§ 5 Applications

In this section we give a number of concrete examples of $\mathcal{L}$-forcing. A major aim will be to answer the question posed at the outset in §1, except that we now suppose $M = L^U_\beta = \langle L_\beta[U], U\rangle$ to be uncountable. We again suppose M to be iterable and, letting $M_i = L^{U_i}_{\beta_i}$ to be the i-th iterate, we assume there is α s.t. U_α is a measure in L^{U_α}. What is the least such α? We again know that U_i is not ω-complete for $i<\alpha$. It follows that $\alpha \leq \omega_1$ and that $cf(\tau_i) = \omega$ for $\tau_i = \kappa_i^{+M_i}$, where κ_i is the critical point of U_i. But then $cf(\tau_\alpha) = cf(\tau_0) = \omega$ if $0<\alpha$. Hence $\tau_\alpha < \omega_2$ if $0<\alpha$ by the covering lemma for L^{U_α}.

We shall first show that α can be any countable ordinal, and then that it can be ω_1.

Thm 1 Assume GCH. Let U be a normal measure on κ. Let $\beta > \kappa$ be ~~regular~~. There is a generic extension of V with:

(a) $\mathcal{P}(\omega)$ is absolute

(b) For each $\alpha < \omega_1$ there is $\bar{M} = \langle \bar{H}, \bar{U}\rangle$ which iterates to $M = \langle H_\beta, U\rangle$ in exactly α many steps.

Thm 2 Assume GCH. Let κ, $M = \langle H, U \rangle$ be as above. There is a generic extension of V with:

(a) $\mathbb{R}(\omega)$ is absolute

(b) There is $\bar{M}$ which iterates to M in ω_1 steps.

Note that in the generic extension of Thm 1 κ has countable cofinality, whereas it has cofinality ω in the extension of Thm 2.

. . . .

In order to prove these theorems we need the interpolation lemma for elementary embeddings. We define:

Def Let $M = \langle |M|, \in, \ldots \rangle$ be a transitive ZFC$^-$ model. Let $\pi : M \prec M'$ where M' is transitive. π is cofinal in M' iff $M' = \bigcup_{u \in M} \pi(u)$.

(Note If π is cofinal in M', $\beta = On \cap M$, $\beta' = On \cap M$, then $\beta' = \sup \pi''\beta$. Conversely, if $M = \langle L_\beta[A], \in, A, \ldots \rangle$ and $\beta' = \sup \pi''\beta$, then π is cofinal in M'.)

<u>Fact 1</u> Let $\pi : M \prec M'$ and set $\tilde{M} = M' | \bigcup_{u \in M} \pi(u)$. Then $\tilde{M} \prec M'$ and $\pi : M \prec \tilde{M}$ cofinally.

(The proof uses the following fact, which is a form of Łoś Thm and is proven by induction on φ:

Let $x_1, \ldots, x_n \in \tilde{M}$, $x_i \in \pi(u_i)$. Then

$\tilde{M} \models \varphi(x_1, \ldots, x_n) \longleftrightarrow$

$\langle x_1, \ldots, x_n \rangle \in \pi(X)$

where $X = \{\langle \vec{z} \rangle \in u_1 \times \ldots \times u_n \mid M \models \varphi(\vec{z})\}$.)

An immediate corollary is:

<u>Fact 2</u> Let $\tau > \omega$ be regular in M, where $\pi : M \prec M'$. Set:

$\bar{H} = H_\tau^M$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$, $\bar{\pi} = \pi \upharpoonright \bar{H}$.

Then $\bar{\pi} : \bar{H} \prec \tilde{H}$ cofinally.

<u>Def</u> Let $\pi : M \prec M'$. Let $\tau > \omega$ be regular in M. π is τ-<u>cofinal</u> in M' iff $M' = \bigcup_{\bar{\bar{u}} < \tau \text{ in } M} \pi(u)$.

(Hence τ-cofinality implies cofinality.)

Def Let $\tau > \omega$ be regular in M, $\bar{H} = H_\tau^M$, $\bar{\pi} : \bar{H} \prec H$ cofinally. By a liftup of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$ s.t. M' is transitive, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : M \prec M'$ τ-cofinally. (Hence $H \subset M'$ and, in fact, $H = H_{\tau'}^{M'}$, where $\tau' = \pi(\tau) = On \cap H$.)

