Since $\pi \upharpoonright \kappa = \mathrm{id}$, we have:
$\nu \in [\langle a, f \rangle] \longleftrightarrow \{u \mid \nu \in f(u)\} \in E_a$.
But $\{\langle u, \nu \rangle \in [\kappa]^{|a|} \times \kappa \mid \nu \in f(u)\} \in \bar{N}$ QED

By Lemma 1.3 we have: $H_\kappa^{\bar N} = H_\kappa^N$, $\pi \upharpoonright H_\kappa^{\bar N} = \mathrm{id}$.

(10) Let $\omega\rho_{\bar N}^{n+1} \leq \kappa < \omega\rho_{\bar N}^{n}$. Then

$$\mathcal{P}(H_\kappa^{\bar N}) \cap \underline{\Sigma}_1^{(n)}(N) \subset \underline{\Sigma}_1^{(n)}(\bar N),$$

proof:

Let $A \subset H_\kappa^{\bar N}$ be $\Sigma_1^{(n)}(N)$ in $[\langle a, f\rangle]$. Let $Az \longleftrightarrow \bigvee x^n A'(x^n, z, [\langle a, f\rangle])$, where A' is $\Sigma_0^{(n)}(N)$. Let $\bar A'$ be $\Sigma_0^{(n)}(\bar N)$ by the same def. Then $Az \longleftrightarrow$

$$\longleftrightarrow \bigvee \nu < \omega\rho_{\bar N}^{n} \bigvee x \in \pi(S_\nu^B)\, A'(x, z, [\langle a, f\rangle])$$

$$\longleftrightarrow \quad " \quad \underbrace{\{u \mid \bigvee x \in S_\nu^B\ \bar A'(x, z, f(u))\} \in E_a}_{\underline{\Sigma}_1^{(n)}(\bar N)}$$

QED(10)

Since $H_{\bar N}^{m} = H_m$ for $\omega\rho_{\bar N}^{m} \leq \kappa$, we conclude:

(11) Let $\omega\rho_{\bar N}^{n} \leq \kappa$. Then

$$\mathcal{P}(H_{\bar N}^{n}) \cap \Sigma_1^{(n)}(N) \subset \Sigma_1^{(n)}(\bar N)$$

Hence:

(12) Claim A holds for $\omega\rho_{\bar N}^{n} \leq \kappa$.

(13) Claim B holds for $\omega\rho_{\bar N}^{n+1} > \alpha$.

prf.
Let $\bar A \subset \omega\rho_{\bar N}^{n}$ be $\Sigma_1^{(n-1)}(\bar N)$ in p and let A be $\Sigma_1^{(n-1)}(N)$ in $\pi(p)$ by the same definition. Assume: $\bar A \notin \bar N$.

<u>Claim</u> $A \cap \omega\rho^n \notin N$.

Suppose not. $[\langle a, f\rangle]$. The statement: $A \cap \omega\rho^n = x$ is expressed by $\bigwedge z^n (z^n \in A \longleftrightarrow z^n \in x)$, which is Π_1^n in $\pi(p)$. Hence $\{u \mid \bar A = f(u)\} \in E_a$. Hence $\bar A \in N$. Contr! QED (13)

The proof of (13) also shows:

(14) π " $P_{\bar N}^n \subset P_N^n$ if $\omega\rho_{\bar N}^{n+1} > \alpha$.

If $\omega\rho_{\bar N}^n \leq \alpha$, then $H_n = H_{\bar N}^n$, $\sigma \upharpoonright H_{\bar N}^n = \mathrm{id}$. Thus if $\bar A$ is $\Sigma_1^{(n-1)}(\bar N)$ in p and A is $\Sigma_1^{(n-1)}(N)$ in $\pi(p)$ by the same def., then $\bar A \cap H_{\bar N}^n = A \cap H_n$. By (9) we conclude $\bar A \cap \omega\rho_{\bar N}^n \notin \bar N \rightarrow A \cap \omega\rho_N^n \notin N$. Thus:

(15) Claim B holds for $\omega\rho_{\bar N}^n \leq \alpha$.

