§ 5 Applications

In this section we give a number of concrete examples of $\mathcal{L}$-forcing. A major aim will be to answer the question posed at the outset in §1, except that we now suppose $M = L^U_\beta = \langle L_\beta[U], U \rangle$ to be uncountable. We again suppose M to be iterable and, letting $M_i = L^{U_i}_{\beta_i}$ to be the i-th iterate, we assume there is α s.t. U_α is a measure in L^{U_α}. What is the least such α? We again know that U_i is not ω-complete for $i < \alpha$. It follows that $\alpha \leq \omega_1$ and that $cf(\tau_i) = \omega$ for $\tau_i = \kappa_i^{+M_i}$, where κ_i is the critical point of U_i. But then $cf(\tau_\alpha) = cf(\tau_0) = \omega$ if $0 < \alpha$. Hence $\tau_\alpha < \omega_2$ if $0 < \alpha$ by the covering lemma for L^{U_α}.

We shall first show that α can be any countable ordinal, and then that it can be ω_1.

Thm 1 Assume GCH. Let U be a normal measure on κ. Let $\beta > \kappa$ be ~~regular~~. There is a generic extension of V with:

(a) $\mathbb{P}(\omega)$ is absolute

(b) For each $\alpha < \omega_1$ there is $\bar{M} = \langle \bar{H}, \bar{U} \rangle$ which iterates to $M = \langle H_\beta, U \rangle$ in exactly α many steps.

Thm 2 Assume GCH. Let κ, $M = \langle H, U\rangle$ be as above. There is a generic extension of V with:

(a) $\mathbb{R}(\omega)$ is absolute

(b) There is $\bar{M}$ which iterates to M in ω_1 steps.

Note that in the generic extension of Thm 1 κ has countable cofinality, whereas it has cofinality ω in the extension of Thm 2.

. . . .

In order to prove these theorems we need the interpolation lemma for elementary embeddings. We define:

Def Let $M = \langle |M|, \in, \ldots\rangle$ be a transitive ZFC$^-$ model. Let $\pi: M \prec M'$ where M' is transitive. π is cofinal in M' iff $M' = \bigcup_{u \in M} \pi(u)$.

(Note If π is cofinal in M', $\beta = \mathrm{On} \cap M$, $\beta' = \mathrm{On} \cap M'$, then $\beta' = \sup \pi''\beta$. Conversely, if $M = \langle L_\beta[A], \in, A, \ldots\rangle$ and $\beta' = \sup \pi''\beta$, then π is cofinal in M'.)

Fact 1 Let $\pi : M \prec M'$ and set $\tilde{M} = M' | \bigcup_{u \in M} \pi(u)$. Then $\tilde{M} \prec M'$ and $\pi : M \prec \tilde{M}$ cofinally.

(The proof uses the following fact, which is a form of Łoz Thm and is proven by induction on φ:

Let $x_1, \ldots, x_n \in \tilde{M}$, $x_i \in \pi(u_i)$. Then

$\tilde{M} \models \varphi(x_1, \ldots, x_n) \longleftrightarrow$

$\langle x_1, \ldots, x_n \rangle \in \pi(X)$

where $X = \{\langle \vec{z} \rangle \in u_1 \times \ldots \times u_n \mid M \models \varphi(\vec{z})\}$.)

An immediate corollary is:

Fact 2 Let $\tau > \omega$ be regular in M, where $\pi : M \prec M'$. Set:

$\bar{H} = H_\tau^M$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$, $\bar{\pi} = \pi \upharpoonright \bar{H}$.

Then $\bar{\pi} : \bar{H} \prec \tilde{H}$ cofinally.

Def Let $\pi : M \prec M'$. Let $\tau > \omega$ be regular in M. π is τ-cofinal in M' iff $M' = \bigcup_{\bar{\bar{u}} < \tau \text{ in } M} \pi(u)$.

(Hence τ-cofinality implies cofinality.)

Def Let $\tau > \omega$ be regular in M, $\bar{H} = H_\tau^M$, $\bar{\pi} : \bar{H} \prec H$ cofinally. By a liftup of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$ s.t. M' is transitive, $\pi \restriction \bar{H} = \bar{\pi}$, and $\pi : M \prec M'$ τ-cofinally. (Hence $H \subset M'$ and, in fact, $H = H_{\tau'}^{M'}$, where $\tau' = \pi'(\tau) = On \cap H$.)

Fact 3 Let $\langle M, \bar{\pi} \rangle$ be as above. Then there is at most one liftup.

proof.
Set $\mathbb{D} = \mathbb{D}_{\bar{\pi}}^M$ = the set of $\langle x, f \rangle$ s.t. $f \in M$, $f : u \to M$ for a $u \in \bar{H}$, and $x \in \bar{\pi}(u)$.
For $\langle x, f \rangle \in \mathbb{D}$ set $k(\langle x, f \rangle) = \pi(f)(x)$.
Clearly k maps $\mathbb{D}$ onto M'. But then if $u_i = \langle x_i, f_i \rangle \in \mathbb{D}$ $(i = 1, \ldots, n)$,
$M' \models \varphi(u_1, \ldots, u_n) \longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X_{\vec{u}})$,
where $X_{\vec{u}} = \{\langle z_1, \ldots, z_n \rangle \mid M \models \varphi(f_1(z_1), \ldots, f_n(z_n))\}$
(hence $X_{\vec{u}} \in \bar{H}$).
This means, in particular, that if $\langle \pi', M'' \rangle$ is a second liftup, then
$k(u) \in k(v) \longleftrightarrow k'(u) \in k'(v)$
$\quad " = " \qquad\qquad " = "$,
where $k'(\langle x, f \rangle) = \pi'(f)(x)$.
Hence we can define $\sigma : M' \xrightarrow{\sim} M''$

by: $\sigma(k(u)) = k'(u)$. Hence $\sigma = \mathrm{id}$, $M' = M''$, since M, M' are transitive. But then $k = k'$ and $\pi(x) = \pi'(x) = k(\langle \emptyset, f_x \rangle)$, where $f_x = \langle x, \emptyset \rangle$. QED (Fact 3)

<u>Fact 4</u> The liftup of $\langle M, \bar{\pi} \rangle$ exists iff E is well founded, where $E \subset \mathbb{D}^2$ is defined by:

$$\langle x, f \rangle E \langle z, g \rangle \longleftrightarrow \langle x, z \rangle \in \bar{\pi}(\{\langle u, v \rangle \mid f(u) \in g(v)\}).$$

prf.

($\rightarrow$) trivial.

($\leftarrow$) Define $I \subset \mathbb{D}^2$ by

$$\langle x, f \rangle I \langle z, g \rangle \longleftrightarrow \langle x, z \rangle \in \bar{\pi}(\{\langle u, v \rangle \mid f(u) = g(v)\}).$$

Similarly set

$$\tilde{A}\langle x, f \rangle \longleftrightarrow x \in \bar{\pi}(\{u \mid f(u) \in A^M\})$$

where A is any predicate in the language of M. Interpreting $\dot{\in}, \dot{=}, A$ by $E, I, \tilde{A}$ we get <u>Łoś theorem</u> in the form:

$$\mathbb{D} \models \varphi(\vec{u}) \longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X_{\vec{u}}),$$

where $u_i = \langle x_i, f_i \rangle \in \mathbb{D}$ $(i = 1, \ldots, n)$.

