§1.2 Iterated Projecta

Let $N = \langle J^{\vec{B}}_{\beta}, \vec{B}' \rangle$ be acceptable. For $n < \omega$ we define the n-th projecta $\rho^n = \rho^n_N$ in such a way that $\rho^0 = \beta$; $\rho^{n+1} \leq \rho^n$ and each ρ^n is a cardinal in N. For each $p \in \bigtimes_{i<n} H^N_{\omega\rho^i}$ we simultaneously define the n-th reduct $N^{n,p}$ s.t. $|N^{np}| = H_{\omega\rho^n}$. For $n \geq 1$ we have: $N^{np} = \langle J^{\vec{B}}_{\rho^n}, A^{np} \rangle$ and we call A^{np} the n-th standard code.

Def $\rho^0 = \beta$; $N^{0,\emptyset} = N$;

$$\rho^{n+1} = \min \left\{ \rho_{N^{n,p}} \;\middle|\; p \in \bigtimes_{i<n} H^N_{\omega\rho^i} \right\};$$

$$N^{n+1,p} = (N^{n,p\restriction n})^{p(n),\rho^{n+1}}.$$

Def $H^n_N = H^N_{\omega\rho^n_N}$; $\Gamma^n_N = \bigtimes_{i<n} H^i_N$ $(n \leq \omega)$

We define P^n, $R^n \subset \Gamma^n$ as follows:

Def $P^0 = \{\emptyset\}$

$P^{n+1} = \{ p \in \Gamma^{n+1} \mid p \upharpoonright n \in P^n \wedge$

$\wedge \rho^{n+1} = \rho_{N^{n, p \upharpoonright n}} \wedge p(n) \in P_{N^{n, p \upharpoonright n}} \}$

R^n then has the same def. with R in place of P. Hence:

$p \in R^n \longleftrightarrow p \in P^n$ and $p(i) \in R_{N^{i, p \upharpoonright i}}$ for $i < n$.

Trivially: $R^n \subset P^n \neq \emptyset$.

Def N is *n-sound* iff $R^n = P^n$

N is *sound* iff N is n sound for $n < \omega$

The downward ext. of imbeddings lemma for iterated projecta is the conjunction of the following three lemmata, which follow trivially from the corresponding lemmata for one projectum:

Lemma 1.1 Let $\pi : \bar{N}^{n\bar{p}} \xrightarrow[\Sigma_0]{} N^{np}$ $(n < \omega)$ where $\bar{p} \in R^n_{\bar{N}}$. There is a unique $\tilde{\pi} \supset \pi$ s.t. $\mathrm{dom}(\tilde{\pi}) = \bar{N}$, $\tilde{\pi}(\bar{p}) = p$ and, setting $\tilde{\pi}_i = \tilde{\pi} \upharpoonright H^i_N$,

$$\tilde{\pi}_i : \bar{N}^{i, \bar{p} \upharpoonright i} \xrightarrow[\Sigma_0]{} N^{i, p \upharpoonright i} \quad \text{for } i \leq n.$$

Moreover, for $i < n$ the map $\tilde{\pi}_i$ is Σ_1 preserving.

<u>Lemma 1.2</u> Let $\pi : \bar{N}^{n\bar{p}} \xrightarrow[\Sigma_l]{} N^{np}$ where $n, l < \omega$, $\bar{p} \in R^n_{\bar{N}}$, $p \in R^n_N$. Let $\tilde{\pi}$ be as above. Then

$$\tilde{\pi}_i : \bar{N}^{i,\bar{p}\restriction i} \xrightarrow[\Sigma_{n-i+l}]{} N^{i,p\restriction i} \quad (i \leq n).$$

<u>Lemma 1.3</u> Let $\pi : M \xrightarrow[\Sigma_0]{} N^{np}$ (M a J-model) ($n < \omega$, $p \in \Gamma^n$). There are unique $\bar{N}, \bar{p}$ s.t. $\bar{p} \in R^n_{\bar{N}}$ and $M = \bar{N}^{n,\bar{p}}$.

These lemmas naturally suggest a notion of "$\Sigma_l^{(n)}$ - preserving imbedding" - i.e. imbeddings $\pi : \bar{N} \xrightarrow[\Sigma_0]{} N$ s.t. $\pi \restriction H^n_{\bar{N}} : \bar{N}^{n\bar{p}} \xrightarrow[\Sigma_l]{} N^{np}$ whenever $\bar{p} \in \Gamma^n_{\bar{N}}$ and $\pi(\bar{p}) = p$.

