§6 Another Look

We now return to Example 1 and consider the case $\beta = \omega_2$ (hence we assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$). Our forcing then adds a cofinal ω-sequence in ω_2 without adding new reals. This is, of course, what Namba forcing was designed to accomplish. When we first developed the forcing $\mathbb{P} = \mathbb{P}_\mathcal{L}$ of Example 1 we were enormously proud of having accomplished the same task in a "completely different" way. Indeed the motivation of the construction and the combinatorics of the proof are different, but the forcing turns out to be the same, for we have: $BA(\mathbb{P}) \simeq BA(\mathbb{N})$, where $\mathbb{N}$ is the set of Namba conditions. We shall now prove this. Until further notice we assume: $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. (We also let β denote ω_2.) We first define the set $\mathbb{N}$ of Namba conditions:

Def $\mathbb{N}$ = the set of subtrees $T \neq \emptyset$ of $\omega_2^{<\omega}$ s.t. $\overline{\{t \mid s \leq_T t\}} = \omega_2$ for all $s \in T$.
$\mathbb{N}$ is partially ordered by: $T \leq T' \leftrightarrow T \subset T'$;

Note Call T a strict Namba tree iff
- $\bigwedge s \in t \bigvee t (s \leq_T t \wedge t$ is a split pt. of $T)$
- If s is a split point of T, then s has ω_2 many immediate successors in T.

$\mathbb{N}$ is sometimes defined as the set of strict Namba trees. This is equivalent, however, since the strict Namba trees are dense in the Namba trees.

Call $f : \omega \to \omega_2$ a branch in T iff $f \restriction n \in T$ for $n < \omega$. If H is $\mathbb{N}$-generic over V, then $h = \bigcup \bigcap H$ is a cofinal map of ω to ω_2. Moreover, $H = \{T \in \mathbb{N} \mid h$ is a branch in $T\}$. We then say that h is a Namba-generic sequence.

Let $\mathbb{P} = \mathbb{P}$ be as in Example 1 with $\beta = \omega_2$. (Thus we assume $2^{\omega} = \omega_1 \wedge 2^{\omega_1} = \omega_2$.)

Lemma 1 Let G be $\mathbb{P}$-generic. Let $h \in V[G]$ s.t. $h : \omega \to \omega_2^V$ cofinally. Then h is Namba-generic.

proof of Lemma 1

Let $\mathring{h}^G = h$. Assume w.l.o.g. that $\Vdash \mathring{h} : \check{\omega} \to \check{\omega}_2$ cofinally.

<u>Claim 1</u> $\Delta_{\mathring{h}} = \{p \mid |p| = \omega_1^{M^p_{|p|}} \wedge$

$\wedge \bigvee \bar{h} \subset M^p_{|p|} \; p \Vdash \mathring{h} = \mathring{\pi}_{|p|,\omega_1} \circ \check{\bar{h}} \}$

is dense in $\mathbb{P}$.

proof.

Let $r \in \mathbb{P}$. We seek $p \leq r$ s.t. $p \in \Delta_{\mathring{h}}$. Let $G \ni r$ be $\mathbb{P}$-generic, $h = \mathring{h}^G$. Then there must be $\alpha < \omega_1$ s.t. $\mathrm{rng}(h) \subset \mathrm{rng}(\pi^G_{\alpha\,\omega_1})$. Hence $h = \pi^G_{\alpha,\omega_1} \circ \bar{h}$ for an $\bar{h} \subset M^G_\alpha$. But then there must be a $p \in G$ s.t. $|p| \geq \alpha$, $|p| = \omega_1^{M^p_{|p|}}$, and $p \Vdash \mathring{h} = \dot{\pi}_{\check{\alpha}\check{\omega}_1} \circ \check{\bar{h}}$. We may the assume $|p| = \alpha$ (otherwise replace $\bar{h}$ by $\pi^p_{\alpha,|p|} \circ \bar{h}$). QED (Claim 1)

<u>Def</u> Let $p \in \Delta_{\mathring{h}}$. Set T^p = the set of $s \in \omega_2^{<\omega}$ s.t. $[\![\varphi_s]\!] \neq \emptyset$ in $BA(\mathbb{P})$, where $\varphi_s =$ $(\check{p} \in \mathring{G} \wedge \bigwedge_{i < |s|} \mathring{h}(\check{i}) = \check{s}_i)$. ($|s| =_{\mathrm{Df}}$ length(s))

<u>Claim 2</u> $T^p \in \mathbb{N}$ for $p \in \Delta_{\mathring{h}}$.

prf. Suppose not. Let $T = T^p$.
Then there is $s \in T$ s.t. $\overline{\overline{T_{(s)}}} < \omega_2$,
where $T_{(s)} = \{ t \in T \mid s \leq_T t \vee t \leq_T s \}$.
Hence $\{ t(i) \mid t \in T_{(s)} \wedge i < |t| \} \subset \gamma < \omega_2$
for some γ. Let G be $\mathbb{P}$-generic s.t. $G \cap [\![\varphi_s]\!] \neq \phi$. Then
$p \in G$ and $\mathring{h}^G(i) = s_i$ for $i < |s|$.
But $\sup \mathring{h}''\omega_1 = \omega_2^V$; hence there is
j s.t. $\mathring{h}^G(j) \geq \gamma$. But then
$V[G] \models \varphi_t$, where $t = \mathring{h}^G \upharpoonright (j+1)$.
Hence $t \in T_{(s)}$ and $t(j) \geq \gamma$.
Contr! QED (Claim 2)

In the following let $p \in \Delta_{\mathring{h}}$ and
let $\bar{M} = M_\alpha^p$, $\alpha = |p|$. Let
$p \Vdash \mathring{\pi}_{\alpha\omega_1}^{\vee} \circ \check{\bar{h}} = \mathring{h}$.

Def For $\nu \in M$ set $M^{(\nu)} = L_\nu^A$, where
$M = L_\beta^A$. Similarly, for $\nu \in \bar{M} = M_\alpha^p =$
$= L_{\beta_\alpha}^{A_\alpha}$ we set $\bar{M}^{(\nu)} = L_\nu^{A_\alpha}$.

Claim 3 Let $\bar{\nu} = \bar{h}(i)$ and let $\nu = s(i)$ for an $s \in T = T^p$. Set:

$f =$ the M-least $f : \omega_1 \xrightarrow{\text{onto}} M^{(\nu)}$

$\bar{f} =$ " $\bar{M}$- " $f : \omega_1 \xrightarrow{\text{onto}} \bar{M}^{(\bar{\nu})}$.

Set $\pi = \pi^{(\bar{\nu},\nu)} = \{\langle \bar{f}(\xi), f(\xi)\rangle \mid \xi < \alpha\}$.