The proof is by induction on φ and is just like the proof of Łoś theorem for ultrapowers. But then if we factor $\mathbb{D}$ by I, we get a well founded model satisfying extensionality. Let $[u]$ be the equivalence class of

$u \in \mathbb{D}$ under I. There is then an $h : \mathbb{D}/I \overset{\sim}{\longleftrightarrow} M'$, where M' is transitive. It follows easily that $\langle M', \pi \rangle$ is the liftup of $\langle M, \overline{\pi} \rangle$, where $\pi(x) = h([\langle \emptyset, f_x \rangle])$, $f_x = \langle x, \emptyset \rangle$.

QED (Fact 4)

By this we get the <u>interpolation lemma</u> for ZFC^- models:

<u>Fact 5</u> Let $\pi : M \prec M'$. Let $\tau > \omega$ be regular in M, $\overline{H} = H_\tau^M$, $\overline{\pi} = \pi \upharpoonright \overline{H}$, $\tilde{H} = \bigcup_{u \in \overline{H}} \pi(u)$. Then the liftup $\langle \tilde{M}, \tilde{\pi} \rangle$ of $\langle M, \overline{\pi} \rangle$ exists. Moreover, there is a unique $\sigma : \tilde{M} \prec M$ s.t. $\sigma\tilde{\pi} = \pi$ and $\sigma \upharpoonright \tilde{H} = \mathrm{id}$.

prf

E is well founded, since
$\langle x, f \rangle E \langle y, g \rangle \longleftrightarrow \pi(f)(x) \in \pi(g)(y)$.
Hence $\langle \tilde{M}, \tilde{\pi} \rangle$ exists. Define $\sigma : \tilde{M} \to M'$ by: $\sigma(\tilde{\pi}(f)(x)) = \pi(f)(x)$. Then σ is elementary by Łoś theorem. It follows easily that $\sigma\tilde{\pi} = \pi$ and $\sigma \upharpoonright \tilde{H} = \mathrm{id}$. Conversely, if σ' has these properties, then
$\sigma'(\tilde{\pi}(f)(x)) = \pi(f)(x)) = \sigma(\tilde{\pi}(f)(x))$

QED (Fact 5)

<u>Note</u> Let $\tau \le \kappa$ where κ is regular in M. If π maps M τ-cofinally to M', then $H^{M'}_{\pi(\kappa)} = \bigcup_{\bar{u} < \tau \text{ in } H^M_\kappa} \pi(u)$. Hence $\pi(u) = \sup \pi''\kappa$ and $\pi \upharpoonright H^M_\kappa$ maps τ-cofinally to $H^{M'}_{\pi(\kappa)}$ if $\tau < \kappa$.

<u>Note</u> If $\pi : M \prec M'$, $\pi' : M' \prec M''$, $\pi(\tau) = \tau'$, then $\pi'\pi$ is τ-cofinal to M'' iff π is τ-cofinal to M' and π' is τ'-cofinal to M''.

<u>Note</u> ZFC^- does <u>not</u> imply that H_τ is a set for all regular τ. Thus in the above definitions and results we have not assumed that $H^M_\tau \in M$. In our applications, however, we shall normally have: $H^M_\tau = L_\tau[A]$ for an $A \in M$. Hence $H^M_\tau \in M$ and $\pi'' H^M_\tau$ lies cofinally in $H^{M'}_{\pi(\tau)}$ iff $\pi''\tau$ lies cofinally in $\pi(\tau)$.

<u>Example 1</u> We assume $\beta \ge \omega_2$ to be regular s.t. $2^\beta = \beta$ and $2^\omega = \omega_1$. We devise an $\mathcal{L}$ s.t. $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ collapses each regular $\gamma \in (\omega_1, \beta]$, making it ω-cofinal. We shall also ensure that, if G is $\mathbb{P}$-generic, then $V[G] = V[x]$ where $x \subset \beta$ is a countable set. (We can take $x = \operatorname{rng}(\dot{\pi}^G_{0,\omega_1})$)

In the following let $\tau = \omega_2$ and assume that $\beta \geq \tau$ is regular, $2^\beta = \beta$, and $2^\omega = \omega_1$. Set $M = L_\beta^A = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$. Set $N = \langle H_{\beta^+}, M, <, \ldots \rangle$, where $<$ well orders N.

Before we tackle the problem mentioned at the outset of this section, we tackle a simpler problem: Is there a forcing extension of V which adds no reals but in which every regular $\gamma \in (\omega_1, \beta]$ becomes ω-cofinal? (A positive answer can, of course, be obtained by first collapsing β onto ω_1 and then applying Namba forcing. However, the problem with we want to answer the problem with our methods. The connections with Namba forcing will be discussed later.)

Let $\mathcal{L}$ be the language on N which, in addition to the basic axioms of §3, has the following axioms:

- $\dot{\pi}_{i\underline{n}}$ is $\dot{\tau}_i$-cofinal in $\underline{M}$ $(i<n)$, where $\dot{\tau}_i =_{\text{Df}} \omega_2^{\dot{M}_i}$
- $\text{rng}(\dot{\pi}_{i+1,\underline{\omega_1}}) =$ the smallest $X \prec \underline{M}$ s.t. $\text{rng}(\dot{\pi}_{i,\underline{\omega_1}}) \cup \{\dot{\alpha}_i\} \subset X$ for $i < \underline{\omega_1}$
- $\dot{B} = \emptyset$ (or better, just omit $\dot{B}$ and the axioms containing it)

Note The second axiom will guarantee that $V[G] = V[\mathrm{rng}(\dot\pi^G_{0\omega_1})]$ for $P_{\mathcal{L}}$-generic G.

Note The first of these axioms says that $\langle \underline{M}, \dot\pi_{i\underline{\omega_1}} \rangle$ is the liftup of $\langle \dot M_i, \dot\pi_{i,\omega_1} \restriction H^{M_i}_{\tau_i} \rangle$. The second axiom guarantees that the entire sequence $\dot M, \dot\pi$ is determined by $\mathrm{rng}(\dot\pi_{\underline{0}\,\underline{\omega_1}})$.