It turns out that these are the imbeddings which are elementary with respect to satisfaction of $\Sigma_l^{(n)}$ formulae, a notion which we now develop.

Σ^* - Relations

Let $\mathcal{L}^*$ be a language with predicate $\in$, $\doteq$, $\vec{\dot{B}}$, $\vec{\dot{B}}'$ and vble v^i of type $i<\omega$. We interpret $\mathcal{L}^*$ in N in the obvious way, letting v^i range over H_N^i ($i<\omega$). (Thus larger indices mean smaller domai

By ind. on $m<\omega$ define sets $\Sigma_k^{(m)}$ of fmlae ($k<\omega$):

$\Sigma_0^{(m)}$ = the smallest set Σ of fmlae s.t.

(a) $x^i \in y^i$, $x^i = y^i$, $\dot{B}_k x^i$, $\dot{B}_k' x^i$ are in Σ

(b) $\Sigma_1^{(n)} \subset \Sigma$ for $n<m$

(c) Σ is closed under $\wedge, \vee, \rightarrow, \longleftrightarrow, \neg$

(d) If φ is in Σ, so are $\bigwedge x^m \in x^i \varphi$, $\bigvee x^m \in y^i \varphi$ for $i \geq m$.

For $k>0$ we then set:

$\Sigma_k^{(m)}$ = the set of fmlae:

$$\bigvee x_1^m \bigwedge x_2^m \ldots \bigvee\!\!\!\!\bigwedge x_k^m \varphi \quad (\varphi \in \Sigma_0^{(m)})$$

$\Pi_k^{(m)}$ = the set of fmlae:

$$\bigwedge x_1^m \bigvee x_2^m \ldots \bigvee\!\!\!\!\bigwedge x_k^m \varphi \quad (\varphi \in \Sigma_0^{(m)})$$

We also set: $\Pi_0^{(m)} = \Sigma_0^{(m)}$.

Finally, set: $\Sigma^* = \Sigma^{(\omega)} = \bigcup_n \Sigma_0^{(n)}$.

When dealing with the $\mathcal{L}^*$ model theory of N we informally use x^i as a variable for elements of H^i_N. Using this notation we define:

<u>Def</u> $R(x_1^{i_1}, \dots, x_p^{i_p})$ is a $\Sigma_k^{(n)}(N)$ <u>relation</u> <u>of type</u> $\langle i_1, \dots, i_p \rangle$ (in the parameters $q_1^{j_1}, \dots, q_m^{j_m}$) iff iff R is defined by a $\Sigma_k^{(n)}$ fmla $\varphi(v_1^{i_1}, \dots, v_p^{i_p})$ (in the parameters $\vec{q}$).

R is $\underline{\Sigma}_k^{(n)}(N)$ iff R is $\Sigma_k^{(n)}(N)$ in some parameters.

Similarly for $\Pi_k^{(n)}(N)$, $\underline{\Pi}_k^{(n)}(N)$.

Also: $\Sigma^*(N) = \Sigma^{(\omega)}(N) = \bigcup_n \Sigma_0^{(n)}(N)$.

(Similarly for $\underline{\Sigma}^*(N)$).

We say that f is a $\Sigma_k^{(n)}$ function <u>to</u> H^i <u>of argument type</u> $\langle i_1, \dots, i_p \rangle$ iff $y^i = f(x_1^{i_1}, \dots, x_p^{i_p})$ is a $\Sigma_k^{(n)}$ relation of type $\langle i, i_1, \dots, i_p \rangle$.

[<u>Note</u> A relation in this sense is not characterized by its graph. It could be identified with a pair consisting of its graph and its type].

<u>Lemma 2.1</u> Let $n, l < \omega$. If $R(x^h, \vec{x})$ is $\Sigma_l^{(n)}$ and $k \geq h$, then $R(x^k, \vec{x})$ is $\Sigma_l^{(n)}$

prf. trivial

We call $R(x^h, \vec{x})$ a <u>specialization</u> of R

<u>Lemma 2.2</u> Let $n, l < \omega$. If $R(x^k, \vec{x})$ is $\Sigma_l^{(n)}$ and $k \geq h \geq n$, then R is a specialization of a relation $R(x^h, \vec{x})$ which is $\Sigma_l^{(n)}$.

prf. trivial.

Thus every $\Sigma_l^{(n)}$ relation is obtainable by specialization from one whose arguments are of type $\leq n$. It is also apparent that the $\Sigma_l^{(n)}$ relations are closed under permutation of arguments and insertion of dummy arguments.