Fact 3 Let $\langle M, \bar{\pi} \rangle$ be as above. Then there is at most one liftup.

proof.
Set $\mathbb{D} = \mathbb{D}_{\bar{\pi}}^M$ = the set of $\langle x, f \rangle$ s.t. $f \in M$, $f : u \to M$ for a $u \in \bar{H}$, and $x \in \bar{\pi}(u)$.
For $\langle x, f \rangle \in \mathbb{D}$ set $k(\langle x, f \rangle) = \pi(f)(x)$.
Clearly k maps $\mathbb{D}$ onto M'. But then if $u_i = \langle x_i, f_i \rangle \in \mathbb{D}$ $(i = 1, \ldots, n)$,
$M' \models \varphi(u_1, \ldots, u_n) \leftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X_{\vec{u}})$,
where $X_{\vec{u}} = \{\langle z_1, \ldots, z_n \rangle \mid M \models \varphi(f_1(z_1), \ldots, f_n(z_n))\}$
(hence $X_{\vec{u}} \in \bar{H}$).
This means, in particular, that if $\langle \pi', M'' \rangle$ is a second liftup, then
$k(u) \in k(v) \leftrightarrow k'(u) \in k'(v)$
$\quad$ " $=$ " $\quad \leftrightarrow \quad$ " $=$ " ,
where $k'(\langle x, f \rangle) = \pi'(f)(x)$.
Hence we can define $\sigma : M' \overset{\sim}{\leftrightarrow} M''$

by: $\sigma(k(u)) = k'(u)$. Hence $\sigma = \mathrm{id}$, $M' = M''$, since M, M' are transitive. But then $k = k'$ and $\pi(x) = \pi'(x) = k(\langle \emptyset, f_x \rangle)$, where $f_x = \langle x, \emptyset \rangle$. QED (Fact 3)

<u>Fact 4</u> The liftup of $\langle M, \bar{\pi} \rangle$ exists iff E is well founded, where $E \subset \mathbb{D}^2$ is defined by:

$$\langle x, f \rangle E \langle z, g \rangle \longleftrightarrow \langle x, z \rangle \in \bar{\pi}(\{\langle u, v \rangle \mid f(u) \in g(v)\}).$$

prf.

$(\rightarrow)$ trivial.

$(\leftarrow)$ Define $I \subset \mathbb{D}^2$ by

$$\langle x, f \rangle I \langle z, g \rangle \longleftrightarrow \langle x, z \rangle \in \bar{\pi}(\{\langle u, v \rangle \mid f(u) = g(v)\}).$$

Similarly set

$$\tilde{A}\langle x, f \rangle \longleftrightarrow x \in \bar{\pi}(\{u \mid f(u) \in A^M\})$$

where A is any predicate in the language of M. Interpreting $\dot{\in}, \dot{=}, A$ by $E, I, \tilde{A}$ we get <u>Łoś theorem</u> in the form:

$$\mathbb{D} \models \varphi(\vec{u}) \longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X_{\vec{u}}),$$

where $u_i = \langle x_i, f_i \rangle \in \mathbb{D}$ $(i = 1, \ldots, n)$.

The proof is by induction on φ and is just like the proof of Łoś theorem for ultrapowers. But then if we factor $\mathbb{D}$ by I, we get a well founded model satisfying extensionality. Let $[u]$ be the equivalence class of

$u \in \mathbb{D}$ under I, There is then an $h : \mathbb{D}/I \overset{\sim}{\longleftrightarrow} M'$, where M' is transitive. It follows easily that $\langle M', \pi \rangle$ is the liftup of $\langle M, \bar{\pi} \rangle$, where $\pi(x) = h([\langle \emptyset, f_x \rangle])$, $f_x = \langle x, \emptyset \rangle$.