We first show:

Lemma 1 $\mathcal{L}$ is consistent.

proof.
Let $X \prec N$ be countable. Then $\alpha = \omega_1 \cap X$ is transitive and $\alpha = \omega_1^{\bar M}$, where $\sigma: \bar N \overset{\sim}{\longleftrightarrow} N|X$ and $\bar N$ is transitive. Let $\sigma(\bar M) = M$. Let $\langle \tilde N, \tilde\sigma \rangle$ be the liftup of $\langle \bar N, \sigma \restriction H_{\omega_2}^{\bar N} \rangle$. Let $\tilde M = \tilde\sigma(\bar M)$. Since $\bar M = H^{\bar N}_{\bar\beta}$ (where $\sigma(\bar\beta) = \beta$), we see that $\tilde M = H^{\tilde N}_{\tilde\beta}$, where $\sigma(\bar\beta) = \tilde\beta$, and $\langle \tilde M, \tilde\sigma \restriction \bar M \rangle$ is the liftup of $\langle \bar M, \sigma \restriction H^{\bar N}_{\omega_2} \rangle$. Note that $\tilde N$ is a ZFC$^-$ model. Moreover by the interpolation lemma there is $\sigma': \tilde N \prec N$ s.t. $\sigma' \tilde\sigma = \sigma$. Clearly $\sigma'(\tilde M) = M$. Let $\tilde{\mathcal{L}}$ be defined from $\tilde M$ in $\tilde N$ as $\mathcal{L}$ was defined from M in N.

Since $\tilde{N}, N$ are ZFC$^-$-models, we have:

$\tilde{\mathcal{L}}$ is consistent $\leftrightarrow \tilde{N} \models \tilde{\mathcal{L}}$ is consistent

$\leftrightarrow N \models \mathcal{L}$ " "

$\leftrightarrow \mathcal{L}$ is consistent,

Thus it suffices to show that $\tilde{\mathcal{L}}$ is consistent. We do this by displaying a model for $\tilde{\mathcal{L}}$. We define $M' = \langle M_i \mid i \leq \omega_1 \rangle$, $\pi = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$ as follows: Define $\langle X_i \mid i \leq \omega_1 \rangle$ by:

$X_0 = \operatorname{rng}(\tilde{\sigma} \upharpoonright \bar{M})$,

$X_{i+1} =$ the smallest $X \prec \tilde{M}$ s.t. $X_i \cup \{\alpha_i\} \subset X$,

where $\alpha_i = \omega_1 \cap X_i$

$X_\lambda = \bigcup_{i<\lambda} X_i$ for $\operatorname{Lim}(\lambda)$

Set: $\pi_{i\omega_1}: M_i' \overset{\sim}{\leftrightarrow} \tilde{M} | X_i$, and $\pi_{ij} = \pi_{j\omega_1}^{-1} \circ \pi_{i\omega_1}$ for $i \leq j$. Then $\langle H_{\beta^{++}}, M', \pi \rangle$ is a model of $\tilde{\mathcal{L}}$. (The axiom $H_{\omega_1} = \underline{H}_{\omega_1}$ follows since $H_{\omega_1} = \sigma(H_{\omega_1}^{\bar{N}}) = \tilde{\sigma}(H_{\omega_1}^{\bar{N}}) = H_{\omega_1}^{\tilde{N}}$, since $H_{\omega_1}^{\bar{N}} \in H_{\omega_2}^{N}$.) QED (Lemma 1)

We then show:

<u>Lemma 2</u> $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is revisable.

proof of Lemma 2.

Let p conform to $N^* = \langle H_\delta, N, <, m \rangle$, where $\delta > \beta^+$ is regular. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{N}, <, m \rangle$ be as in §3 Lemma 3. Let $\bar{\mathcal{L}}$, $\bar{\mathbb{P}} = \mathbb{P}_{\bar{\mathcal{L}}}$ be defined in $\bar{N}^*$ as $\mathcal{L}$, $\mathbb{P}$ were defined in N^*. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ (where $\bar{G} \in V$). This gives us $M^{\bar{G}}, \pi^{\bar{G}}$. Define a new model $\tilde{\mathfrak{A}} = \langle |\mathfrak{A}|, \in^{\mathfrak{A}}, \tilde{M}, \tilde{\pi} \rangle$ by:

$$\tilde{M}_i = \begin{cases} M_i^{\mathfrak{A}} & \text{for } i \geq |p| \\ M_i^{\bar{G}} & \text{for } i \leq |p| \end{cases}$$

$$\tilde{\pi}_{ij} = \begin{cases} \pi_{ij}^{\mathfrak{A}} & \text{for } |p| \leq i \leq j \\ \pi_{|p|j}^{\mathfrak{A}} \pi_{i|p|}^{\bar{G}} & \text{for } i \leq |p| \leq j \\ \pi_{ij}^{\bar{G}} & \text{for } i \leq j \leq |p|, \end{cases}$$

$\tilde{\mathfrak{A}}$ is easily seen to be a model of $\mathcal{L}(\tilde{p})$, where $\tilde{p} = \langle\langle M^{\bar{G}}, \pi^{\bar{G}}, \emptyset\rangle, F^p\rangle$. Hence $\tilde{p} \in \mathbb{P}$. QED (Lemma 2)

Now let G be $\mathbb{P}$-generic. ~~It~~ It is easily seen that each $\pi_{i\omega_1}^G$ is τ_i-cofinal in M, since π_{ij}^G is τ_i-cofinal in M_j^G for $i \leq j < \omega_1$ (where $\tau_i = \omega_1^{M_i^G}$).

But then if $\omega_2 \leq \gamma \in M$ is regular and $\pi^G_{i\omega_1}(\bar{\gamma}) = \gamma$ it follows easily that $\pi^G_{i\omega_1} \upharpoonright H^{M_i^G}_{\bar{\gamma}}$ is τ_i-cofinal in H^M_γ. Hence every regular $\gamma \in (\omega_1, \beta]$ is ω-cofinal in $V[G]$.

In the special case $\beta = \omega_2$, $\mathbb{P}$ simply makes ω_1 ω-cofinal. It is known, of course, that Namba forcing accomplishes the same, although the methods seem very different. To our great surprise, we discovered that forcing with $\mathbb{P}$ is the same as adding a Namba sequence. We defer the proof of that fact to the next chapter, however, and return to the problems stated at the outset. We are now in a position to prove Theorem 1.

We in fact show:

<u>Lemma 3</u> Let κ be measurable. Let U be a normal ultrafilter on κ and let $\beta > \kappa$ be regular s.t. $2^{\beta} = \beta$. There is a generic extension $V[G]$ in which

- $\mathcal{P}(\omega)$ is absolute
- For each $\alpha < \omega_1$ there is $\langle H', U' \rangle$ which iterates to $\langle H_\beta, U \rangle$ in exactly α steps

prf. of Lemma 3.

Set $M = L^A_\beta = \langle L_\beta[A], A\rangle$ s.t. $L_\beta[A] = H_\beta$ and U is M-definable. Set $N = \langle H_{\beta^+}, M, <, \ldots\rangle$, where $<$ well orders N.

There is then a generic extension $V[G]$ in which H_{ω_1} is absolute and in which there is a countable transitive $\bar{M}$ and a $\sigma : \bar{M} \prec M$ which is $\bar{\tau}$-cofinal, where $\bar{\tau} = \omega_1^{\bar{M}}$, $\tau = \omega_1^M$. (Thus $\langle M, \sigma\rangle$ is the liftup of $\langle \bar{M}, \sigma \restriction H^{\bar{M}}_{\bar{\tau}}\rangle$.)