Let $\langle a, \bar{a}\rangle \in F^p$. Then

$\pi : \langle \bar{M}^{(\bar{\nu})}, \bar{a} \cap \bar{M}^{(\bar{\nu})}\rangle \prec \langle M^{(\nu)}, a \cap M^{(\nu)}\rangle$.

proof.

Let G be $\mathbb{P}$-generic with $[\![\varphi_s]\!] \cap G \neq \emptyset$. Then $\pi^G_{\alpha\omega_1}(\bar{f}) = f$. Hence

$\pi^G_{\alpha\omega_1}(\bar{f}(\xi)) = f(\xi)$ for $\xi < \alpha$ & hence

$\pi = \pi^G_{\alpha\omega_1} \restriction \bar{M}^{(\bar{\nu})}$, where

$\pi^G_{\alpha\omega_1}(\langle \bar{M}^{(\bar{\nu})}, \bar{a} \cap \bar{M}^{(\bar{\nu})}\rangle) = \langle M^{(\nu)}, a \cap M^{(\nu)}\rangle$,

since $\pi^G_{\alpha\omega_1}(\bar{\nu}) = \nu$ and

$\pi^G_{\alpha\omega_1} : \langle \bar{M}, \bar{a}\rangle \prec \langle M, a\rangle$. QED (Claim 3)

Claim 4 Let $T \leq T^p$ in N. There is $q \leq p$ s.t. $q \Vdash \mathring{h}$ is a branch in $\check{T}$.

proof of Claim 4.

Let $N^* = \langle H_\delta, M, <, p, T, \mathbb{P}, \mathbb{N}, \ldots \rangle$ where $\delta > \beta^+$. Let $p' \leq p$ conform to N^*.
Let $\bar{N}^* = \bar{N}^*(p'; N^*) = \langle \tilde{N}, \tilde{M}, <, \tilde{p}, \tilde{T}, \tilde{\mathbb{P}}, \tilde{\mathbb{N}}, \ldots \rangle$.
Let $\bar{H}$ be $\tilde{\mathbb{N}}$ - generic over $\bar{N}^*$ s.t. $\tilde{T} \in \bar{H}$. Set $\tilde{h} = \bigcup \bar{H}$. Then $\tilde{h} : \omega \to \tilde{\beta}$ cofinally, where $\tilde{\beta} = On \cap \tilde{M}$. (Note that $\tilde{M} = M^p_{|p|}$.) Obviously Claim 3 holds relativized to $\bar{N}^*$. Thus, for $\bar{\nu} = \bar{h}(i)$, $\tilde{\nu} = \tilde{h}(i)$ we can define $\pi^{(\bar{\nu}, \nu)}$ as in Claim 3. It is easily seen that if $\bar{h}(i) < \bar{h}(j)$, then

$$\pi^{(\bar{h}(i), \tilde{h}(i))} \subset \pi^{(\bar{h}(j), \tilde{h}(j))}.$$

Thus, setting $\pi = \bigcup_{i<\omega} \pi^{(\bar{h}(i), \tilde{h}(i))}$,

we have:

$$\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle \tilde{M}, \tilde{a} \rangle \text{ cofinally}$$

for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\tilde{p}}$. Since $\langle \tilde{M}, \tilde{a} \rangle$ is a ZFC$^-$ model, we conclude:

(1) $\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle \tilde{M}, \tilde{a} \rangle$ for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\tilde{p}}$.

Moreover:

(2) $\text{rng}(\pi) =$ the smallest $X \prec \tilde{M}$ s.t. $\text{rng}(\tilde{h}) \cup \alpha \subset X$ ($\alpha = \bar{p}_{\dagger}|p|$)

proof of (2)

($\supset$) is trivial. But if $\alpha \cup \mathrm{rng}(\tilde{h}) \subset X \prec \tilde{M}$, then $f_{\tilde{h}(i)} \in X$ for all $i < \omega$. Hence

$$\mathrm{rng}(\pi^{(\tilde{h}(i), \tilde{h}(i))}) = f_{\tilde{h}(i)}{}''\alpha \subset X.$$

QED (2)

Now let $\tilde{\alpha} = |p'|$ (hence $\tilde{\alpha} = \omega_1^{\bar{N}^*}$). Since $\bar{H}$ is $\bar{\mathbb{N}}^*$-generic over $\bar{N}^*$, $\bar{N}^*[\tilde{h}]$ is a ZFC$^-$ model. In $\bar{N}^*[\tilde{h}]$ we define $\langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle$, $\langle \tilde{\pi}_{ij} \mid i \leq j \leq \tilde{\alpha} \rangle$ as follows:

For $\xi < \tilde{\alpha}$ let X_ξ = the smallest $X \prec \tilde{M}$ s.t. $\xi \cup \mathrm{rng}(\tilde{h}) \subset X$. Set:

$C = \{\xi \geq \alpha \mid \xi = (X_\xi \cap \tilde{\alpha})\}$. Then $\tilde{\alpha} \in C$ and $C \cap \tilde{\alpha}$ is club in $\tilde{\alpha}$. Set:

$$\tilde{C} = C \cup \{\omega_1^{M_i^p} \mid i \leq \alpha\} \quad (\alpha = |p|).$$

For $\alpha \leq i \leq \tilde{\alpha}$ set: $\tilde{\pi}_i : \tilde{M}_i \stackrel{\sim}{\longleftrightarrow} X_{\tilde{\alpha}_i}$, where $\tilde{M}_i$ is transitive. For $i \leq \alpha$ set; $\tilde{M}_i = M_i^p$, $\tilde{\tilde{\pi}}_i = \tilde{\pi}_\alpha \pi_{i\alpha}^p$. We then set:

$$\tilde{\tilde{\pi}}_{ij} = \tilde{\pi}_j^{-1} \circ \tilde{\pi}_i \text{ for } i \leq j \leq \tilde{\alpha}.$$

Define q by: $M^q = \langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle$, $\pi^q = \langle \tilde{\tilde{\pi}}_{ij} \mid i \leq j \leq \tilde{\alpha} \rangle$, $F^q = F^{p'}$.

But then:

(3) $q \in \mathbb{P}$

proof.

Let $\mathfrak{A}$ model $\mathcal{L}(p')$. Change $\mathfrak{A}$ to $\tilde{\mathfrak{A}}$ by replacing $\dot{M}_i^{\mathfrak{A}}, \dot{\pi}_{ij}^{\mathfrak{A}}$ by $\tilde{M}_i, \tilde{\pi}_{ij}$ for $i \le j \le \tilde{\alpha}$ – i.e.

$$\dot{M}_i^{\tilde{\mathfrak{A}}} = \begin{cases} \dot{M}_i^{\mathfrak{A}} & \text{for } i \ge \tilde{\alpha} \\ \tilde{M}_i & \text{for } i \le \tilde{\alpha} \end{cases}$$

$$\dot{\pi}_{ij}^{\tilde{\mathfrak{A}}} = \begin{cases} \dot{\pi}_{ij}^{\mathfrak{A}} & \text{for } \tilde{\alpha} \le i \le j \\ \dot{\pi}_{\tilde{\alpha} j}^{\mathfrak{A}} \circ \tilde{\pi}_{i\tilde{\alpha}} & \text{for } i \le \tilde{\alpha} \le j \\ \tilde{\pi}_{ij} & \text{for } i \le j \le \tilde{\alpha} \end{cases}$$

Then $\tilde{\mathfrak{A}}$ models $\mathcal{L}(q)$. QED (3)

But then:

(4) $\le$.

prf.

