§4 More on revivability

We now develop the general theory of revivable forcings further. In particular, we shall prove the fact mentioned after the proof of §3 Lemma 3.1 and examine its consequences. The reader who is more interested in applications can skip directly to the next chapter.

We recall that for any set of conditions $\mathbb{P}$ the canonical complete Boolean algebra $BA(\mathbb{P})$ is defined as the set of $X \subset \mathbb{P}$ s.t. $X = \neg\neg X$, where $\neg X =_{df}$
$=_{df} \{p \mid \bigwedge q \in X \; p \perp q\}$ for $X \subset \mathbb{P}$.

(Thus $\neg X = \{p \mid \bigwedge p' \leq p \;\; p' \notin X\}$ if X is closed under $\leq$. In particular $\neg X$ is closed under $\leq$, which easily gives:
$\neg\neg X = \{p \mid \bigwedge p' \leq p \; \bigvee p'' \leq p' \; \bigvee q \in X \; p'' \leq q\} =$
$= \{p \mid \bigwedge p' \leq p \; \bigvee q \in X \; p' \parallel q\}$. $\mathbb{B} = BA(\mathbb{P})$ is a complete BA with the complement operation $\neg$ and the operations:

$$\bigcap^{\mathbb{B}} A = \bigcap A \;, \quad \bigcup^{\mathbb{B}} A = \neg\neg \bigcup A\,.)$$

For $q \in \mathbb{P}$ we set: $[q] = [q]^{\mathbb{B}} = \neg\neg\{q\}$.
(Hence $[q]$ = the smallest $b \in \mathbb{B}$ s.t. $q \in b$ $= \{p \mid \bigwedge p' \leq p \;\; p' \parallel q\}$)

Lemma 1 Let $\mathbb{B} = BA(\mathbb{P})$ where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ where $\mathcal{L}$ satisfies the basic axioms. Then $[p] \subset [q] \leftrightarrow \mathcal{L}(p) \vdash \mathcal{L}(q)$

prf.

$(\leftarrow)$ $\mathcal{L}(p) \vdash \mathcal{L}(q) \rightarrow \bigwedge p' \leq p\ \mathcal{L}(p) \vdash \mathcal{L}(q)$

$\rightarrow \bigwedge p' \leq p\ \mathrm{con}(\mathcal{L}(p') \cup \mathcal{L}(q)) \leftrightarrow$

$\leftrightarrow \bigwedge p' \leq p\ p' \| q \leftrightarrow [p] \subset [q]$

$(\rightarrow)$ Let $\mathcal{L}(p) \nvdash \mathcal{L}(q)$. We find $p' \leq p$ s.t. $\mathcal{L}(p') \cup \mathcal{L}(q)$ is inconsistent (hence $p' \perp q$). Let $\mathfrak{A}$ be solid s.t. $\mathfrak{A} \models \mathcal{L}(p)$, $\mathfrak{A} \not\models \mathcal{L}(q)$. Pick $\alpha \geq |p|, |q|$ s.t. for all $a \in R^p \cup R^q$ there is $a^* = \dot\pi^{\mathfrak{A}\,-1}_{\alpha,\omega_1}{}''a$ with $\dot\pi^{\mathfrak{A}}_{\alpha,\omega_1} : \langle \dot M^{\mathfrak{A}}_\alpha, a^* \rangle < \langle M, a \rangle$.

Define $p' \leq p$ by: $M^{p'} = \langle \dot M^{\mathfrak{A}}_i \mid i \leq \alpha \rangle$, $\pi^{p'} = \langle \dot\pi^{\mathfrak{A}}_{ij} \mid i \leq j \leq \alpha \rangle$, $\dot B^{p'} = \dot\pi^{\mathfrak{A}\,-1}_{\alpha,\omega_1}{}''\dot B^{\mathfrak{A}}$, and $F^{p'} = \{\langle a, a^* \rangle \mid a \in R^p \cup R^q\}$.

Clearly $p' \in \mathbb{P}$, since $\mathfrak{A} \models \mathcal{L}(p')$. But then $p' \leq q$. We must show:

<u>Claim</u> $p' \perp q$

<u>Case 1</u> $M^q \neq \langle \dot M^{\mathfrak{A}}_i \mid i \leq |q| \rangle$ or $\pi^q \neq \langle \dot\pi^{\mathfrak{A}}_{ij} \mid i \leq j \leq |q| \rangle$ or $b^q \neq b_{|q|} =_{\mathrm{df}} \dot\pi^{\mathfrak{A}\,-1}_{|q|,\omega_1}{}''\dot B^{\mathfrak{A}}$.

This immediately gives: $\neg con(\mathcal{L}(q) \cup \mathcal{L}(p'))$.

<u>Case 2</u> Case 1 fails.