Another obvious consequence of our definitions is:

<u>Lemma 3.1</u> Let $n, l < \omega$. $R(\vec{x}^{n+1}, \ldots, \vec{x}^0)$ is $\Sigma_l^{(n+1)}(N)$ iff the relation

$$R_{\vec{x}} = \{ \langle \vec{x}^{n+1} \rangle \mid R(\vec{x}^{n+1}, \vec{x}) \}$$

is uniformly $\Sigma_l(\langle H^{n+1}, \vec{Q}_{\vec{x}} \rangle)$, where each $Q_{i\vec{x}}$ has the form

$$Q_{i\vec{x}} = \{ \langle \vec{z}^{n+1} \rangle \mid Q_i(\vec{z}^{n+1}, \vec{x}) \},$$

and $Q_i(\vec{z}^{n+1}, \vec{x})$ is $\Sigma_1^{(n)}(N)$.

<u>Note</u> The reduction in Lemma 3.1 is uniform for all N, with the $\Sigma_1^{(n)}$ definitions of the Q_i depending only on the $\Sigma_l^{(n+1)}$ def. of R.

By a straightforward induction on n, using Lemma 3.1:

<u>Lemma 3.2</u> Let $n < \omega$, $1 \le l < \omega$. $R(\vec{x}^n, \ldots, \vec{x}^0)$ is $\Sigma_l^{(n)}(N)$ iff

iff $R_{\vec{x}} = \{ \langle \vec{x}^n \rangle \mid R(\vec{x}^n, \vec{x}) \}$ is uniformly $\Sigma_l(N^{n, p(\vec{x})})$, where $p(\vec{x}) = \langle \langle \vec{x}^{n-1} \rangle, \ldots, \langle \vec{x}^0 \rangle \rangle$.

Similarly:

<u>Lemma 3.3</u> Let $n < \omega$. If $R(\vec{x}^n, \ldots, \vec{x}^0)$ is $\Sigma_0^{(n)}(N)$, then $R_{\vec{x}}$ is uniformly rudimentary in $N^{n, p(\vec{x})}$

<u>Def</u> $\pi : \bar{N} \xrightarrow[\Sigma_\ell^{(n)}]{} N$ iff $\pi : \bar{N} \to N$ and whenever $\varphi(v^{j_1}, \ldots, v^{j_m})$ is a $\Sigma_\ell^{(n)}$ fmla with $j_1, \ldots, j_m \le n$ and $x_i \in H_{\bar{N}}^{j_i}$ for $i = 1, \ldots, m$, then $\pi(x_i) \in H_N^{j_i}$ and

$$\bar{N} \models \varphi[\vec{x}] \longleftrightarrow N \models \varphi[\pi(\vec{x})].$$

(Hence $\pi'' H_{\bar{N}}^{j} \subset H_N^{j}$ for $j \le n$. For $\ell > 0$ or $j < n$ it follows that $\pi^{-1} {}'' H_N^{j} \subset H_{\bar{N}}^{j}$, since $\bigvee v^j\, v^j = v^0$ is a $\Sigma_\ell^{(n)}$ fmla.)

Lemmas 3.2, 3.3 then give us:

<u>Lemma 4.1</u> Let $n, \ell < \omega$. Then $\pi : \bar{N} \xrightarrow[\Sigma_\ell^{(n)}]{} N$ iff $\pi : \bar{N} \to N$ and whenever $\bar{p} \in \Gamma_{\bar{N}}^{n}$, $p = \pi(\bar{p})$, then $p \in \Gamma_N^n$ and $\pi \upharpoonright H_{\bar{N}}^n : \bar{N}^{n\bar{p}} \xrightarrow[\Sigma_\ell]{} N^{np}$.

(<u>Note</u> For $\ell = 0$ the proof uses the fact that for J-models, $\pi : \bar{M} \xrightarrow[\Sigma_0]{} M$ implies $\pi : \bar{M} \xrightarrow[\text{rud}]{} M$).

In order to reformulate the extension of imbedding lemma in terms of $\Sigma_\ell^{(n)}$ imbeddings we prove:

<u>Lemma 4.2</u> Let $\tilde\pi : \bar N \to N$ s.t.