[This can be accomplished by the previously defined $\mathcal{L}$-forcing. We can also do it by first generically collapsing β to ω_2 and then applying Namba forcing.]

We then have:

(1) for every $\alpha < \omega_1$ there is $\bar{M}, \sigma$ with the above properties s.t. $\omega_1^{\bar{M}} > \alpha$ and σ extends to a $\sigma' : \bar{N} \prec N$, where $\bar{N}$ is a transitive end extension of $\bar{M}$ and $\sigma'(\bar{M}) = M$.

prf. Fix $\bar{M}', \sigma'$ with the above properties. Let $X \prec N$ be the smallest $X \prec N$ s.t. $\mathrm{rng}(\sigma') \cup \{\alpha\} \subset X$. Let $\sigma' : \bar{N} \overset{\sim}{\longleftrightarrow} X$, where $\bar{N}$ is transitive. Let $\sigma'(\bar{M}) = M$ and let $\sigma = \sigma' \upharpoonright \bar{M}$. Then $\sigma'^{-1}\sigma : \bar{M}' \prec \bar{M}$ is $\omega_1^{\bar{M}'}$ cofinal and $\sigma' : \bar{M} \prec M$ is $\omega_1^{\bar{M}}$ cofinal. Clearly $\omega_1^{\bar{M}} = \omega_1 \cap X > \alpha$.

QED(1)

From now on let $\bar{M}, \sigma, \bar{N}, \sigma'$ be as in (1). As we showed at the end of §1, the infinitary language $\mathcal{L}$ on N which says that there exists $\langle M', U' \rangle$ which iterates to $\langle M, U \rangle$ in exactly α many steps is consistent. But then the corresponding language $\bar{\mathcal{L}}$ on $\bar{N}$ is consistent. Since $\bar{N}$ is countable, we can find a solid model of $\bar{\mathcal{L}}$. This gives us $\langle \bar{M}', \bar{U}' \rangle$ which iterates to $\langle \bar{M}, \bar{U} \rangle$ in exactly α-many steps (where $\sigma'(\bar{U}) = U$). We then use σ to "lift" $\langle \bar{M}', \bar{U}' \rangle$ to a $\langle M', U' \rangle$ which iterates to $\langle M, U \rangle$

Let $\langle \bar{M}_i, \bar{U}_i \rangle$ be the i-th iterate of $\langle \bar{M}', \bar{U}' \rangle$ $(i \le \alpha)$ with iteration maps $\bar{\pi}_{ij}$ $(i \le j \le \alpha)$.

(2) The liftup $\langle M_i, \sigma_i \rangle$ of $\langle \bar{M}_i, \sigma \restriction H_{\bar{\tau}}^{\bar{M}_i} \rangle$ exists (where $\bar{\tau} = \omega_2^{\bar{M}} = \omega_2^{\bar{M}_i}$).

proof.

We know that $\langle M_\alpha, \sigma_\alpha \rangle = \langle M, \sigma \rangle$ exists. We must show that E_i is well founded where

$\langle x, f \rangle E_i \langle y, g \rangle \longleftrightarrow \langle x, y \rangle \in \sigma(\{\langle u, v \rangle \mid f(u) \in g(v)\})$

where $\langle x, f \rangle, \langle y, g \rangle \in \mathbb{D}^{M_i}_{\sigma \restriction H_{\bar{\tau}}^{M_i}}$. But

Then

$\langle x, f \rangle E_i \langle y, g \rangle \longleftrightarrow \langle x, \bar{\pi}_{i\alpha}(f) \rangle E_\alpha \langle y, \bar{\pi}_{i\alpha}(g) \rangle$,

where E_α is well founded. QED(2)

By Łos Theorem for liftups (as in the proof of Fact 4) we can then define:

(3) $\pi_{ij} : M_i \prec M_j$ by:

$\pi_{ij}(\sigma_i(f)(x)) = \sigma_j(\bar{\pi}_{ij}(f)(x))$

for $\langle x, f \rangle \in \mathbb{D}^{M_i}_{\sigma \restriction H_{\bar{\tau}}^{M_i}}$.

But then there is U_i which is M_i-definable by the same definition as U from M. Hence U_i is a normal measure in $\langle M_i, U_i \rangle$.

Since $\langle M_\alpha, U_\alpha\rangle = \langle M, U\rangle$, it suffices to show that $\langle\langle M_i, U_i\rangle | i \le \alpha\rangle$ is the iteration of $\langle M_0, U_0\rangle$ with iteration maps π_{ij} $(i \le j \le \alpha)$. We first show:

(4) $M_\lambda = \bigcup_{i<\lambda} \mathrm{rng}(\pi_{i\lambda})$ for $\mathrm{Lim}(\lambda)$, $\lambda \le \alpha$

prf.

Let $x = \sigma_\lambda(f)(z) \in M_\lambda$, where $f \in \bar{M}_\lambda$, $f : u \to \bar{M}_\lambda$, $z \in \sigma_\lambda(u)$. Then $f = \bar{\pi}_{i\lambda}(f')$ for an $i < \lambda$. Hence $x = \pi_{i\lambda}(\sigma_i(f')(z))$ QED(4)

(5) $\pi_{ij} \restriction H^{M_i}_{\kappa_i} = \mathrm{id}$, where $\kappa_i = \mathrm{crit}(U_i)$

prf

Let $x = \sigma_i(f)(z) \in H^{M_i}_{\kappa_i}$. Then $f : u \to H^{\bar{M}_i}_{\bar{\kappa}_i}$, where $\bar{\kappa}_i = \mathrm{crit}(\bar{U}_i)$. Hence $f \in H^{\bar{M}_i}_{\bar{\kappa}_i}$ and $\bar{\pi}_{ij}(f) = f$, since $\bar{\pi}_{ij} \restriction H^{M_i}_{\bar{\kappa}_i} = \mathrm{id}$. But since $H^{\bar{M}_i}_{\bar{\kappa}_i} = H^{\bar{M}_j}_{\bar{\kappa}_i}$, we have $\sigma_i \restriction H^{\bar{M}_i}_{\bar{\kappa}_i} = \sigma_j \restriction H^{\bar{M}_j}_{\bar{\kappa}_j}$. Hence $\pi_{ij}(x) =$

$= \pi_{ij}(\sigma_i(f)(z)) = \sigma_j \bar{\pi}_{ij}(f)(z) = \sigma_i(f)(z) = x.$

QED(5)

(7) M_{i+1} = the Σ_0-closure of $\mathrm{rng}(\pi_{i,i+1}) \cup \{\kappa_i\}$ in M_{i+1}.

prf.

Let $x = \sigma_{i+1}(f)(z) \in M_{i+1}$, $f \in \overline{M}_{i+1}$, $f : u \to \overline{M}_{i+1}$, $z \in \sigma(u)$ where $u \in H^{\overline{M}}_{\overline{\tau}}$.