QED (Fact 4)

By this we get the <u>interpolation lemma</u> for ZFC$^-$ models:

<u>Fact 5</u> Let $\pi : M \prec M'$. Let $\tau > \omega$ be regular in M, $\bar{H} = H_\tau^M$, $\bar{\pi} = \pi \upharpoonright \bar{H}$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$. Then the liftup $\langle \tilde{M}, \tilde{\pi} \rangle$ of $\langle M, \bar{\pi} \rangle$ exists. Moreover, there is a unique $\sigma : \tilde{M} \prec M$ s.t. $\sigma \tilde{\pi} = \pi$ and $\sigma \upharpoonright \tilde{H} = \mathrm{id}$.

prf

E is well founded, since $\langle x, f \rangle E \langle y, g \rangle \longleftrightarrow \pi(f)(x) \in \pi(g)(y)$,

Hence $\langle \tilde{M}, \tilde{\pi} \rangle$ exists. Define $\sigma : \tilde{M} \to M'$ by: $\sigma(\tilde{\pi}(f)(x)) = \pi(f)(x)$. Then σ is elementary ~~by~~ Łos theorem. It follows easily that $\sigma \tilde{\pi} = \pi$ and $\sigma \upharpoonright \tilde{H} = \mathrm{id}$, Conversely, if σ' has these properties, then

$$\sigma'(\tilde{\pi}(f)(x)) = \pi(f)(x) = \sigma(\tilde{\pi}(f)(x))$$

QED (Fact 5)

<u>Note</u> Let $\tau \leq \kappa$ where κ is regular in M. If π maps M τ-cofinally to M', then $H^{M'}_{\pi(\kappa)} = \bigcup_{\bar{u} < \tau \text{ in } H^M_\kappa} \pi(u)$. Hence $\pi(u) = \sup \pi''\kappa$ and $\pi \upharpoonright H^M_\kappa$ maps τ-cofinally to $H^{M'}_{\pi(\kappa)}$ if $\tau < \kappa$.

<u>Note</u> If $\pi : M \prec M'$, $\pi' : M' \prec M''$, $\pi(\tau) = \tau'$, then $\pi'\pi$ is τ-cofinal to M'' iff iff π is τ-cofinal to M' and π' is τ'-cofinal to M''.

<u>Note</u> ZFC$^-$ does <u>not</u> imply that H_τ is a set for all regular τ. Thus in the above definitions and results we have not assumed that $H^M_\tau \in M$. In our applications, however, we shall normally have: $H^M_\tau = L_\tau[A]$ for an $A \in M$. Hence $H^M_\tau \in M$ and $\pi''H^M_\tau$ lies cofinally in $H^{M'}_{\pi(\tau)}$ iff $\pi''\tau$ lies cofinally in $\pi(\tau)$.

<u>Example 1</u> We assume $\beta \geq \omega_2$ to be regular s.t. $2^\beta = \beta$ and $2^\omega = \omega_1$. We devise an $\mathcal{L}$ s.t. $\mathbb{P} = \mathbb{P}_\mathcal{L}$ collapses each regular $\gamma \in (\omega_1, \beta]$, making it ω-cofinal. We shall also ensure that, if G is $\mathbb{P}$-generic, then $V[G] = V[x]$ where $x \subset \beta$ is a countable set. (We can take $x = \mathrm{rng}(\dot{\pi}^G_{0,\omega_1})$)

In the following let $\tau = \omega_2$ and assume that $\beta \geq \tau$ is regular, $2^\beta = \beta$, and $2^\omega = \omega_1$. Set $M = L_\beta^A = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$. Set $N = \langle H_{\beta^+}, M, <, \ldots \rangle$, where $<$ well orders N.

Before we tackle the problem mentioned at the outset of this section, we tackle a simpler problem: Is there a forcing extension of V which adds no reals but in which every regular $\gamma \in (\omega_1, \beta]$ becomes ω-cofinal? (A positive answer can, of course, be obtained by first collapsing β onto ω_1 and then applying Namba forcing. However, the problem with we want to answer the problem with our methods. The connections with Namba forcing will be discussed later.)

Let $\mathcal{L}$ be the language on N which, in addition to the basic axioms of §3, has the following axioms:

- $\dot\pi_{i\underline{n}}$ is $\dot\tau_i$-cofinal in $\underline{M}$ $(i < n)$, where $\dot\tau_i =_{Df} \omega_2^{\dot M_i}$
- $\mathrm{rng}(\dot\pi_{i+1, \underline{\omega_1}}) =$ the smallest $X \prec \underline{M}$ s.t. $\mathrm{rng}(\dot\pi_{i, \underline{\omega_1}}) \cup \{\dot\alpha_i\} \subset X$ for $i < \underline{\omega_1}$
- $\dot B = \emptyset$ (or better, just omit $\dot B$ and the axioms containing it)