$\pi \restriction H^i_{\bar N} : \bar N^{i,\bar p ri} \xrightarrow[\Sigma_1]{} N^{i,pri}$ for $i<n$

and $\pi \restriction H^n_{\bar N} : \bar N^{n,\bar p} \xrightarrow[\Sigma_\ell]{} N^{n,p}$, where

$\bar p \in R^n_{\bar N}$, $p = \pi(\bar p)$, $\ell < \omega$. Then

$$\pi : \bar N \xrightarrow[\Sigma^{(n)}_\ell]{} N .$$

proof.

Let $\bar q \in \Gamma^n_{\bar N}$, $q = \pi(\bar q)$. By ind on $i \le n$ we show:

$A^{i,\bar q ri}_{\bar N}$ is rud in $\bar N^{i,\bar p ri}$ in a parameter x and $A^{i,qri}_N$ is rud in $N^{i,pri}$ in $\pi(x)$ by the same rud definition.

The conclusion follows by Lemma 5.1.

QED (Lemma 5.2)

This enables us to reformulate Lemma 1.1 as

<u>Lemma 1.1'</u> Let $\pi : \bar N^{n\bar p} \xrightarrow[\Sigma_\ell]{} N^{np}$ $(\ell, n < \omega)$ where $\bar p \in R_{\bar N}$. There is a unique $\tilde\pi \supset \pi$ s.t. $\tilde\pi(\bar p) = p$ and $\tilde\pi : \bar N \xrightarrow[\Sigma^{(n)}_\ell]{} N$.

<u>Lemma 4.3</u> Let $\pi : \bar{N} \longrightarrow_{\Sigma_1^{(n)}} N$. Then $\pi^{-1} " P_N^n \subset P_{\bar{N}}^n$.

prf.
Suppose not. Let $p \in P_N^n$, $\bar{p} = \pi^{-1}(p) \notin P_{\bar{N}}^n$. Then $n > 0$. Let A be $\Sigma_1^{n-1}(N)$ in p s.t. $A \cap \omega\rho \notin N$ ($\rho = \rho_N^n$). Let $\bar{A}$ have the same definition in $\bar{p}$ over $\bar{N}$. Then $\bar{a} = \bar{A} \cap \omega\bar{\rho} \in \bar{N}$ ($\bar{\rho} = \rho_{\bar{N}}^n$). Let $a = \pi(\bar{a})$. Then

(*) $\bigwedge z^n ((z^n \in On \wedge z \in \bar{A}) \longleftrightarrow z^n \in \bar{a})$

holds in $\bar{N}$. (*) is $\Pi_1^{(n)}(\bar{N})$ in $\bar{a}, p$. Hence the same $\Pi_1^{(n)}$ statement holds of a, p in N. Hence $A \cap \omega\rho = a \cap \omega\rho \in N$. Contr!

QED (Lemma 4.3)

Note that if $p \in P_N^n$, then $q = \langle p, \overbrace{0, -, 0}^{n} \rangle \in P_N^n$, since A^{np} is uniformly r.e.d in A^{nq}. Since there is m s.t. $\rho = \rho^\omega = _{\mathrm{Df}} \min\{\rho^m \mid m < \omega\}$, we conclude that $P^* \neq \emptyset$, where:

Def P^*_N = the set of $p \in N$ s.t. $\langle p, 0, -, 0\rangle \in P^n_N$ for all n.

(Similarly for R^*_N). Then

Corollary 4.4 Let $\pi : \bar{N} \to_{\Sigma^*} N$. Then $\pi^{-1}{}'' P^*_N \subset P^*_{\bar{N}}$.

Note P^* is the set of p s.t. for each n there is A which is $\Sigma_1^{(n)}$ in p and $A \cap \omega\rho^{n+1} \notin N$.

$\Sigma_1^{(n)}$ relations are not, in general, closed under substitution of $\Sigma_1^{(n)}$ functions. There are, however, important cases where such substitution is possible. One such is:

Lemma 5.1 Let $p \leq n < \omega$, $1 \leq \ell < \omega$. Let $R(\vec{x}^n, \dots, \vec{x}^0)$ be $\Sigma_\ell^{(n)}(N)$. Let $\vec{F}^n, \dots, \vec{F}^0$ be s.t. each $F^i_j(\vec{z}^p, \dots, \vec{z}^0)$ is a $\Sigma_1^{(i)}$ map to H^i_N. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_\ell^{(n)}(N)$.