Let $f = \overline{\pi}_{i,i+1}(g)(\overline{\kappa}_i)$. Then $x =$
$= \sigma_{i+1}(\overline{\pi}_{i,i+1}(g)(\overline{\kappa}_i))(z) = (\pi_{i,i+1}\sigma_i(g)(\kappa_i))(z)$,

where $z \in \mathrm{rng}(\pi_{i,i+1})$ by (5), QED (7)

(8) Let $X \in \mathcal{P}(\kappa_i) \cap M_i$. Then
$$X \in U_i \leftrightarrow \kappa_i \in \pi_{i,i+1}(X)$$

prf.

Let $X = \sigma_i(f)(z)$ as before. Then

$$X \in U_i \leftrightarrow z \in \sigma_i(\{v \mid f(v) \in \overline{U}_i\})$$
$$\leftrightarrow z \in \sigma_{i+1}(\{v \mid \overline{\kappa}_i \in \overline{\pi}_{i,i+1}(f)(v)\}$$
$$\leftrightarrow \kappa_i \in \pi_{i,i+1}(f)(z) = \pi_{i,i+1}(X)$$

since $\sigma_{i+1} \restriction H^{\overline{M}}_{\overline{\tau}} = \sigma_i \restriction H^{\overline{M}}_{\overline{\tau}} = \sigma \restriction H^{\overline{M}}_{\overline{\tau}}$,

and $f(v) \in \overline{U}_i \leftrightarrow \overline{\kappa}_i \in \overline{\pi}_{i,i+1}(f(v)) = \overline{\pi}_{i,i+1}(f)(v)$

for $v \in \mathrm{dom}(f) \in H^{\overline{M}}_{\overline{\tau}}$, QED (8)

(1) - (8) establish the desired conclusion.

QED (Lemma 3)

We note that this result holds not only for normal measures but also for "sufficiently closed" extenders. We adopt the following notion of extenders:

Let M be a transitive ZFC$^-$ model.

Def Let M be a transitive ZFC$^-$ model. E is an extender at κ of length γ on M iff $\kappa \in M$ and:

(a) $\gamma > \kappa$ is closed under Gödel's tuple function $\langle\ \rangle$.

(b) $\kappa \in M$ and $E = \langle E_\xi \mid \xi < \gamma \rangle$, where each E_ξ is an ultrafilter on $\mathcal{P}(\kappa)^M$

(c) The transitive ultrapower $M' = \mathrm{Ult}(M, E)$ exists – i.e. there is $\pi : M \prec M'$ s.t.

- M' = the Σ_0 - closure of $\gamma \cup \mathrm{rng}(\pi)$ in M'
- $X \in E_\xi \longleftrightarrow \xi \in \pi(X)$ for $X \in \mathcal{P}(\kappa)^M$, $\xi < \gamma$
- $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{M'}$.

Note π, M' are uniquely determined by these conditions + π is called the canonical embedding. We also write $\pi : M \xrightarrow[E]{} M'$ to mean that $M' = \mathrm{Ult}(M, E)$ and π is the canonical embedding.

Łoś Thm for ultrapowers says:

$M' \models \varphi(\pi(f_1)(\xi_1), \ldots, \pi_n(f_n)(\xi_n)) \longleftrightarrow$

$\longleftrightarrow \{\langle \eta_1, \ldots, \eta_n \rangle \mid M \models \varphi(f_1(\eta_1), \ldots, f_n(\eta_n))\} \in E_{\langle \xi_1, \ldots, \xi_n \rangle}$

where $\xi_1, \ldots, \xi_n < \gamma$ and $f_i \in M$, $f_i : \kappa \to M$ for $i = 1, \ldots, n$. (Here $\langle\ \rangle$ is Gödel's tuple function on On.)

Def Let E be an extender of length γ on M. E is nicely closed iff iff $({}^{\omega_1}\gamma)^M = ({}^{\omega_1}\gamma) \cap \mathrm{Ult}(M, E)$.

We then get:

Lemma 4 Let E be a nicely closed extender of length γ on V. Let $\beta > \gamma$ be regular s.t. $2^\beta = \beta$. There is a generic extension $V[G]$ in which

- $\mathcal{H}(\omega)$ is absolute
- For each $\alpha < \omega_1$ there is $\langle H', E' \rangle$ which iterates to $\langle H_\beta, E \rangle$ in exactly α steps.

prf.
Set $M = L^A_\beta$, where $L_\beta[A] = H_\beta$ and E is M-definable.
Set $N = \langle H_{\beta^+}, M, <, \ldots \rangle$ where $<$ wellorders N,
Work in the same generic extension as before.
Let $\alpha < \omega_1$. (1) holds just as before. Let $\bar{M}, \sigma, \bar{N}, \sigma'$ be as in (1). We see as before that the infinitary language $\bar{\mathcal{L}}$ on $\bar{N}$ which says that some $\langle \bar{M}', \bar{E}' \rangle$ iterates in exactly α many steps to $\langle \bar{M}, \bar{E} \rangle$ is consistent. (Here $\bar{E}$ is the extender on $\bar{N}$ (hence on $\bar{M}$) which is defined in $\bar{N}$ as E was defined on N.) We then use σ to "lift" $\langle \bar{M}', \bar{E}' \rangle$ to $\langle M', E' \rangle$ as before.
Let $\langle \bar{M}_i, \bar{E}_i \rangle$ be the i-th iterate of $\langle \bar{M}', \bar{E}' \rangle$ $(i \leq \alpha)$ with iteration maps $\bar{\pi}_{ij}$ $(i \leq j \leq \alpha)$,

(2) and (3) follow exactly as before. But then there is E_i which is M_i-definable by the same definition as E in M. It follows that E_i is an extender on $\langle M_i, E_i\rangle$. Since $\langle M_\alpha, E_\alpha\rangle = \langle M, E\rangle$, it suffices to show that $\langle\langle M_i, E_i\rangle \mid i \leq \alpha\rangle$ is the iteration of $\langle M_0, E_0\rangle$ with iteration maps π_{ij} $(i \leq j \leq \alpha)$.

(4), (5) follow exactly as before.