<u>Note</u> The second axiom will guarantee that $V[G] = V[\mathrm{rng}(\dot{\pi}^{G}_{0\omega_1})]$ for $\mathbb{P}_{\mathcal{L}}$-generic G,

<u>Note</u> The first of these axioms says that $\langle \underline{M}, \dot{\pi}_{i\underline{\omega_1}} \rangle$ is the liftup of $\langle \dot{M}_i, \dot{\pi}_{i,\omega_1} \restriction H^{\dot{M}_i}_{\tau_i} \rangle$.
The second axiom guarantees that the entire sequence $\dot{M}, \dot{\pi}$ is determined by $\mathrm{rng}(\dot{\pi}_{\underline{0}\,\underline{\omega_1}})$.

We first show:

<u>Lemma 1</u> $\mathcal{L}$ is consistent.

proof.
Let $X \prec N$ be countable. Then $\alpha = \omega_1 \cap X$ is transitive and $\alpha = \omega_1^{\bar{M}}$, where $\sigma : \bar{N} \longleftrightarrow N|X$ and $\bar{N}$ is transitive.
Let $\sigma(\bar{M}) = M$. Let $\langle \tilde{N}, \tilde{\sigma} \rangle$ be the liftup of $\langle \bar{N}, \sigma \restriction H^{\bar{N}}_{\omega_2} \rangle$. Let $\tilde{M} = \tilde{\sigma}(\bar{M})$.
Since $\bar{M} = H^{\bar{N}}_{\bar{\beta}}$ (where $\sigma(\bar{\beta}) = \beta$), we see that $\tilde{M} = H^{\tilde{N}}_{\tilde{\beta}}$, where $\sigma(\bar{\beta}) = \tilde{\beta}$, and $\langle \tilde{M}, \tilde{\sigma} \restriction \bar{M} \rangle$ is the liftup of $\langle \bar{M}, \sigma \restriction H^{\bar{N}}_{\omega_2} \rangle$.
Note that $\tilde{N}$ is a ZFC$^-$ model. Moreover by the interpolation lemma there is $\sigma' : \tilde{N} \prec N$ s.t. $\sigma' \tilde{\sigma} = \sigma$. Clearly $\sigma'(\tilde{M}) = M$. Let $\tilde{\mathcal{L}}$ be defined from $\tilde{M}$ in $\tilde{N}$ as $\mathcal{L}$ was defined from M in N.

Since $\tilde{N}, N$ are ZFC$^-$-models, we have:

$\tilde{\mathcal{L}}$ is consistent $\leftrightarrow$ $\tilde{N} \models \tilde{\mathcal{L}}$ is consistent

$\leftrightarrow N \models \mathcal{L}$ " "

$\leftrightarrow \mathcal{L}$ is consistent,

Thus it suffices to show that $\tilde{\mathcal{L}}$ is consistent. We do this by displaying a model for $\tilde{\mathcal{L}}$. We define $M' = \langle M_i' \mid i \leq \omega_1 \rangle$, $\pi = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$ as follows: Define $\langle X_i \mid i \leq \omega_1 \rangle$ by:

$X_0 = \mathrm{rng}(\tilde{\sigma} \upharpoonright \bar{M})$,

$X_{i+1} =$ the smallest $X \prec \tilde{M}$ s.t. $X_i \cup \{\alpha_i\} \subset X$,

where $\alpha_i = \omega_1 \cap X_i$

$X_\lambda = \bigcup_{i<\lambda} X_i$ for $\mathrm{Lim}(\lambda)$

Set: $\pi_{i\omega_1} : M_i' \overset{\sim}{\leftrightarrow} \tilde{M} | X_i$, and $\pi_{ij} = \pi_{j\omega_1}^{-1} \circ \pi_{i\omega_1}$ for $i \leq j$. Then $\langle H_{\beta^{++}}, M', \pi \rangle$ is a model of $\tilde{\mathcal{L}}$.

(The axiom $H_{\omega_1} = \underline{H}_{\omega_1}$ follows since $H_{\omega_1} = \sigma(H_{\omega_1}^{\bar{N}}) = \tilde{\sigma}(H_{\omega_1}^{\bar{N}}) = H_{\omega_1}^{\tilde{N}}$, since $H_{\omega_1}^{\bar{N}} \in H_{\omega_2}^{N}$.) QED (Lemma 1)

We then show:

<u>Lemma 2</u> $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is revisable.

proof of Lemma 2.