$M^p = M^q \upharpoonright (|p|+1)$, $\pi^p = \pi^q \upharpoonright (|p|+1)^2$ by the construction of q. But if $\langle a, \bar{a} \rangle \in F^p$, then, since $p \le p'$, there is a' s.t. $\langle a, a' \rangle \in F^{p'}$ and $\pi^{p'}_{|p|,|p'|} : \langle M^p_{|p|}, \bar{a} \rangle \prec \langle M^{p'}_{|p'|}, a' \rangle$.

But then, whenever $G \ni p'$ is $\mathbb{P}$-generic, we have:

$\pi^G_{|p'|\,|p|} : \langle M^{p'}_{|p'|}, a' \rangle \prec \langle M, a \rangle$.

Since $\pi^G_{|p'|,|p|}$ extends uniquely to $\sigma : \bar{N}^* \prec N^*$ with $\pi^{a}_{|p'|,|p|} \cup F^{p'} \subset \sigma$, it follows that $F^{p'}(a') = \sigma(a') = a$, hence $a' = \tilde{a} = \sigma^{-1}(a)$. Thus we have $\pi^{q}_{|p|,|q|} : \langle M^{p}_{|p|}, \bar{a} \rangle \prec \langle M^{p'}_{|p'|}, \tilde{a} \rangle$ by (1), where $\langle a, \tilde{a} \rangle \in F^{q}$.

QED(4)

It remains only to show:

(5) $q \Vdash \overset{\circ}{h}$ is a branch in $\check{T}'$.

Let $G \ni q$ be $\mathbb{P}$-generic. Let $h = \overset{\circ}{h}^{G}$. Then $\pi^{G}_{\tilde{\alpha}\omega_1}$ extends to a unique $\sigma : \bar{N}^* \prec N^*$ s.t. $F^q \subset \sigma$. Since $p \in G$, we know that $h = \pi^{G}_{\alpha\omega_1} \circ \bar{h}$. But $\tilde{h} = \pi^{G}_{\alpha\tilde{\alpha}} \circ \bar{h}$. Hence $h = \pi^{G}_{\tilde{\alpha}\omega_1} \circ \tilde{h}$. Since $\tilde{h}$ is a branch in $\tilde{T}'$, h must be a branch in $T' = \sigma(\tilde{T}')$.

QED (Claim 4)

We now prove Lemma 1. Let G be $\mathbb{P}$-generic, $h = \overset{\circ}{h}^{G}$. We must show that
$$H = \{T \mid h \text{ is a branch in } T\}$$
is $\mathbb{N}$-generic.

Let Δ be dense in $\mathbb{N}$. It suffices to show that Δ^* is dense in $\mathbb{P}$, where:

$\Delta^* = \{p \mid \forall T \in \Delta \; p \Vdash \mathring{h} \text{ is a branch in } \check{T}\}$

is dense in $\mathbb{P}$. Let $r \in \mathbb{P}$. By Claim 1 there is $p \leq r$ s.t. $p \in \Delta_{\mathring{h}}$. Let $T \leq T_p$ s.t. $T \in \Delta$. By Claim 4 there is $p' \leq p$ s.t. $p' \Vdash \mathring{h}$ is a branch in $\check{T}$.

QED (Lemma 1)

Let $\mathbb{A} = BA(\mathbb{N})$, $\mathbb{B} = BA(\mathbb{P})$. We wish to show that $\mathbb{A} \simeq \mathbb{B}$. For $a \in \mathbb{A}$ let $\mathbb{A}|a$ be the restriction of $\mathbb{A}$ to $\{a' \mid a' \subset a\}$. Then $\mathbb{A}|a$ is a complete BA and $1\!\!1_{\mathbb{A}|a} = a$. We also have: $\mathbb{A}|a = BA(\mathbb{P} \wedge a)$. Similarly for $\mathbb{B}$. As a corollary of Claim 4 in the foregoing proof we have:

<u>Lemma 2.1</u> Let $p \in \Delta_{\mathring{h}}$. Then

$\mathbb{B} | [p]_{\mathbb{P}} = \mathbb{A} | [T_p]_{\mathbb{N}}$.

(Here $[p]_{\mathbb{P}}$ = the smallest $b \in \mathbb{B}$ s.t. $p \in b$. Similarly for $[T]_{\mathbb{N}}$.)

Prf. of Lemma 2.1

Let $G \ni p$ be $\mathbb{P}$-generic. Set:
$H_G = \{T \in \mathbb{N} \mid \mathring{h}^G$ is a branch in $T\}$. Then H_G is $\mathbb{N}$-generic by Claim 4. But $\mathring{h}^G = \bigcup \bigcap H_G \in V[H_G]$ and $\langle M_i^G \mid i \leq \omega_1 \rangle$, $\langle \pi_{ij}^G \mid i \leq j \leq \omega_1 \rangle$ are uniformly $V[H_G]$-definable in $M, p, \mathring{h}^G$. But then G is uniformly $V[H_G]$-definable in M, p, H_G, since

$q \in G \longleftrightarrow (M^q = M^G \upharpoonright (|q|+1) \wedge \pi^q = \pi^G \upharpoonright (|q|+1)^2 \wedge$
$\wedge \bigwedge \langle a, \bar{a} \rangle \in F^q (\pi^G_{|q|, \omega_1} : \langle M^G_{|q|}, \bar{a} \rangle \prec \langle M, a \rangle)$.

Hence there is a canonical $\ddot{G} \in V^{\mathbb{N}}$ s.t. $\ddot{G}^{H_G} = G$, whenever $G \ni p$ is $\mathbb{P}$-generic. For $a \in \mathbb{B} | [p]_{\mathbb{P}}$ set

$\sigma(a) = \sigma([\![\check{a} \cap \mathring{G} \neq \emptyset]\!]_{\mathbb{P}}) = [\![\check{a} \cap \ddot{G} \neq \emptyset]\!]_{\mathbb{N}}$

σ is easily seen to be a homomorphism of $\mathbb{B} | [p]$ into $\mathbb{A} | [T_p]$. But σ is injective, since if $\sigma(a) = \emptyset$, then $a \cap G = \emptyset$ for all $\mathbb{P}$-generic $G \ni p$. Hence $a = \emptyset$. It remains only to show that σ is onto. It is enough to show that $[T] \in \text{rng}(\sigma)$ for each $T \leq T_p$ in $\mathbb{N}$. Let $a = [\![\check{T} \in \mathring{H}_G]\!]_{\mathbb{P}}$.