This proof shows:

(16) π " $P_{\bar N}^n \subset P_N^n$ if $\omega\rho_{\bar N}^n \leq \alpha$.

Finally!

(17) Claim B holds for $\omega\rho_{\bar N}^{n+1} \leq \kappa < \omega\rho_{\bar N}^{n}$.

prf.

Let $A \subset \omega\rho^{n+1}$ be $\Sigma_1^{(n)}(N)$ s.t. $A \notin N$. A is $\Sigma_1(\langle H_n, D\rangle)$ where D is $\Sigma_1^{(n-1)}(N)$. But then $D \notin N$, since $A \notin N$. Since $H_n = |J_{\rho^n}^B|$, there is an f p.r. in N mapping $\omega\rho^n$ onto H_n. Set:
$D' = \{\nu \mid f(\nu) \in D\}$. Then D' is $\Sigma_1^{(n-1)}(N)$ and $D' \notin N$, since otherwise $D \in N$. QED(17).

Finally, we note that by (14), (16):

(18) $\pi'' P_{\bar N}^{*} \subset P_{N}^{*}$.

QED (Lemma 2).

<u>Corollary 2.1</u> Let $\pi : \bar N \to_{E}^{*} N$.

(a) $\pi : \bar N \to_{Q^*} N$

(b) $\pi : \bar N \to_{\Sigma_2^{(n)}} N$ for $\omega\rho_{\bar N}^{n+1} > \kappa$

(c) Let $\omega\rho^n = \omega\rho^m > \kappa$ in $\bar N$ for $m \geq n$. Then $\omega\rho^n = \omega\rho^m$ in N for $m \geq n$ and $\pi : \bar N \to_{\Sigma_\omega^{(n)}} N$.

proof of Cor 2.1

(a), (b) are immediate from (5) + its proof in the pf. of Lemma 2. To prove (c) note that $\Gamma_m' = \Gamma_m$ and hence $H_m = H_m'$ for $m \geq n$. But then every $\Sigma_\omega^{(n)}$ condition is equivalent to a $\Sigma_1^{(m)}$ condition for an $m \geq n$ (uniformly over $\bar{N}, N$). Hence $\bar{\pi}$ is $\Sigma_\omega^{(n)}$-preserving. QED (2.1)

We now extend π to an operation on arbitrary elements of Γ. Note first that every good $\Sigma_1^{(n)}$ fcn has a definition which is <u>functionally absolute</u> in the sense that it defines a good $\Sigma_1^{(n)}$ fcn over any acceptable N (of the same structural type). To see this, note first that a $\Sigma_1^{(n)}$ fcn $y^n = F(\vec{x}^n, \ldots, \vec{x}^0)$ has a func. absolute def. of the form:

$$y^n = h_{N^n, p(\vec{x})}(i, \langle \vec{x}^n \rangle),$$

where $p(\vec{x}) = \langle \langle \vec{x}^{n-1} \rangle, \ldots, \langle \vec{x}^0 \rangle \rangle$.

But every good fcn is obtained by substitution and argument permutation in such fcns.

Def If $f \in \Gamma$ but $f \notin \bar{N}$, let f be a good $\Sigma_1^{(n)}(\bar{N})$ fcn in the parameter p by a functionally absolute definition φ. Set:
$\pi(f) =$ the good $\Sigma_1^{(n)}(N)$ fcn defined by φ over $\pi(p)$.

It is clear that $\pi(f) \in \Gamma$ and that $\pi(f)$ is independent of the choice of φ, p.

Since $[\langle a, id\rangle] = a$ by Lemma 1.3, we easily get:

Corollary 2.2 $[\langle a, f\rangle] = \pi(f)(a)$.

Thus $\pi : \bar{N} \xrightarrow{*}_{E} N$ satisfies the characterization (a)–(c) given at the outset. It would be easy to show uniqueness. (In fact, we can weaken (a) to: $\pi : \bar{N} \longrightarrow_{\Sigma_1^{(n)}} N$ if $\kappa < \omega\rho_{\bar{N}}^{n+1}$, and still show uniqueness).