Let p conform to $N^* = \langle H_\delta, N, <, m \rangle$, where $\delta > \beta^+$ is regular. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, <, m \rangle$ be as in §3 Lemma 3. Let $\bar{\mathcal{L}}$, $\bar{\mathbb{P}} = \mathbb{P}_{\bar{\mathcal{L}}}$ be defined in $\bar{N}^*$ as $\mathcal{L}$, $\mathbb{P}$ were defined in N^*. Let $\mathfrak{N}$ be a solid model of $\mathcal{L}(p)$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ (where $\bar{G} \in V$). This gives us $M^{\bar{G}}, \pi^{\bar{G}}$. Define a new model $\tilde{\mathfrak{N}} = \langle |\mathfrak{N}|, \in^{\mathfrak{N}}, \tilde{M}, \tilde{\pi} \rangle$ by:

$$\tilde{M}_i = \begin{cases} M_i^{\mathfrak{N}} & \text{for } i \geq |p| \\ M_i^{\bar{G}} & \text{for } i \leq |p| \end{cases}$$

$$\tilde{\pi}_{ij} = \begin{cases} \pi_{ij}^{\mathfrak{N}} & \text{for } |p| \leq i \leq j \\ \pi_{|p|j}^{\mathfrak{N}} \pi_{i|p|}^{\bar{G}} & \text{for } i \leq |p| \leq j \\ \pi_{ij}^{\bar{G}} & \text{for } i \leq j \leq |p|, \end{cases}$$

$\tilde{\mathfrak{N}}$ is easily seen to be a model of $\mathcal{L}(\tilde{p})$, where $\tilde{p} = \langle\langle M^{\bar{G}}, \pi^{\bar{G}}, \emptyset \rangle, F^p \rangle$.

Hence $\tilde{p} \in \mathbb{P}$. QED (Lemma 2)

Now let G be $\mathbb{P}$-generic. It is easily seen that each $\pi_{i\omega_1}^G$ is τ_i-cofinal in M, since π_{ij}^G is τ_i-cofinal in M_j^G for $i \leq j < \omega_1$ (where $\tau_i = \omega_1^{M_i^G}$).

But then if $\omega_2 \leq \gamma \in M$ is regular and $\pi^G_{i\omega_1}(\bar{\gamma}) = \gamma$ it follows easily that $\pi^G_{i\omega_1} \upharpoonright H^{M^G_i}_{\bar{\gamma}}$ is τ_i-cofinal in H^M_γ. Hence every regular $\gamma \in (\omega_1, \beta]$ is ω-cofinal in $V[G]$.

In the special case $\beta = \omega_2$, $\mathbb{P}$ simply makes ω_1 ω-cofinal. It is known, of course, that Namba forcing accomplishes the same, although the methods seem very different. To our great surprise, we discovered that forcing with $\mathbb{P}$ is the same as adding a Namba sequence. We defer the proof of that fact to the next chapter, however, and return to the problems stated at the outset. We are now in a position to prove Theorem 1.

We in fact show:

Lemma 3 Let κ be measurable. Let U be a normal ultrafilter on κ and let $\beta > \kappa$ be regular s.t. $2^{\beta} = \beta$. There is a generic extension $V[G]$ in which

- $\mathcal{P}(\omega)$ is absolute
- For each $\alpha < \omega_1$ there is $\langle H', U' \rangle$ which iterates to $\langle H_\beta, U \rangle$ in exactly α steps

prf. of Lemma 3.

Set $M = L_\beta^A = \langle L_\beta[A], A\rangle$ s.t. $L_\beta[A] = H_\beta$ and U is M-definable. Set

$N = \langle H_{\beta^+}, M, <, \ldots\rangle$, where $<$ well orders N.

There is then a generic extension $V[G]$ in which H_{ω_1} is absolute and in which there is a countable transitive $\bar{M}$ and a $\sigma : \bar{M} \prec M$ which is $\bar{\tau}$-cofinal, where $\bar{\tau} = \omega_1^{\bar{M}}$, $\tau = \omega_1^M$. (Thus $\langle M, \sigma\rangle$ is the liftup of $\langle \bar{M}, \sigma \upharpoonright H^{\bar{M}}_{\bar{\tau}}\rangle$.)