We claim: $\sigma(a) = [T]$, or in other words:

$\sigma(a) \cap H \neq \emptyset \leftrightarrow T \in H$ for $\mathbb{N}$-generic T with $T_p \in H$

If not there is $T' \leq T_p$ which forces the negation of this equivalence. Let $G \ni p$ be $\mathbb{P}$-generic s.t. $T' \in H_G$. Then $G = \breve{G}^H$, where $H = H_G$. Hence $\sigma(a) \cap H \neq \emptyset \leftrightarrow a \cap \breve{G}^H = a \cap G \neq \emptyset \leftrightarrow T \in H$.

Contr! QED (Lemma 2.1)

Using this we prove:

<u>Lemma 2.2</u> $\mathbb{A} \simeq \mathbb{B}$ ($\mathbb{A} = BA(\mathbb{N})$, $\mathbb{B} = BA(\mathbb{P})$)

prf.
We first note some facts about $\mathbb{N}$. We recall that the strict Namba trees are dense in $\mathbb{N}$.

(1) Let $T \in \mathbb{N}$ be strict. Then

$$\mathbb{A}|[T] \simeq \mathbb{A}$$

proof.
$\mathbb{A}|[T] \simeq BA(\{T' \mid T' \leq T\})$. But forcing with subtrees of T is the same as forcing with subtrees of the set S of split points in T.
But $S \simeq 2^{<\omega_1}$. QED(1)

(2) Let $a \in \mathbb{A} \setminus \{\emptyset\}$. Then $\mathbb{A}|a \simeq \mathbb{A}$

prf. of (2)

$\mathbb{A}|a = BA(\{T \mid T \in a\})$. But $\{T \mid T \in a\}$ then collapses 2^{ω_2}, hence cannot satisfy the 2^{ω_2}-chain condition. Thus there is an max antichain $\langle T_\nu \mid \nu < 2^{\omega_2}\rangle$ in $\{T \mid T \in a\}$. We may w.l.o.g assume that each T_ν is strict. Similarly there is such a maximal antichain $\langle T'_\nu \mid \nu < 2^{\omega_2}\rangle$ in $\mathbb{N}$. Let $\sigma_i : \mathbb{A}|[T_\nu] \overset{\sim}{\longleftrightarrow} \mathbb{A}|[T'_\nu]$. Then $a = \bigcup_\nu [T_\nu]$, $1\!\!1 = \bigcup_\nu [T'_\nu]$ and we can define $\sigma : \mathbb{A}|a \overset{\sim}{\longleftrightarrow} \mathbb{A}$ by $\sigma(b) = \bigcup_{\nu < 2^{\omega_2}} \sigma_\nu(b \cap [T_\nu])$. QED(2)

Since Δ_h° is dense in $\mathbb{P}$, there is a max. antichain $\langle p_\nu \mid \nu < 2^{\omega_2}\rangle$ in $\mathbb{P}$ s.t. $p_\nu \in \Delta_h^\circ$ for all ν. Hence we may pick $\sigma_\nu : \mathbb{B}|[p_\nu] \overset{\sim}{\longleftrightarrow} \mathbb{A}|[T'_{p_\nu}]$ by Lemma 2.1. We then define $\sigma : \mathbb{B} \overset{\sim}{\longleftrightarrow} \mathbb{A}$ by $\sigma(b) = \bigcup_\nu \sigma_\nu(b \cap [p_\nu])$.

QED (Lemma 2.2)

We now show that, even if $\beta > \omega_2$, the forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of Example 1 is equivalent to a variant of Namba forcing. We define:

<u>Def</u> Let $\beta > \omega_1$ be regular. By a <u>Namba amoeba</u> on β we mean a subtree T of $\mathcal{P}_{\omega_2}(\beta)^{<\omega}$ s.t. if $s \in T$, then

(a) $s(i) \subset s(j)$ for $i \leq j < |s|$

(b) If $u \in \mathcal{P}_{\omega_2}(\beta)$, then

$$\{t \mid s \leq_T t \wedge \bigvee i \; u \subset t(i)\} \neq \emptyset .$$

From now on we let $\mathbb{N}$ be the set of Namba amoebas ordered by:

$$T \leq T' \longleftrightarrow T \subset T' \quad \text{for } T, T' \in \mathbb{N}.$$

We develop the main properties of Namba amoebas with a view to proving Lemma 3 below.

<u>Def</u> In any forcing extension of V we call h a <u>neat sequence</u> iff $h : \omega \longrightarrow \mathcal{P}_{\omega_2}(\beta)^V$, $h(i) \subset h(j)$ for $i \leq j < \omega$, and for every $u \in \mathcal{P}_{\omega_2}(\beta)^V$ there is i s.t. $u \subset h(i)$.

It is easily seen that if G is $\mathbb{N}$-generic and $h = \bigcup \bigcap G$, then h is a neat sequence. We shall also show that $\mathbb{N}$ does not add new reals.

We shall prove:

<u>Lemma 3</u> Let G be $\mathbb{P}$-generic over V.
(a) $V[G] = V[h]$, where h is a neat sequence.
(b) If $h \in V[G]$ is a neat sequence, then h is $\mathbb{N}$-generic over V and $V[G] = V[h]$

<u>Note</u> We do not know whether $BA(\mathbb{P}) \simeq BA(\mathbb{N})$.

<u>Def</u> Let $T \in \mathbb{N}$. $s \in T$ is a <u>big split point</u> in T iff $\mathrm{card}(\{u \mid s^\frown\langle u\rangle \in T\}) \geq \beta$.

(1) Let $s \in T$. Then there is a big split point $t \geq s$ in T.

prf. Suppose not.
By ind. on n there are fewer than β $t \in T$ s.t. $s \leq_T t$ and $|t| \leq n$. Hence $\overline{\overline{T}}_{(s)} < \beta$, which contradicts (b) in the definition of Namba amoeba.
(Here $T_{(s)} = T \mid \{t \mid s \leq_T t \vee t \leq_T s\}$)

As in the case of ordinary Namba trees we get an amalgamation lemma for Namba amoebas:

Def By an amalgamation sequence we mean a sequence $\langle\langle T_u, s_u\rangle \mid u \in \beta^{<\omega}\rangle$ s.t.

(a) $T_u \in \mathbb{N}$ and $s_u \in T_u$ is a big split pt. in T_u s.t. $T_{u^\frown\langle i\rangle} \subset T_{u(s_u)}$

(b) $s_u \subsetneqq s_v$ if $u \subsetneqq v$

(c) There is a 1-1 enumeration $\langle s_u^i \mid i<\beta\rangle$ of the immediate successors of s_u s.t. $s_u^i \leq_{T_u} s_{u^\frown\langle i\rangle}$ for $i<\beta$.