[Note The F^i_j may be partial on $\bigtimes_{h=1}^{p} H^h_N$. As usual, $R(\vec{F}(\vec{z}))$ is taken to hold of $\vec{z}$ iff the $\vec{F}(\vec{z})$ exist and R holds of $\vec{F}(\vec{z})$].

proof of Lemma 5.1. By ind. on n.
The case $n=0$ is trivial. Let $n=m+1$.
Let $R_{\vec{x}} = \{\langle \vec{x}^n \rangle \mid R(\vec{x}^n, \vec{x})\}$ be uniformly
$\Sigma_\ell(H^n, \vec{Q}_{\vec{x}})$, where

$$Q_{i,\vec{x}} = \{\langle u^n \rangle \mid Q_i(\vec{u}^n, \vec{x})\}$$

and Q_i is $\Sigma_1^{(m)}(N)$. Let $Q_i(\vec{u}^n, \vec{x})$ be a specialization of $Q_i(\vec{u}^m, \vec{x})$, where this is $\Sigma_1^{(m)}(N)$. By the induction hypothesis $Q_i(\vec{u}^m, \vec{F}'(\vec{z}))$ is $\Sigma_1^{(m)}(N)$, where $\vec{F}' = \vec{F}^m, \ldots, \vec{F}^0$.
Hence, by specialization, so is $Q_i'(\vec{u}^n, \vec{z}) \longleftrightarrow Q_i(\vec{u}^n, \vec{F}'(\vec{z}))$. Let $R'_{\vec{z}}$ be $\Sigma_\ell(H^n, \vec{Q}'_{\vec{z}})$ by the same definition as $R_{\vec{x}}$ over $\langle H^n, \vec{Q}_{\vec{x}} \rangle$. Clearly

$$R'_{\vec{z}}(\vec{x}) \longleftrightarrow R(\vec{x}, \vec{F}(\vec{z}))$$

whenever $\vec{F}'(\vec{z})$ exist. Hence:

$$R(\vec{x}, \vec{F}'(\vec{z})) \longleftrightarrow (\underbrace{\vec{F}'(\vec{z}) \text{ exist}}_{\Sigma_\ell^{(m)}} \wedge \underbrace{R'_{\vec{z}}(\vec{x})}_{\Sigma_\ell^{(n)}}).$$

But:

$$R(\vec{F}(\vec{z})) \longleftrightarrow \bigvee \vec{x}^n \left(\left(\bigwedge_i x_i^n = F_i^n(\vec{z}) \right) \wedge R(\vec{x}^n, \vec{F}'(\vec{z})) \right)$$

QED (Lemma 5.1)

(Note It is apparent from the proof that the defining $\Sigma_\ell^{(n)}$ fmla of $R(\vec{F}(\vec{z}))$ depends – uniformly for all N – only on the defining fmlae of R, $\vec{F}$.)

One consequence of this is that $\Sigma_1^{(n)}$ relations are essentially characterized by their graphs: Let $R(x^{i_1}, \dots, x^{i_m})$ and $R(x^{j_1}, \dots, x^{j_m})$ have the same graph, where $i_1, \dots, i_m, j_1, \dots, j_m \leq n$. Then one is $\Sigma_1^{(n)}$ iff the other is. To see this, note that we can convert one to the other by composition with the identity function $y^i = x^j$. It follows in particular that a relation is $\Sigma_1^{(n)}$ iff the relation with the same graph and arguments of type 0 is $\Sigma_1^{(n)}$.

In order to extend this result we define

Def The *good* $\Sigma_1^{(n)}(N)$ forms comprise the smallest class s.t.

(a) Each partial $\Sigma_1^{(i)}(N)$ map $F(x_1^{j_1}, \dots, x_p^{j_p})$ to H_N^i is good $(i, j_1, \dots, j_p \leq n)$

(b) If $F(x_1^{j_1}, \dots, x_p^{j_p})$ is good and $G_i(\vec{z})$ is a good map to H^{j_i} $(i = 1, \dots, p)$ (the arguments of $G_i(\vec{z})$ all being of type $\leq n$), then $F(\vec{G}(\vec{z}))$ is good.

Lemma 5.2 Let $R(x_1^{j_1}, \dots, x_p^{j_p})$ be $\Sigma_\ell^{(n)}(N)$ $(n < \omega, 1 \leq \ell < \omega, j_1, \dots, j_p \leq n)$. Let $F_i(\vec{z})$ be a good $\Sigma_1^{(n)}(N)$ map to H^{j_i} $(i = 1, \dots, p)$. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_\ell^{(n)}(N)$.

prf.
$R(\vec{F}(\vec{z}))$ results from iterated application of Lemma 5.1 (and permutations of arguments. QED (5.2)

The remark at the end of Lemma 5.1 obviously applies to 5.2 as well.