We note that the fact that $(\gamma^{\omega_1})^M = (\gamma^{\omega_1})^{Ult(M,E)}$ if $Ult(M,E)$ exists is expressible by $\langle M, E\rangle \models \varphi$ for a certain statement φ. (This is left to the reader. Then each $\langle \bar{M}_i, \bar{E}_i\rangle$ has the same property. By this we get:

(6) Let $f \in \bar{M}_i$, $f : \omega_1 \xrightarrow{\bar{M}_i} \bar{\gamma}_i$ (where $\bar{\gamma}_i$ is the length of $\bar{E}_i$). Then $\sigma_i(f) = \sigma_{i+1}(f)$. Moreover $\sigma_i \restriction (\bar{\gamma}_i + 1) = \sigma_{i+1} \restriction (\bar{\gamma}_i + 1)$.

proof:

Since σ_i, σ_{i+1} are liftups of $\langle \bar{M}_i, \sigma_i \restriction \bar{\mathcal{I}}\rangle$, $\langle \bar{M}_{i+1}, \sigma_i \restriction \bar{\mathcal{I}}\rangle$ resp., where $\bar{\mathcal{I}} = \omega_2^{\bar{M}} = \omega_2^{\bar{M}_i} = \omega_2^{\bar{M}_{i+1}}$, we know that each element of $\bar{\gamma}_i$ has the form $\sigma_i(f)(\xi)$

where $f_i : \omega_1^{\bar M} \to \bar\gamma_i$ in $\bar M_i$ and $\xi < \omega_1$. But Then $\bar\gamma_i$ is also the set of $\sigma_{i+1}(f)(\xi)$ for the same collection of f, since $(\bar\gamma_i\ \omega_1)^{\bar M_i} = (\bar\gamma_i\ \omega_1)^{\bar M_{i+1}}$. But then

$$\sigma_i(f)(\xi) < \sigma_i(g)(\zeta) \longleftrightarrow$$

$$\longleftrightarrow \langle \xi, \zeta \rangle \in \sigma(\{\langle \mu, \tau\rangle \mid f(\mu) < g(\tau)\})$$

$$\longleftrightarrow \sigma_{i+1}(f)(\xi) < \sigma_{i+1}(g)(\zeta)$$

for $f, g : \omega_1^{\bar M} \to \bar\gamma_i$ in $\bar M_i$, $\xi, \zeta < \omega_1$.

But then there is an isomorphism $h : \langle \bar\gamma_i, < \rangle \xrightarrow{\sim} \langle \bar\gamma_i, < \rangle$ defined by $h(\sigma_i(f)(\xi)) \longleftrightarrow \sigma_{i+1}(f)(\xi)$. Hence $h = id$ and $\sigma_i(f)(\xi) = \sigma_{i+1}(f)(\xi)$. It follows easily that $\sigma_i \upharpoonright (\bar\gamma_i + 1) = \sigma_{i+1} \upharpoonright (\bar\gamma_i + 1)$.

QED(6).

The proofs of (7), (8) are virtually as before, using (6). To prove (7) we let $x = \sigma_{i+1}(f)(\xi) \in M_{i+1}$, where $f \in \bar M_{i+1}$, $f : \omega_1^{\bar m} \to \bar M_{i+1}$, $\xi < \omega_1$. Let $f = \bar\pi_{i,i+1}(g)(\mu)$, where $g : \bar\kappa_i \to \bar M_i$, $g \in \bar M_i$, $\mu < \bar\gamma_i$. Then

$x = \sigma_{i+1}(\bar\pi_{i,i+1}(g)(\mu))(\xi) =$

$= (\pi_{i,i+1}\sigma_i(g)(\sigma_i(\mu))(\xi)$, where

$\sigma_i(\mu) < \bar{\delta}_i$ and $\xi < \omega_1$ (hence $\pi_{i,i+1}(\xi) = \xi$).
This proves (7). We reformulate (8) as:

(8') Let $X \in \mathcal{P}(\kappa_i) \cap M_i$, $\mu < \delta_i$. Then

$$X \in E_\mu \longleftrightarrow \mu \in \pi_{i,i+1}(X).$$

proof

Let $X = \sigma_i(f)(\xi)$, $f \in \bar{M}_i$, $f : \omega_1^{\bar{M}} \to \mathcal{P}(\bar{\kappa}_i)^{M_i}$,
and $\mu = \sigma_i(h)(\zeta)$, $h \in \bar{M}_i$, $h : \omega_1^{\bar{M}} \to \bar{\delta}_i$,
and $\xi, \zeta < \omega_1$. Then

$$X \in E_\mu \longleftrightarrow \langle \xi, \zeta \rangle \in \sigma_i(\{\langle \xi', \zeta' \rangle \mid f(\xi') \in \bar{E}_{i, h(\zeta')}\})$$

$$\longleftrightarrow \langle \xi, \zeta \rangle \in \sigma_{i+1}(\{\langle \xi', \zeta' \rangle \mid h(\zeta') \in \bar{\pi}_{i,i+1}(f)(\xi')\})$$

$$\longleftrightarrow \underset{\mu}{\underset{\shortparallel}{\sigma_{i+1}(h)(\zeta)}} \in \pi_{i,i+1}\underset{X}{\underset{\shortparallel}{\sigma_i(f)(\xi)}}$$

$$\mu \in \pi_{i,i+1}(X).$$

QED (Lemma 4)

We now turn to the proof of Thm 2. We note that the proof of Thm 1 gives:

<u>Fact 6</u> Let $\bar{U} \subset \bar{M}$ be a normal measure on $\bar{\kappa}$ which is $\bar{M}$-definable. Let $\langle \bar{M}_i, \bar{U}_i \rangle$ be the i-th iterate of $\langle \bar{M}, \bar{U} \rangle$ with iteration maps $\bar{\pi}_{ij}$ $(i \leq j \leq \alpha)$. Let $\bar{\tau} \leq \bar{\kappa}$ be regular in $\bar{M}$. Set $\bar{H} = H^{\bar{M}}_{\bar{\tau}}$. (Hence $\bar{H} = H^{\bar{M}_i}_{\bar{\tau}}$ and $\bar{\pi}_{ij} \restriction \bar{H} = \mathrm{id}$.) Let $\sigma : \bar{H} \prec H$ cofinally and let $\langle M_i, \sigma_i \rangle$ be the liftup of $\langle \bar{M}_i, \sigma \rangle$. Let U_i be definable over M_i as $\bar{U}$ was defined over M. Then $\langle M_i, U_i \rangle$ is the i-th iterate of $\langle M_0, U_0 \rangle$. Moreover, if π_{ij} $(i \leq j \leq \alpha)$ are the iteration maps, then $\pi_{ij} \sigma_i = \sigma_j \bar{\pi}_{ij}$ $(i \leq j \leq \alpha)$.

<u>Note</u> By a straightforward modification of the proof of Lemma 4, the corresponding statement holds for $\langle \bar{M}, \bar{E} \rangle$, where $\bar{E} \subset \bar{M}$ is an extender of length $\bar{\gamma}$ on $\bar{M}$ which is $\bar{M}$ definable and $\bar{E}$ is nicely closed wrt. $\bar{\tau}$ — i.e. $(\bar{\gamma}^{\bar{\tau}})^{\bar{M}} = (\bar{\gamma}^{\bar{\tau}})^{\mathrm{Ult}(\bar{M}, \bar{E})}$.

<u>Note</u> If $\bar{M}, \bar{\tau}, \bar{\kappa}, \bar{M}_i, \ldots, M, M_i, \sigma_i$ are as in Fact 6, and $\bar{\kappa}_i = \mathrm{crit}(\bar{U}_i)$, then $\sigma_i \restriction (\bar{\kappa}_i + 1) = \sigma_j \restriction (\bar{\kappa}_i + 1)$ for $i \leq j$. This is because $\sigma_i \restriction \bar{H} = \sigma_j \restriction \bar{H}$, where $H = H^{\bar{M}_i}_{\bar{\kappa}_i^+} = H^{\bar{M}_j}_{\bar{\kappa}_i}$, since $\langle H, \sigma_i \restriction \bar{H} \rangle = \langle H, \sigma_j \restriction \bar{H} \rangle$ is the liftup of $\langle \bar{H}, \sigma \restriction H^{\bar{M}}_{\bar{\tau}} \rangle$, where $H = H^{M_i}_{\kappa_i^+} = H^{M_j}_{\kappa_i^+}$.