(d) If $v \in \mathcal{P}_{\omega_2}(\beta)$, then $\forall i\, \forall n\; v \subset s_{u^\frown\langle i\rangle}(n)$.

(2) Let $\langle\langle T_u, s_u\rangle \mid u \in \beta^{<\omega}\rangle$ be an amalgamation sequence. Then

$$\bigcap_{n<\omega} \bigcup_{|u|=n} T_u = \bigcup_{h:\omega\to\beta} \bigcap_{n<\omega} T_{h\restriction n}$$

is a Namba amoeba.

Note If such a sequence is defined for $|u|<n$, it can be extended to $|u|\leq n$,

Note If $T^* = \bigcap_{n<\omega} \bigcup_{|u|=n} T_u$, then the s_u $(u\in\beta^{<\omega})$ are exactly the split points of T^*. Hence every split pt. of T^* is a big split pt.

Using this we get:

(3) Let G be $\mathbb{N}$-generic over V. Then $P(\omega)$ is absolute in $V[G]$.

prf.

Let $\Vdash \mathring{f} : \check{\omega} \to \check{2}$. It suffices to show:

<u>Claim</u> $\Delta = \{T \mid \exists f \; T \Vdash \mathring{f} = \check{f}\}$ is dense in $\mathbb{N}$

Let $T \in \mathbb{N}$. We first construct an amalgamation sequence $\langle T_u, s_u \rangle$ $(u \in \beta^{<\omega})$ s.t. $T_u \leq T$, $T_u \Vdash \mathring{f}(|\check{u}|) = \check{m}$ for some m, by (We construct $\langle\langle T_u, s_u\rangle \mid u \in \beta^n\rangle$ by induction on n.) Let $T^* = \bigcap_n \bigcup_{|u|=n} T_u$.

Then $T^*_{(s_u)} \Vdash \mathring{f}(|\check{u}|) = \check{m}$ for some $m < 2$ for all $u \in \beta^{<\omega}$. For each $f : \omega \to 2$ define a game G_f by:

In the i-th step I chooses $v_i \in P_{\omega_2}(\beta)$ s.t. $v_i \supset v_h$ for all $h < i$.

II then chooses $\xi_i < \beta$ s.t. $v_i \subset s_{\xi_0 \cdots \xi_i}(h)$ for an $h < |s_{\xi_0 \cdots \xi_i}|$.

II wins iff $T^*_{(s_{\xi_0 \cdots \xi_{i-1}})} \Vdash \mathring{f}(\check{i}) = \check{f(i)}$ for all $i < \omega$.

Clearly, I can only win at a finite stage. Hence one player has a winning strategy.

<u>Subclaim</u> II has a winning strategy.

Suppose not. For each $f:\omega \to 2$ let S_f be I's winning strategy. Set:

$$S(\xi_0,\dots,\xi_{i-1}) = \bigcup_{f:\omega\to 2} S_f(\xi_0,\dots,\xi_{i-1}).$$ Then

$S(u) \in \mathcal{P}_{\omega_2}(\beta)$ for $u \in \beta^{<\omega}$, since $2^\omega = \omega_1$.

But then S wins <u>every</u> G_f. Now let

ξ_i = the least ξ s.t. $S(\xi_0,\dots,\xi_{i-1}) \subset s_{\xi_0\dots\xi_{i-1},\xi}$.

Let $f(n)$ = that m s.t. $T^*_{(s_{\xi_0\dots\xi_{n-1}})} \Vdash \mathring{f}(\check{n}) = \check{m}$.

Then II wins G_f against S with the play $\langle \xi_i \mid i<\omega \rangle$. Contr! QED (Subclaim)

Now let S be II's winning strategy for G_f. Let $\tilde{T}$ be the maximal Namba amoeba (i.e. the set of all $v: n \to \mathcal{P}_{\omega_2}(\beta)$ s.t. $n<\omega$ and $v_i \subset v_l$ for $i \le l < n$). Then $\tilde{T}$ is the tree of all possible plays by I. For $v = \langle v_0,\dots,v_{n-1}\rangle \in \tilde{T}$ set $S(v) = \langle S(v_0,\dots,v_{j-1}) \mid j<n\rangle$.

Then $s_{S(v)}$ is a point of length $|v|$ in T^*.

It is easily seen that

$$T' = \{s \mid \bigvee v \in \tilde{T}\ s \le_{T^*} s_{S(v)}\}$$

is a Namba amoeba. But

$$T'_{(s_{S(v)})} \subset T^*_{(s_{S(v)})} \Vdash \mathring{f}(|\check{v}|) = \check{f(|v|)}.$$

Hence $T' \Vdash \mathring{f} = \check{f}$. QED (3)

We are assuming $2^\omega = \omega_1$ and $2^\beta = \beta$ in V.
If G is $\mathbb{N}$-generic over V, then β has cofinality ω in $V[G]$, where $\mathcal{P}(\omega)$ is absolute in $V[G]$. Hence 2^β is not a cardinal in $V[G]$, ~~by~~ §4 Lemma 4.1
Since $\overline{\overline{\mathbb{N}}} = 2^\beta$ in V, however, $2^{\beta+}$ remains regular. Hence $cf(2^{\beta V}) = \omega_1$ in $V[G]$

We note that we could also have used the set of strict Namba amoebas in the above proof, where we define:

<u>Def</u> T is a <u>strict</u> Namba amoeba iff T is a Namba amoeba and whenever s is a split point of T, then for each $v \in \mathcal{P}_{\omega_2}(\beta)$ we have $v \subset u$ for a u s.t. $s^\frown\langle u\rangle \in T$.

<u>Open question</u> Does forcing with strict Namba amoebas yield the same model?
We doubt it.

Now let $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ be as in Example 1.
Recalling the definition of neat sequence we first show:

(4) ~~Let~~ Let G be $\mathbb{P}$-generic over V. Then $V[G]$ contains a neat sequence.

prf.
Letting $M^G = \langle M_i \mid i \leq \omega_1 \rangle$, $\pi^G = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$,
we know that $\langle M_{\omega_1}, \pi_{0\omega_1} \rangle$ is the liftup of $\langle M_0, \pi_{0,\omega_1} \restriction H^{M_i}_{\omega_2} \rangle$. Hence each $x \in M = M_{\omega_1}$ is M-definable in parameters from $\omega_1 \cup \mathrm{rng}(\pi_{0\omega_1})$.

Let $\langle u_i \mid i < \omega \rangle \in V$ s.t. u_i is finite and $M_0 = \bigcup_i u_i$, $u_i \subset u_j$ for $i \leq j < \omega$. Set:

v_i = the set of $\xi < \beta$ s.t. $\{\xi\}$ is M-definable in parameters from $\omega_1 \cup \pi_{0,\omega_1}$"$u_i$.