(Note Good functions are closed under composition.)

Finally we note:

<u>Lemma 5.3</u> Let $R(y^n, x^{i_1}, \ldots, x^{i_n})$ be $\Sigma_1^{(n)}$ $(\vec{j} \leq n)$. There is a $\Sigma_1^{(n)}$ function F to H^n s.t.

(a) $\mathrm{dom}(F) = \{y^n \mid R(y^n, \vec{x})\}$

(b) $\bigvee y^n R(y^n, \vec{x}) \longleftrightarrow R(F(\vec{x}), \vec{x})$.

proof.
We prove it for $R(y^n, \vec{x}^n, \ldots, \vec{x}^0)$.
Set: $p(\vec{x}) = \langle\langle \vec{x}^0 \rangle, \ldots, \langle \vec{x}^{n-1} \rangle\rangle$. Then the claim holds for appropriate i:

$$F(\vec{x}) = h_{N^n, p(\vec{x})}(i, \langle \vec{x}^n \rangle).$$

QED (5.3)

<u>Def</u> $\Sigma_\omega^{(n)} = \bigcup_{m<\omega} \Sigma_m^{(n)}$

<u>Def</u> $Q^{(n)}$ = the set of fmlae

$$Qx^n \varphi = \bigwedge z^n \bigvee x^n (z^n \in x^n \wedge \varphi),$$

where φ is $\Sigma_1^{(n)}$.

$\pi : \bar{N} \longrightarrow_{Q^{(n)}} N$ then has the obvious meaning.

We set: $Q^* = \bigcup_n Q^{(n)}$.

<u>Lemma 5.4</u> There is a good $\Sigma_1^{(m)}(N)$ fcn F s.t. if $p \in \Gamma_N^{m+1}$, then each $x \in N$ has the form $F(u,p)$, where $u \in H_N^{m+1}$.

prf. Ind. on m.

$m = 0$: $F(u,p) \simeq h_N((u)_0, \langle (u)_1, p_0 \rangle)$

$m = m+1$: Let $x = G(v, p \restriction m)$ where $v \in H_N^m$ and G is a good $\Sigma_1^{(m)}(N)$ map. Then

$$v = H(u,p) \simeq_{Df} h_{N^m, p \restriction m}((u)_0, \langle (u)_1, p_m \rangle)$$

where $u \in H_N^{m+1}$. Hence $x = G(H(u,p), p \restriction m)$.

QED (Lemma 5.4)

<u>Note</u> This gives a new proof of Lemma 4.:

<u>Cor 5.5</u> Let $p \in \Gamma_N^m$. Then

$\underline{\Sigma}_\ell^{(m)}(N) \supset \underline{\Sigma}_\ell(N)$ for $\ell \geq 1$.

Thus if N is sound we have:

$$\underline{\Sigma}_\omega^{(m)}(N) = \underline{\Sigma}_\omega(N).$$

($\underline{\Sigma}_\omega^{(m)} \subset \underline{\Sigma}_\omega$ always holds, but the converse may not.)

<u>Note</u> In subsequent sections we sometimes write:

$$\pi : M \longrightarrow_{\Sigma_0^{(n)}} N \quad \text{cofinally}$$

to mean:

$$\pi : M \longrightarrow_{\Sigma_0^{(n)}} N \text{ and } H_N^n = \bigcup \pi'' H_M^n .$$

This clearly implies:

$$\pi \restriction H_M^n : M^{np} \longrightarrow_{\Sigma_0} N^n \quad \text{cofinally}$$

for all p;

hence: $\pi : M \longrightarrow_{\Sigma_1} N$.

Functional absoluteness

Every good $\Sigma_1^{(n)}$ fcn $f(x^{j_1}, \ldots, x^{j_m})$ $(j_1, \ldots, j_m \leq n)$ has a $\Sigma_1^{(n)}$ definition which is functionally absolute in the sense that it defines a $\Sigma_1^{(n)}$ fcn over every acceptable structure of the appropriate structural type. To see this, note that a $\Sigma_1^{(n)}$ fcn $y^n = F(\vec{x}^n, \ldots, \vec{x}^0)$ has a functionally absolute definition of the form:

$$y^n = h_{N^n, p(\vec{x})}(i, \vec{x}^n)$$

where $p(\vec{x}) = \langle \vec{x}^{n-1}, \ldots, \vec{x}^0 \rangle$ ($p(\vec{x}) = \emptyset$ if $n = 0$). But every good fcn is obtained by composition and argument permutation from such functions.