By Cor 2.2 and (2) in the proof of Lemma 2 we have the following form of Łoś Theorem:

<u>Cor 2.3</u> Let $\langle a, f_1 \rangle, \ldots, \langle a, f_n \rangle \in D$. Let φ be a $\Sigma_0^{(n)}$ fmla, where $\omega\rho^n_{\bar{N}} > \kappa$ or a $\Sigma_1^{(n)}$ fmla, where $\omega\rho^{n+1}_{\bar{N}} > \kappa$. Then:

$$N \models \varphi[\pi(\vec{f})(a)] \longleftrightarrow \{u \mid \bar{N} \models \varphi[\vec{f}(u)]\} \in$$

Hence:

<u>Cor 2.4</u> Let $\vec{f}$ be as above + let φ be a $\Sigma_1^{(n)}$-fmla where $\omega\rho^{n+1}_{\bar{N}} \leq \kappa < \omega\rho^n_{\bar{N}}$. Then $\{a \mid N \models \varphi[\pi(\vec{f})(a)]\}$ is $\underline{\Sigma}_1^{(m)}(\bar{N}$

prf.

Let $\varphi = \bigvee z^n \psi$ where ψ is $\Sigma_0^{(n)}$.

$$N \models \varphi[\pi(\vec{f})(a)] \longleftrightarrow$$

$$\longleftrightarrow \bigvee u \in H^n_{\bar{N}} \underbrace{N \models \chi[\pi(u), \pi(\vec{f})(a)]}_{\underline{\Sigma}_1^{(m)}(\bar{N})}$$

where $\chi(u, \vec{x}) = \bigvee z \in u\, \psi(z, \vec{x})$.

QED (Cor 2.4)

Since $g \in \Gamma(\bar{N}, \kappa)$ whenever $g \in N$ and $g : [\kappa]^m \to \bar{N}$, we conclude:

<u>Cor 2.4.1</u> Let $\vec{f}, \varphi$ be as above. $\{\langle a, \vec{g}\rangle \mid N \models \varphi[\pi(\vec{f})(a), \pi(\vec{g})(a)]\}$ is $\underset{=}{\Sigma}_1^{(n)}(\bar{N})$

By Cor 2.3, Cor 2.4 and the fact that $H^m_{\bar{N}} = H^m_N$ for $\omega\rho^m \leq \kappa$, we may conclude:

<u>Cor 2.5</u> Let $\vec{f}$ be as above + let φ be Σ^*. Then
$\{a \mid N \models \varphi[\pi(\vec{f})(a)]\}$ is $\underset{=}{\Sigma}^*(\bar{N})$.

<u>Cor 2.5.1</u> Let $\vec{f}, \varphi$ be as above. Then $\{\langle a, \vec{g}\rangle \mid N \models \varphi[\pi(\vec{f})(a), \pi(\vec{g})(a)]\}$ is $\underset{=}{\Sigma}^*($

Another consequence is:

<u>Cor 2.5.2</u> $\mathcal{P}(\kappa) \cap \underset{=}{\Sigma}^*(\bar{N}) = \mathcal{P}(\kappa) \cap \underset{=}{\Sigma}^*(N)$

Since $H^m_{\bar{N}} = H^m_N$ for $\omega\rho^m \leq \kappa$, Cor 2.4.1 also gives us:

<u>Cor 2.5.3</u> $\mathcal{P}(\kappa) \cap \underset{=}{\Sigma}_1^{(n)}(\bar{N}) = \mathcal{P}(\kappa) \cap \underset{=}{\Sigma}_1^{(n)}(N)$ for $\omega\rho^{n+1} \leq \kappa$.

<u>Note</u> In the subsequent sections we shall work with normal ultrafilters rather than extenders. Hence the definitions & theorems of this section must be restated accordingly. Thus $\Gamma(\kappa, \bar{N}) = D$ is a set of $f: \kappa \to \bar{N}$. The relation e is defined by: $f e g$ iff iff $\{\nu < \kappa \mid f(\nu) \in g(\nu)\} \in E$.
The assertion of Cor 2.5 becomes
$\{\nu < \kappa \mid N \models \varphi[\pi(\vec{f})(\nu)]\}$ is $\underset{\sim}{\Sigma}^*(\bar{N}$