Then $\langle v_i \mid i < \omega \rangle$ is a neat sequence.
(To see this, note that if $v \in \mathcal{P}_{\omega_2}(\beta)^V$, then $\{v\}$ is M-definable in parameters from $\omega_1 \cup u_i$ for some i,
Hence $v \subset v_i$.) QED (4)

We also note:

(5) Let $h = \langle v_i \mid i < \omega \rangle \in V[G]$ be a neat sequence. Then $V[G] = V[h]$

proof.
Pick $\xi < \omega_1$ s.t. $v_i \in \mathrm{rng}(\pi^G_{\xi,\omega_1})$ for $i < \omega$.
Then $\mathrm{rng}(\pi^G_{\xi,\omega_1})$ = the smallest $X \prec M$ s.t. $\alpha_\xi \cup \{v_i \mid i < \omega\} \subset X$.

But $M^G \upharpoonright ((\omega_1+1)-3)$, $\pi^G \upharpoonright ((\omega_1+1)-3)^2$ are definable from M, $\mathrm{rng}(\pi^G_{3,\omega_1})$. Hence $M^G, \pi^G \in V[h]$, where G is definable from M^G, π^G by §4 Cor. 2.8. Hence $V[G] = V[h]$. QED (5)

<u>Def</u> a neat sequence h is $\mathbb{N}$-<u>generic</u> over V iff $G_h = \{T \in \mathbb{N} \mid h \text{ is a branch in } T\}$ is $\mathbb{N}$-generic over V.

(<u>Note</u> If h is $\mathbb{N}$-generic, then $h = \bigcup \bigcap G_h$. Hence $V[G_h] = V[h]$. Conversely, if G is $\mathbb{N}$-generic and $h = \bigcup \bigcap G$, then $G = G_h$.)

Lemma 3(a) follows by (4), (5), as does the last clause in Lemma 3(b). We must still prove:

<u>Sublemma 3.1</u> Let G be $\mathbb{P}$-generic over V and $h \in V[G]$ a neat sequence. Then h is $\mathbb{N}$-generic over V.

We shall closely imitate the proof of Lemma 1.

Let $h = \mathring{h}^G$. We assume wlog that $\Vdash \mathring{h}$ is a neat sequence.

<u>Claim 1</u> $\Delta_{\mathring{h}}$ is dense in $\mathbb{P}$, where $\Delta_{\mathring{h}}$ = the set of $p \in \mathbb{P}$ s.t. $|p| = \omega_1^{M^p_{|p|}}$ and $\bigvee \bar{h} \in M^p_{|p|}\ p \Vdash \mathring{h} = \mathring{\pi}_{|p|,\omega_1} \circ \check{\bar{h}}$.

proof of Claim 1: Exactly as before.

<u>Def</u> Let $p \in \Delta_h^\circ$. Set:

$T^p = s \in \mathbb{P}_{\omega_2}(\beta)^{<\omega}$ s.t. $[\![\varphi_s]\!] \neq \emptyset$ in $BA(\mathbb{P})$,

where $\varphi_s = (\check{p} \in \mathring{G} \wedge \bigwedge_{i<|s|} \mathring{h}(\check{i}) = \check{s}_i)$

<u>Claim 2</u> $T^p \in IN$ for $p \in \Delta_h^\circ$.

prf. Suppose not.

Let $T = T^p$. Then there is $s \in T$ and $v \in \mathbb{P}_{\omega_2}(\beta)$ s.t. for all $t \geq_T s$ we have $\bigwedge_{j<|t|} v \not\subset t(j)$.

Let G be $\mathbb{P}$-generic s.t. $G \cap [\![\varphi_s]\!] \neq \emptyset$.

Then $p \in G$ and $h \upharpoonright |s| = s$, where $h = \mathring{h}^G$. But h is neat. Hence there is $j \geq |s|$ s.t. $v \subset h(j)$. Let $t = h \upharpoonright j+1$.

Then $t \in T^p$ and $v \subset t(j)$. Contr!

QED (Claim 2)

In the following let $p \in \Delta_h^\circ$, $\alpha = |p|$, where $p \Vdash \mathring{\pi}_{\check{\alpha}\check{\omega}_1} \circ \check{\bar{h}} = \mathring{h}$.

<u>Def</u> For $u \in M$ s.t. $\bar{\bar{u}} \leq \omega_1$ set:

M^u = the smallest $X \prec L_\nu^A$ s.t. $u \subset X$, where $\nu = \sup(u)$ and $M = L_\beta^A$.

Similarly for $u \in \bar{M} = M_\alpha^p$ s.t. $\bar{M} \models \bar{\bar{u}} \leq \alpha$ we set M^u = the smallest $X \prec L_\nu^{\bar{A}}$ s.t. $u \in X$, where $\nu = \sup(u)$, $\bar{M} = L_{\bar{\beta}}^{\bar{A}}$.

Claim 3 Let $\bar{u} = \bar{h}(i)$, $u = s(i)$, where $s \in T^p$.

Set: f = the M-least $f : \omega_1 \xrightarrow{\text{onto}} M^u$

$\bar{f}$ = " $\bar{M}$- " $f : \alpha \xrightarrow{\text{onto}} \bar{M}^{\bar{u}}$

Set $\pi = \pi^{\bar{u}u} = \{\langle \bar{f}(\xi), f(\xi)\rangle \mid \xi < \alpha\}$.

Let $\langle a, \bar{a}\rangle \in F^p$. Then

$\pi : \langle \bar{M}^{\bar{u}}, \bar{a} \cap \bar{M}^{\bar{u}}\rangle \prec \langle M^u, a \cap M^u\rangle$.

prf. Exactly as in the proof of Lemma 1

Claim 4 Let $T \leq T^p$ in $\mathbb{N}$. There is $q \leq p$ s.t. $q \Vdash \mathring{h}$ is a branch in $\check{T}$.

prf. We imitate the earlier proof. Let $N^* = \langle H_\delta, M, <, p, T, \mathbb{P}, \mathbb{N}, \ldots\rangle$ where $\delta > \beta^+$. Let $p' \leq p$ conform to N^*. Set: $\bar{N}^* = \bar{N}^*(p', N^*) = \langle \tilde{N}, \bar{M}, <, \tilde{p}, \tilde{T}, \tilde{\mathbb{P}}, \tilde{\mathbb{N}}, \ldots\rangle$ s.t. $\tilde{T} \in \bar{H}$.

Let $\bar{H}$ be $\tilde{\mathbb{N}}$-generic over $\bar{N}^*$, s.t. $\tilde{T} \in \bar{H}$. Set $\tilde{h} = \bigcup \cap \bar{H}$. Then $\tilde{h}$ is a neat sequence in $\bar{N}^*$. But Claim 3 holds relativized to $\bar{N}^*$. Thus if $\bar{u} = \bar{h}(i)$, $\tilde{u} = \tilde{h}(i)$, we can define $\pi^{\bar{u}\tilde{u}}$ as in Claim 3. It is easily seen that if $\bar{h}(i) \in \bar{M}^{\bar{h}(j)}$, then $\pi^{\bar{h}(i), \tilde{h}(i)} \subset \pi^{\bar{h}(j), \tilde{h}(j)}$.