<u>Note</u> By the preceding note Fact 6 has a converse. Let U be an M-definable measure on κ in M and suppose that $\langle M, U\rangle$ has iterates $\langle M_i, U_i\rangle$ $(i \leq \alpha)$ with iteration maps π_{ij} $(i \leq j \leq \alpha)$. Let $\sigma : \bar{M} \prec M$ and let $\bar{U}$ be defined over $\bar{M}$ like U over M. It is well known that $\langle \bar{M}, \bar{U}\rangle$ then has iterates $\langle \bar{M}_i, \bar{U}_i\rangle$ $(i \leq \alpha)$ ~~and~~ that there are <u>unique</u> maps $\sigma_i : \bar{M}_i \prec M_i$ with the properties: $\sigma_0 = \sigma$, $\sigma_j \bar{\pi}_{ij} = \pi_{ij}\sigma_i$, and $\sigma_i \restriction (\bar{\kappa}_i + 1) = \sigma_j \restriction (\bar{\kappa}_i + 1)$ for $i \leq j \leq \alpha$, where $\bar{\pi}_{ij} : \bar{M}_i \prec \bar{M}_j$ are the iteration maps and $\bar{\kappa}_i = \mathrm{crit}(\bar{U}_i)$. If we then assume σ to $\bar{\tau}$-cofinal in M for some $\bar{\tau} \leq \bar{\kappa}_0$, it follows by the previous note that σ_i is $\bar{\tau}$-cofinal in M_i for $i \leq \alpha$.

(A similar converse can be formulated for the case that E is an M-definable extender which is nicely closed w.r.t. τ, where $\tau = \sigma(\bar{\tau}) \leq \kappa = \mathrm{crit}(E)$.)

<u>Example 2</u> We now construct an $\tilde{\mathcal{L}}$ s.t. $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ proves Theorem 2. We shall in fact show:

<u>Lemma 5</u> Let κ, U, β be as in Lemma 3. There is a generic extension $V[G]$ in which

- $\mathbb{R}(\omega)$ is absolute
- There is $\langle H', U' \rangle$ which iterates to $\langle H_\beta, U \rangle$ in exactly ω_1 steps.

proof.
Define $M = L_\beta^A = \langle H_\beta, A \rangle$, $N = \langle H_{\beta^+}, N, <, \ldots \rangle$ as before. Let $\tilde{\mathcal{L}}$ be the language on N which, in addition to the basic axioms, contains the axioms:

- $\dot{B} = \phi$
- $\underline{\omega_2} = \sup \dot{\pi}_{i \underline{\omega_1}} {}'' \dot{\tau}_i$, where $\dot{\tau}_i =_{Df} \omega_2^{\dot{M}_i}$ for $i \leq \underline{\omega_1}$
- Let $\langle M', \sigma \rangle$ be the lift up of ~~Let~~ U' be defined over M' as $\underline{U}$ over $\underline{M}$. The $\langle M', U' \rangle$ iterates to $\dot{M}_j$ in $j-i$ steps. Moreover, if $h : M' \to \dot{M}_j$ is the iteration map, then $\dot{\pi}_{ij} = h\sigma$ $(i \leq j \leq \underline{\omega_1})$.

We first show:
<u>Lemma 5.1</u> $\tilde{\mathcal{L}}$ is consistent.

prf.
Let $\langle N', U' \rangle$ be the ω_1-th iterate of $\langle N, U \rangle$ with iteration map k.

Let $M' = k(M)$. Then $\langle M', U' \rangle$ is the ω_1-th iterate of $\langle M', U' \rangle$ with iteration map $k \restriction M$. Let $\tilde{\mathcal{L}}'$ be N'-definable as $\tilde{\mathcal{L}}$ was defined in N. Then

$$k : \langle N, M, \tilde{\mathcal{L}} \rangle \prec \langle N', M', \mathcal{L}' \rangle$$

and it suffices to show:

<u>Claim</u> $\tilde{\mathcal{L}}'$ is consistent.

Pf.

Let $\mathcal{L}$ be the language on N constructed in Example 1 and let G be $\mathbb{P}_{\mathcal{L}}$-generic over V. We construct a model of $\tilde{\mathcal{L}}'$ in $V[G]$. Let $M^G = \langle M_i \mid i \le \omega_1 \rangle$, $\pi^G = \langle \pi_{ij} \mid i \le j \le \omega_1 \rangle$ be defined as before. Then $\pi_{ij} : M_i \prec M_j$ is $\tau_i = \omega_1^{M_i}$-cofinal for $i \le j \le \omega_1$ and $M = M_{\omega_1}$. Let U_i be defined over M_i as U over M. Since $\pi_{i\omega_1} : \langle M_i, U_i \rangle \prec \langle M_{\omega_1}, U_{\omega_1} \rangle$, $\langle M_i, U_i \rangle$ is iterable. Let $\langle M_{i,l}, U_{i,l} \rangle$ be the l-th iterate of $\langle M_i, U_i \rangle$ for $l \le i \le \omega_1$. Let $k^i_{lj} : M_{il} \prec M_{ij}$ be the iteration map ($l \le j \le i \le \omega_1$). Set $H_i =_{Df} H^{M_i}_{\tau_i}$, where $\tau_i = \omega_1^{M_i}$.

Since $\pi_{ij} : \langle M_i, U_i \rangle \prec \langle M_j, U_j \rangle$, there are maps $\pi^l_{ij} : \langle M_{il}, U_{il} \rangle \prec \langle M_{jl}, U_{jl} \rangle$ $(l \le i \le j \le \omega_1)$ uniquely defined by the conditions:

$\pi^0_{ij} = \pi_{ij}$, $\pi^l_{ij} k^i_{hl} = k^j_{hl} \pi^h_{ij}$, and

$\pi^h_{ij} \upharpoonright (\kappa_{ih} + 1) = \pi^l_{ij} \upharpoonright (\kappa_{ih} + 1)$, where

$\kappa_{ih} = \mathrm{crit}(U_{ih})$ $(h \le l \le i \le j \le \omega_1)$.

