Measures of Order Zero

These notes exposit the full core model theory for measures of order zero. §1 and §2 supersede the notes I made available at the Oberwolfach fine structure meeting in 1988, for I have since made numerous corrections and amendments.

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§1 Fine Structure

§1.1 Projecta

By a J – *model* we mean an amenable structure of the form:

$$N = \langle J_\beta^{\vec{B}}, \vec{B}' \rangle = \langle |J_\beta^{\vec{B}}|, \vec{B}, \vec{B}' \rangle.$$

We note some facts about J-models:

Fact 1 N is closed under TC and $TC \upharpoonright N$ is uniformly $\Delta_1(N)$

Def N is *acceptable* iff whenever $\xi < \beta$, $\omega \le \tau \le \omega\xi$ and $\mathbb{P}(\tau) \cap J_{\xi+1}^{\vec{B}} \not\subset J_\xi^{\vec{B}}$, then $\overline{\overline{\omega\xi}}^{J_{\xi+1}^{B}} \le \tau$.

"N is acceptable" is a Q-condition*.

Hence for J-models $\bar{N}, N$:

Fact 2 $\pi : \bar{N} \longrightarrow_{\Sigma_1} N$, N acc. $\to \bar{N}$ acc.

Fact 3 $\pi : \bar{N} \longrightarrow_{Q} N$, $\bar{N}$ acc $\to$ N acc.

*/ By a Q-condition we mean a condition of the form $Qx\varphi = \bigwedge z \bigvee x \ni z\, \varphi$, where φ in Σ_1.

Projecta:

Def Let $N = \langle J_\beta^{\vec{B}}, \vec{B}' \rangle$ be acceptable.

$\rho = \rho_N$ (the Σ_1 projectum of N) =

= the least $\rho \leq \beta$ s.t. $\underset{=}{\Sigma}_1(N) \cap \mathbb{P}(\omega\rho) \not\subset$

(Hence $a \subset \tau < \omega\rho$, $a \in \underset{=}{\Sigma}_1(N) \rightarrow a \in N$,

Lemma 1.1 $\omega\rho$ is a cardinal in N

prf. Straightforward.

Note $\omega\rho = \rho$ for $\rho > 1$

Note We regard $\omega\beta = On \cap |N|$ as a cardinal in N.

Lemma 1.2 $\tau < \omega\rho \longrightarrow \mathbb{P}(\tau) \cap N \subset J_\rho^{\vec{B}}$

prf. By Lemma 1.1 + def. of acceptability

Lemma 1.3 $\omega\rho$ is a Σ_1 cardinal in N.

prf. Suppose not.

We use a diagonal argument. Let $\tau < \omega\rho$, $f \in \underset{=}{\Sigma}_1(N)$ a partial map of τ onto $J_\rho^{\vec{B}}$. Set: $a = \{\nu \in dom(f) \mid \nu \notin f(\nu)\}$

Then $a \in \underset{=}{\Sigma}_1(N) \cap \mathbb{P}(\tau) \subset J_\rho^{\vec{B}}$ by Lemma 1.2.

Let $a = f(\nu)$, $\nu < \tau$. Then $\nu \in a \longleftrightarrow \nu \notin a$.

Contr! QED

Lemma 1.4 $H^N_{\omega\rho} = |J^{\vec{B}}_\rho|$

prf.

$(\supset)$ triv

$(\subset)$ Let $x \in H_{\omega\rho}$, $\overline{\overline{TC(x)}} \leq \tau < \omega\rho$ in N

Let $a \subset \tau$ code x. Then $a \in J^{\vec{B}}_\rho$.

But J^A_ρ is p.r. closed by Lemma 1.3.

Hence $x \in J^{\vec{B}}_\rho$. QED

Note The proofs of Lemmas 1.1 – 1.4 go through for any $\rho \leq \rho_N$ s.t. $\omega\rho$ is a cardinal in N.

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Standard Codes

Let $p \in N$.

$A^p = A^p_N = \{\langle i, x\rangle \in \omega \times H_{\omega\rho} \mid N \models \varphi_i[x, p]\}$,

where $\langle \varphi_i \rangle$ is a fixed rec. enumeration of the Σ_1 fmlae.

$N^p = \langle J^{\vec{B}}_\rho, A^p \rangle$

$= \langle H_{\omega\rho}, \overrightarrow{B \cap H_{\omega\rho}}, A^p \rangle$

Hence N^p is a J-model.

$N^{p,\delta} = \langle H_{\omega\delta}, \overrightarrow{B \cap H_{\omega\delta}}, A^p \cap H_{\omega\delta} \rangle$

for $\delta \leq \rho$ ... N

Fact 1 Let $A' = A^p \cap (\omega \times (\omega\rho))$.

Then $H^N_{\omega\rho} = J^{A'}_\rho$

Fact 2 Let $A'' = A^p \cap (\omega \times [\omega\rho]^{<\omega})$,

Then $H^N_{\omega\rho} = J^{A''}_\rho$ and $J^{A''}_\rho$ is acceptable.

(Sim for $\rho \leq \rho_N$, $\omega\rho$ a card. in N)

Good Parameters

Def Let N be acceptable, $\rho = \rho_N$.

P_N = the set of $p \in N$ s.t. there is $B \subset On_N$ with: (a) B is $\Sigma_1(N)$ in p

(b) $B \cap \omega\rho \notin N$

Lemma 2.1 Let $p \in N$, $A = A^p_N$. Then

$p \in P_N \longleftrightarrow A \cap (\omega \times \omega\rho) \notin N$.

prf. Straightforward

Def Let N be acceptable, $\rho = \rho_N$.

R_N = the set of $r \in N$ s.t. $|N| = h_N(\omega\rho \cup \{r\})$.

Lemma 2.2 $R_N \subset P_N \neq \emptyset$

prf. Straightforward.

The (downward) <u>extension of imbeddings lemma</u> is the conjunction of the following three lemmata, whose proofs are exactly as in the J_α case.

<u>Lemma 3.1</u> Let $\pi : \bar{N}^{\bar{p}} \xrightarrow[\Sigma_0]{} N^p$ where $\bar{p} \in R_{\bar{N}}$. There is a unique $\tilde{\pi} \supset \pi$ s.t. $\tilde{\pi} : \bar{N} \xrightarrow[\Sigma_0]{} N$ and $\tilde{\pi}(\bar{p}) = p$. Moreover, $\tilde{\pi} : \bar{N} \xrightarrow[\Sigma_1]{} N$.

<u>Lemma 3.2</u> Let $\pi, \tilde{\pi}, \bar{N}^{\bar{p}}, N^p$ be as above with $\pi : \bar{N}^{\bar{p}} \xrightarrow[\Sigma_\ell]{} N^p$ $(\ell < \omega)$. Let $\bar{p} \in R_{\bar{N}}$ and $p \in R_N$. Then $\tilde{\pi} : \bar{N} \xrightarrow[\Sigma_{\ell+1}]{} N$.

<u>Lemma 3.3</u> Let $\pi : M \xrightarrow[\Sigma_0]{} N^p$ where M is a J-model. There are unique $\bar{N}, \bar{p}$ s.t. $\bar{p} \in R_{\bar{N}}$ and $\bar{N}^{\bar{p}} = M$.