But $\bigcup_i \bar{M}^{\bar{h}(i)} = \bar{M}$ since $\bigcup_i \bar{h}(i) = \beta$.

Similarly $\bigcup_i \tilde{M}^{\tilde{h}(i)} = \tilde{M}$. Hence we have

$\pi = \bigcup_i \pi^{\bar{h}(i), \tilde{h}(i)} : \langle \bar{M}, \bar{a} \rangle \prec_{\Sigma_0} \langle \tilde{M}, \tilde{a} \rangle$

cofinally for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\tilde{P}}$. Since $\langle \tilde{M}, \tilde{a} \rangle$ is a ZFC model, we conclude

(1) $\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle \tilde{M}, \tilde{a} \rangle$ for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\tilde{P}}$.

(2) $\operatorname{rng}(\pi)$ = the smallest $X \prec \tilde{M}$ s.t. $\operatorname{rng}(\tilde{h}) \cup \alpha \subset X$.

The proof of (2) is as before.

Now let $\tilde{\alpha} = |p'| = \omega_1^{\bar{N}^*}$. Since $\bar{H}$ is $\bar{\mathbb{N}}$-generic over N^* and $\tilde{h} = \bigcup \cap \tilde{\mathbb{N}}$, $\bar{N}^*[\tilde{h}]$ is a ZFC$^-$ model. In $\bar{N}^*[\tilde{h}]$ we define $\langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle$, $\langle \tilde{\pi}_{ij} \mid i \leq j \leq \tilde{\alpha} \rangle$ exactly as before, noting that $\tilde{M}_i = M_i^p$, $\tilde{\pi}_{ij} = \pi_{ij}^p$ for $i \leq j \leq \alpha$. As before, we define q by:

$M^q = \langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle$, $\pi^q = \langle \tilde{\pi}_{ij} \mid i \leq j \leq \tilde{\alpha} \rangle$, $F^q = F^{p'}$. Just as before we then get:

(3) $q \in \mathbb{P}$

(4) $q \leq p$

(5) $q \Vdash \mathring{h}$ is a branch in T.

QED (Sublemma 3.1) Lemma 3 then follows exactly as Lemma 1 did. QED

We now again let $\beta = \omega_1$. There is a variant of Namba forcing which Shelah calls Nm′ and we shall consequently call $\mathbb{N}'$. We shall show that $\mathbb{N}'$ is reshapable and that, in fact, $BA(\mathbb{N}') \simeq BA(\mathbb{P})$ where $\mathbb{P}$ is a variant of Example 1. We first define $\mathbb{N}'$ and develop its properties.

<u>Def</u> $\mathbb{N}'$ = the set of $T \in \mathbb{N}$ s.t. for some $s \in T$ we have:

- $T = T_{(s)}$
- If $t \in T$ and $|t| \geq |s|$, then t has ω_2 many immediate successors.

s is then unique and is called the <u>stem</u> of T, denoted by $stm(T)$. (Thus T consists of a single stem followed by a tree isomorphic to $\omega_2^{<\omega}$.)

If G is $\mathbb{N}'$-generic and $b = \bigcup \bigcap G$, then b is a branch in $\omega_2^{<\omega}$ and $G = G_b = \{T \in \mathbb{N}' \mid b$ is a branch in $T\}$

We then say that b is $\mathbb{N}'$-generic.

It is known that forcing with $\mathbb{N}'$ adds no new reals. Magidor and Shelah have shown, however, that forcing with $\mathbb{N}'$ adds no $\mathbb{N}$-generic sequence $b: \omega \to \omega_1$ and conversely. We sketch the argument:

<u>Lemma 4.1</u> Let b be $\mathbb{N}'$-generic over V. Let $F \in V$, $F: \omega_2 \to \omega_2$. Then

(*) $\bigvee n \bigwedge i \geq n \quad \gamma_i > \sup_{h<i} F(\gamma_h)$,

where $b = \langle \gamma_i \mid i < \omega \rangle$.

proof.

We show that the set of conditions which force (*) lies dense in $\mathbb{N}'$. Let $T \in \mathbb{N}'$, $s = stm(T)$, $n = |s|$. Set

$T' = \{ t \in T \mid \bigwedge i \geq n \ s(i) > \sup_{h<i} F(s(h)) \}$.

Then $T' \in \mathbb{N}'$, $T' \leq T$ in $\mathbb{N}'$ and every branch thru $\mathbb{N}'$ satisfies (*)

QED(4.1)

<u>Lemma 4.2</u> Let $b = \langle \gamma_i \mid i < \omega \rangle$ be $\mathbb{N}$-generic over V. For some $F: \omega_2 \to \omega_2$ in V there are arbitrarily large $i < \omega$ s.t.

$\gamma_i \leq \sup_{h<i} F(\gamma_h)$.

prf. of 4.2

Let $T \in \mathbb{N}$ be (w.l.o.g.) a strict Namba tree. We construct $T' \leq T$, all of whose branches must satisfy the conclusion. As usual, let $N = \langle H_{\omega_3}, M, <, \ldots \rangle$, where $M = L^A_{\omega_2} = H_{\omega_2}$ and $<$ well orders N.

For $\xi < \omega_2$ set:

Y_ξ = the least $Y \prec N$ s.t. $\omega_1 \cup$

$F(\xi) = \omega_2 \cap Y_\xi$.

Then $\xi < F(\xi) < \omega_2$. Note that if $s \in F(\xi)^{<\omega}$ is a split pt. of T, then

(1) $\sup\{i < F(\xi) \mid s^\frown\langle i\rangle \in T\} = F(\xi)$

For $s \in T$ let $\langle l_0^s, \ldots, l_{m_s-1}^s\rangle$ be the monotone enumeration of the $l < |s|$ s.t. $s \upharpoonright l$ is a split point of T.

Set: T' = the set of $s \in T$ s.t.

$$s(l_i^s) \leq \sup_{h < l_i^s} F(s(h))$$

whenever i is odd.

<u>Claim</u> $T' \in \mathbb{N}$.

prf.

Clearly $\emptyset \in T'$. Now let $s \in T'$.

We show that s can be extended to $t \in T'$ which has ω_2 immediate successors in T'.

<u>Case 1</u> m_s is even.

Let $t \geq s$ in T be minimal s.t. t is a split point in T. Then all immediate successors of t lie in T'.

<u>Case 2</u> $m = m_s$ is odd.