Set: $\tilde{M}_i = M_{ii}$ + define $\tilde{\pi}_{ij} : \tilde{M}_i \prec \tilde{M}_j$ by $\tilde{\pi}_{ij} = k^j_{ij} \pi^i_{ij}$. Then:

$$\begin{tikzcd}
M_k \arrow[r] & M_{ki} \arrow[r] & M_{kj} \arrow[r] & \tilde{M}_k \\
M_j \arrow[u, "\pi_{jk}"] \arrow[r] & M_{ji} \arrow[u, "\pi^i_{jk}"] \arrow[r] & \tilde{M}_j \arrow[u, "\pi^j_{jk}"] \arrow[ur, "\tilde{\pi}_{jk}"'] & \\
M_i \arrow[u, "\pi_{ij}"] \arrow[r] & \tilde{M}_i \arrow[u, "\pi^i_{ij}"] \arrow[ur, "\tilde{\pi}_{ij}"'] & & \\
M_0 \arrow[u, "\pi_{0i}"] \arrow[ur, "\tilde{\pi}_{0i}"'] & & & \\
\tilde{M}_0 \arrow[u, equal] & & &
\end{tikzcd}$$

(The horizontal maps are the iteration maps.)

Since π_{ij} is τ_i-cofinal, it follows by the remarks following Fact 6 that each π^l_{ij} is τ_i-cofinal. But then $\langle H^{V[G]}_{\beta^+}, \langle \tilde{M}_i \mid i \le \omega_1 \rangle, \langle \tilde{\pi}_{ij} \mid i \le j \le \omega_1 \rangle \rangle$ is a model of $\tilde{\mathcal{L}}'$. QED (Lemma 5.1)

Now let G be $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ - generic. Let $M^G = \langle \tilde{M}_i \mid i \leq \omega_1 \rangle$, $\pi^G = \langle \tilde{\pi}_{ij} \mid i \leq j \leq \omega_1 \rangle$. Set $H_i = H_{\tau_i}^{\tilde{M}_i}$, where $\tau_i = \omega_2^{\tilde{M}_i}$.

Let $\langle M_1', \sigma' \rangle =$ the liftup of $\langle M_0, \tilde{\pi}_{0,\omega_1} \restriction H_0 \rangle$. Then $\langle M', u' \rangle$ iterates to $\langle M, u \rangle$ in exactly ω_1 many steps.

It remains only to show that $\mathbb{R}(\omega)$ is absolute in $V[G]$. This, of course, follows from:

<u>Lemma 5.2</u> $\mathbb{P} = \mathbb{P}_{\tilde{\mathcal{L}}}$ is revisable.

proof.

Let $p \in \mathbb{P}$ conform to $N^* = \langle H_\delta, N, <, \ldots \rangle$, where $\delta \geq \beta^+$ is a regular cardinal. Let $\bar{N}^*$ be defined as usual. Let $\bar{\mathcal{L}}$ be defined in $\bar{N}^*$ as $\tilde{\mathcal{L}}$ was defined in N^*. Let $\bar{\mathbb{P}} = \mathbb{P}_{\bar{\mathcal{L}}}$ and let $\bar{G}$ be $\bar{\mathbb{P}}$ - generic over $\bar{N}^*$. Define q by: $q = \langle\langle M^{\bar{G}}, \pi^{\bar{G}}, \emptyset \rangle, F^p \rangle$.

<u>Claim</u> $q \in \mathbb{P}$.

We construct a solid model of $\tilde{\mathcal{L}}(q)$. Let $\mathfrak{A}$ be a solid model of $\tilde{\mathcal{L}}(p)$.

Define a new model $\tilde{\mathfrak{A}}$ by:

$\tilde{\mathfrak{A}} = \langle |\mathfrak{A}|, \tilde{M}, \tilde{\pi} \rangle$, where:

$$\tilde{M}_i = \begin{cases} M_i^{\mathfrak{A}} & \text{for } |p| \le i \le \omega_1 \\ M_i^{\bar{G}} & \text{for } i \le |p| \end{cases}$$

$$\tilde{\pi}_{ij} = \begin{cases} \pi_{ij}^{\mathfrak{A}} & \text{for } |p| \le i \le j \le \omega_1 \\ \pi_{hj}^{\mathfrak{A}} \pi_{ih}^{\bar{G}} & \text{for } i \le |p| \le j \le \omega_1 \\ \pi_{ij}^{\bar{G}} & \text{for } i \le j \le |p| \end{cases} .$$

Then $\tilde{\mathfrak{A}} \models \varphi_q$; where $\tilde{\mathcal{L}}(q) = \mathcal{L} + \varphi_q$.

It suffices to show:

<u>Claim</u> $\tilde{\mathfrak{A}} \models \tilde{\mathcal{L}}$.

Let $i \le j$ and let $\langle M', \sigma \rangle$ be the liftup of $\langle \tilde{M}_i, \tilde{\pi}_{ij} \upharpoonright H_i \rangle$, where $H_i = H_{\tau_i}^{\tilde{M}_i}$, $\tau_i = \omega_2^{\tilde{M}_i}$. We must show that M' iterates to $\tilde{M}_j$ in exactly $j-i$ steps and that $\tilde{\pi}_{ij} = k\sigma$, where k is the iteration map. If $j \le |p|$ or $|p| < i$ this is clear, so let $i < |p| < j$. Let $\langle M_0, \sigma_0 \rangle$ be the liftup of $\langle \tilde{M}_i, \tilde{\pi}_{i,|p|} \upharpoonright H_i \rangle$. Then M_0 iterates to $\tilde{M}_{|p|}$ in $|p| - 1$ steps with iteration map k_0. Moreover

$\tilde{\pi}_{i,|p|} = k_0 \sigma_0$. Similarly if $\langle M_1, \sigma_1 \rangle$ is the lift up of $\langle \tilde{M}_{|p|}, \tilde{\pi}_{|p|j}, \upharpoonright H_{|p|} \rangle$, then M_1 iterates to $\tilde{M}_j$ in $j-|p|$ steps with iteration map k_1, and $\tilde{\pi}_{|p|j} = k_1 \sigma_1$. By Fact 6 it follows, however, that if $\langle M', \sigma' \rangle$ is the liftup of $\langle M_0, \tilde{\pi}_{|p|j}, \upharpoonright H_{|p|} \rangle$, then M' iterates to M_1 in $|p|-j$ steps. If k' is the iteration map, then $k'\sigma' = \sigma_1 k_0$. Hence M' iterates to $\tilde{M}_j$ in $(|p|-i) + (j-|p|) = j-i$ steps with iteration map $k_1 \circ k'$. But then $\tilde{\pi}_{ij} = \tilde{\pi}_{|p|j} \circ \tilde{\pi}_{|p|i} = k_1\sigma_1 \circ k_0\sigma_0 = k_1 k'\sigma'\sigma_0$, where $\langle M', \sigma'\sigma_0 \rangle$ is the liftup of $\langle \tilde{M}_i, \tilde{\pi}_i, \upharpoonright H_i \rangle$.

QED (Lemma 5)

A modification of the proof again yields:

<u>Lemma 6</u> Let E be a nicely closed extender of length γ on V. Let $\beta > \gamma$ be regular s.t. $2^{<\beta} = \beta$. There is a generic extension $V[G]$ in which:

- $\mathcal{H}(\omega)$ is absolute
- There is $\langle H', E' \rangle$ which iterates to $\langle H_\beta, E \rangle$ in exactly ω_1 steps.

The proof is left to the reader.