[This can be accomplished by the previously defined $\mathcal{L}$-forcing. We can also do it by first generically collapsing β to ω_2 and then applying Namba forcing.]

We then have:

(1) for every $\alpha < \omega_1$ there is $\bar{M}, \sigma$ with the above properties s.t. $\omega_1^{\bar{M}} > \alpha$ and σ extends to a $\sigma' : \bar{N} \prec N$, where $\bar{N}$ is a transitive end extension of $\bar{M}$ and $\sigma'(\bar{M}) = M$.

prf. Fix $\bar{M}', \sigma'$ with the above properties. Let $X \prec N$ be

the smallest $X \prec N$ s.t. $\mathrm{rng}(\sigma') \cup \{\alpha\} \subset X$.
Let $\sigma' : \bar{N} \overset{\sim}{\longleftrightarrow} X$, where $\bar{N}$ is transitive. Let $\sigma'(\bar{M}) = M$ and let $\sigma = \sigma' \restriction \bar{M}$. Then $\sigma'^{-1}\sigma : \bar{M}' \prec \bar{M}$ is $\omega_1^{\bar{M}'}$ cofinal and $\sigma' : \bar{M} \prec M$ is $\omega_1^{\bar{M}}$ cofinal. Clearly $\omega_1^{\bar{M}} = \omega_1 \cap X > \alpha$.

QED(1)

From now on let $\bar{M}, \sigma, \bar{N}, \sigma'$ be as in (1). As we showed at the end of §1, the infinitary language $\mathcal{L}$ on N which says that there exists $\langle M', U' \rangle$ which iterates to $\langle M, U \rangle$ in exactly α many steps is consistent. But then the corresponding language $\bar{\mathcal{L}}$ on $\bar{N}$ is consistent. Since $\bar{N}$ is countable, we can find a solid model of $\bar{\mathcal{L}}$. This gives us $\langle \bar{M}', \bar{U}' \rangle$ which iterates to $\langle \bar{M}, \bar{U} \rangle$ in exactly α-many steps (where $\sigma'(\bar{U}) = U$). We then use σ to "lift" $\langle \bar{M}', \bar{U}' \rangle$ to a $\langle M', U' \rangle$ which iterates to $\langle M, U \rangle$

Let $\langle \bar{M}_i, \bar{U}_i \rangle$ be the i-th iterate of $\langle \bar{M}', \bar{U}' \rangle$ $(i \leq \alpha)$ with iteration maps $\bar{\pi}_{ij}$ $(i \leq j \leq \alpha)$.

(2) The liftup $\langle M_i, \sigma_i \rangle$ of $\langle \bar{M}_i, \sigma \restriction H_{\bar{\tau}}^{\bar{M}_i} \rangle$ exists (where $\bar{\tau} = \omega_2^{\bar{M}} = \omega_2^{\bar{M}_i}$).

proof.
We know that $\langle M_\alpha, \sigma_\alpha \rangle = \langle M, \sigma \rangle$ exists.
We must show that E_i is well founded where

$\langle x, f \rangle E_i \langle y, g \rangle \longleftrightarrow \langle x, y \rangle \in \sigma(\{\langle u, v \rangle \mid f(u) \in g(v)\})$
where $\langle x, f \rangle, \langle y, g \rangle \in \mathbb{D}^{M_i}_{\sigma \restriction H_{\bar{\tau}}^{M_i}}$. But

Then
$\langle x, f \rangle E_i \langle y, g \rangle \longleftrightarrow \langle x, \bar{\pi}_{i\alpha}(f) \rangle E_\alpha \langle y, \bar{\pi}_{i\alpha}(g) \rangle$,
where E_α is well founded. QED(2)
By Łoś Theorem for liftups (as in the proof of Fact 4) we can then define:

(3) $\pi_{ij} : M_i \prec M_j$ by:
$$\pi_{ij}(\sigma_i(f)(x)) = \sigma_j(\bar{\pi}_{ij}(f)(x))$$
for $\langle x, f \rangle \in \mathbb{D}^{M_i}_{\sigma \restriction H_{\bar{\tau}}^{M_i}}$.

But then there is U_i which is M_i-definable by the same definition as U from M. Hence U_i is a normal measure in $\langle M_i, U_i \rangle$.