Let $t \geq s$ in T be as above. Let

$$\alpha = \sup_{h<|t|} F(t(h)) = F(\sup_{h<|t|} t(h)).$$

Then $t \in \alpha^{<\omega}$ and there is $i < \alpha$ s.t. $t^\frown\langle i\rangle \in T$ by (1). Hence, letting $t' = t^\frown\langle i\rangle$ we have: $l_m^{t'} = |t'|$ and $t'(l_m^{t'}) \leq \sup_{h<|t'|} F(t'(h)) = \alpha$. Thus $t' \in T'$ and $m_{t'} = m+1$ is even. We can then apply Case 1.

QED (Claim)

Thus $T' \leq T$ in $\mathbb{N}$ has the desired property. QED (4.2)

We know, however, that if b is IN-generic then <u>every</u> $b' \in V[b]$ which is a cofinal ω-sequence in ω_2^V is IN-generic. Hence:

<u>Lemma 4.3</u> Let b be IN-generic. Then $V[b]$ contains no IN′-generic sequence.

But then we get the converse:

<u>Lemma 4.4</u> Let b′ be IN′-generic. Then $V[b']$ contains no IN-generic sequence.

proof.
Let $b \in V[b']$ be IN-generic. Let $f \in V[b]$ biject ω_1 onto ω_2^V. Then $f^{-1} \circ b' \in H_{\omega_1} \subset V$. But then $b' = f \circ (f^{-1} \circ b) \in V[b]$, contradicting Lemma 4.3. QED(4.4).

We now develop some other basic properties of IN′. We have a <u>weak amalgamation</u> <u>lemma</u>:

Lemma 4.5 Let $T \in \mathbb{N}'$, $s = \text{stm}(T)$, Let $\langle T_u \mid u \in \omega_2^{<\omega}\rangle$ be s.t.,

$T_u \in \mathbb{N}'$ and, letting $s_u = \text{stm}(T_u)$, we have: $T_\emptyset = T$, $|s_u| = |s| + u$,

$T_{u^\frown\langle i\rangle} \subset T_u$ for $i < \omega_2$,

$s_{u^\frown\langle i\rangle} \neq s_{u^\frown\langle j\rangle}$ for $i < j < \omega_1$.

Then $T' \in \mathbb{N}'$, where

$$T' = \bigcap_{n<\omega} \bigcup_{|u|=n} T_u = \bigcup_{f:\omega\to\omega_2} \bigcap_{n<\omega} T_{f\restriction n}$$

The proof is left to the reader.

We also have the <u>refinement lemma</u>:

Lemma 4.6 Let $T \in \mathbb{N}'$. Let $f : T \to \omega_1$. There is $\bar{T} \leq T$ in $\mathbb{N}'$ s.t. for all $s, s' \in \bar{T}$: $|s| = |s'| \to f(s) = f(s')$

proof.
For each $g : \omega \to \omega_1$ we play a game G_g defined by Shelah: Let $s = \text{stm}(T)$.

~~At~~ In the i-th move, player I picks an $\alpha_i < \omega_2$. Player II must then pick a ξ_i s.t. $\xi_i \geq \alpha_i$ and $s^\frown\langle \xi_0, \ldots, \xi_i \rangle \in T$ and $f(s^\frown\langle \xi_0, \ldots, \xi_i \rangle) = g(i)$. If at any point II cannot move, then I wins. Otherwise I wins. Thus one of the players has a winning strategy.

<u>Claim</u> There is g for which II has a winning strategy.

pf. Suppose not.
Let S_g be I's winning strategy for the game G_g. Set $S(t) = \bigcup_{g:\omega\to\omega_1} S_g(t)$.

Then $S(t) < \omega_2$. (We assume of course $2^\omega = \omega_1$.) Then S wins all of the games. Now pick successively ξ_i $(i < \omega)$ s.t. $s^\frown\langle \xi_0, \ldots, \xi_i \rangle \in T$ and $\xi_i > S(s^\frown\langle \xi_0, \ldots, \xi_{i-1} \rangle)$. Set $g(i) = f(\xi_i)$. This play wins G_g, defeating S. Contr! QED (Claim)

Let S be a winning strategy for II for G_g. Let T' be the tree of all $t \leq s^\frown S(\vec{\alpha})$ where $\vec{\alpha}$ is any finite sequence of plays by I. Then $T' \leq T$ has the desired property. QED (4.6)

(Note Using the weak amalgamation lemma and refinement lemma in tandem, it is not too hard to show that $\mathbb{N}'$ adds no new reals.)

The following can be regarded as a strengthening of Lemma 4.1:

<u>Lemma 4.6</u> Let W be a transitive ZFC$^-$ model s.t. $2^\omega = \omega_1 \wedge 2^{\omega_1} = \omega_2$ in W and $\alpha = (2^{\omega_2})^W$ exists and is countable in V. Let $\bar{\mathbb{N}} = \mathbb{N}'^W$. Let $F : \omega_2^W \to \omega_2$. For each $T \in \bar{\mathbb{N}}$ there is an $\mathbb{N}$-generic $G \ni T$ s.t. for $b = \langle \gamma_i \mid i < \omega \rangle = \bigcup \bigcap G$:

$$\bigvee m \bigwedge i \geq m \; \gamma_i > \sup_{h<i} F(\gamma_h).$$

(<u>Note</u> We do <u>not</u> require $F \in W$)

(<u>Note</u> If $F \in W$ it follows from this that $\Vdash_{\bar{\mathbb{N}}} \bigvee m \bigwedge i \geq m \; \mathring{\gamma}_i > \sup_{h<i} \check{F}(\mathring{\gamma}_h)$.)

Lemma 4.6 follows from:

<u>Lemma 4.7</u> Let $T \in IN'$, $s = stm(T)$, $n = |s|$. Let Δ be dense in IN'. There is $T' \leq T$ in IN' with $s' = stm(T')$ s.t. $\bigwedge i \geq n\,(i < |s'| \to s'(i) > \sup_{h<i} F(s'(h)))$, and $T' \in \Delta$.

proof.

For $t \in T$, $|t| \geq n$ define:

$$f(t) = \begin{cases} 1 \text{ if there is } T' \in \Delta \text{ with } T' \leq T \text{ and } t = stm(T') \\ 0 \text{ if not.} \end{cases}$$

By the refinement lemma there is $T' \leq T$ s.t. $stm(T') = s$ and $f(t) = g(|t|)$ for all $t \in T'$ s.t. $|t| \geq n$, (here $g: \omega \to \omega_2$).

But then there is $m \geq n$ s.t. $g(m) = 1$, since there is certainly a $T'' \leq T'$ s.t. $T'' \in \Delta$. We can certainly pick $t \in T'$ s.t. $|t| = m$ and

$$\bigwedge i \geq n\,(i < m \to t(i) > \sup_{h<i} F(t(h))).$$

Since $f(t) = 1$ there is $T' \leq T$ s.t. $t = stem(T')$ and $T' \in \Delta$.

QED (4.7)