Then there is $\langle a, \bar{a}\rangle \in F^q$ s.t.

$$\pi^{\dot{}\,\mathfrak{A}}_{|q|, \omega_1} : \langle M^q_{|q|}, \bar{a}\rangle \not\prec \langle M, a\rangle,$$

since $\mathfrak{A} \not\models \mathcal{L}(q)$. Letting a^* be as above, we then have: $\langle a^*, a\rangle \in F^{p'}$ and

$$\pi^{p'}_{|q|, |p'|} : \langle M^q_{|q|}, \bar{a}\rangle \not\prec \langle M^{p'}_{|p'|}, a^*\rangle,$$

Hence $\neg con(\mathcal{L}(q) \cup \mathcal{L}(p'))$. QED (Lemma 1.1

Now let G be $\mathbb{P}$-generic over V, where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$, $\mathbb{B} = BA(\mathbb{P})$. Define M^G, π^G, B^G as in Lemma 2.1. It is easily seen that M^G, π^G, B^G are, in fact, the G-interpretations of fixed canonical $\mathbb{P}$-names $M, \pi, B \in N^{*\mathbb{P}}$, where $N^* = \langle H_\delta, N, <, m\rangle$ for a regular $\delta > \beta^+$. This enables us to interpret the sentences of the language $\mathcal{L}$

in $N^*\mathbb{P}$ by interpreting $\mathring{M}, \mathring{\pi}, \mathring{B}$ as M, π, B and $\underline{x}$ as $\check{x}$ for $x \in N$. (δ can be any regular $\delta > \beta^+$.) Let $[\![\varphi]\!] \in \mathbb{B}$ be the interpretation of φ. As usual we write: $\Vdash \varphi \longleftrightarrow_{\mathrm{Df}} [\![\varphi]\!] = 1$. We can then define a new language $\mathcal{L}^*$ with the same formulae, but taking $\{\varphi \mid \Vdash \varphi\}$ as the new set of axioms. We can, of course, not expect all theorems of $\mathcal{L}$ to become theorems of $\mathcal{L}^*$, but many will, and in fact we shall have: $\mathbb{P} = \mathbb{P}_{\mathcal{L}^*}$. ($\mathcal{L}^* \subset N$ is, of course, not necessarily N-definable, but it is N'-definable, where $N' = \langle N, \mathcal{L}^* \rangle$ is a ZFC$^-$ model.) It is straightforward to see that:

<u>Lemma 2.1</u> Assume that $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ adds no new reals. Then all basic axioms are in $\mathcal{L}^*$.

We recall that $p \in \mathbb{P} \longleftrightarrow \mathrm{con}(\mathcal{L}(p))$, where $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. In order to show: $\mathbb{P} = \mathbb{P}_{\mathcal{L}^*}$ we therefore need:

$con(\mathcal{L}+\varphi_p) \leftrightarrow con(\mathcal{L}^*+\varphi_p)$ for $p \in \tilde{\mathbb{P}}$.

We first note:

Lemma 2.1 $con(\mathcal{L}^*+\psi) \leftrightarrow [\![\psi]\!] \neq \emptyset$ for all $\mathcal{L}$-sentences ψ.

Pf.

$con(\mathcal{L}^*+\psi) \leftrightarrow \mathcal{L}^* \nvdash \neg\psi \leftrightarrow [\![\neg\psi]\!] \neq 1 \leftrightarrow$

$\leftrightarrow [\![\psi]\!] \neq \emptyset.$ QED (2.2)

Note If G is $\mathbb{P}$-generic over N^* we can interpret $\mathcal{L}$-sentences in $N^*[G]$ by letting $\underline{x}$ stand for x $(x \in N)$ and $\dot{M}, \dot{\pi}, \dot{B}$ stand for M^G, π^G, B^G. Using this interpretation, Lemma 2.2 tells us that the following are equivalent:

- $\mathcal{L}^*+\psi$ is consistent
- $\mathcal{L}^*+\psi$ has a sound model in any collapse of 2^β to ω
- $\mathcal{L}^*+\psi$ has a model of the form $N^*[G]$, G being $\mathbb{P}$-generic over N^*, in any collapse of 2^δ to ω
- $[\![\psi]\!] \neq \emptyset$.

We now prove:

Lemma 2.3 Let $p \in \tilde{\mathbb{P}}$. Then

$$p \in \mathbb{P} \leftrightarrow \mathcal{L}^*(p) \text{ is consistent.}$$

Pf

$(\rightarrow)$ Let $p \in \mathbb{P}$ & let $G \ni p$ be $\mathbb{P}$-generic over N^*. Then $N^*[G]$ models $\mathcal{L}^*+\varphi_p$ by §3 Lemma 2. QED $(\rightarrow)$

($\leftarrow$) Suppose not. Then $[\![\varphi_p]\!] \neq \emptyset$. Let $q \in [\![\varphi_p]\!]$ (hence $q \in \mathbb{P}$). We may also assume w.l.o.g. that $|q| \geq |p|$. Let $G \ni q$ be $\mathbb{P}$-generic. Then $N^*[G] \models \varphi_p$. Hence $M_i^p = M_i^q = M_i^G$, $\pi_{ij}^p = \pi_{ij}^q = \pi_{ij}^G$ for $i \leq j \leq |p|$ and $b^p = (\pi^q_{|q|,|p|})^{-1}$ "b^q. It follows that whenever $\mathfrak{A}$ is a solid model of $\mathcal{L}(q)$, there is $\langle a, \bar{a} \rangle \in F^p$ s.t. $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1} : \langle M^p_{|p|}, \bar{a} \rangle \not\prec \langle M, a \rangle$, since otherwise we would have $\mathfrak{A} \models \mathcal{L}(p)$ + hence $p \in \mathbb{P}$.

Set: $\Delta =_{\text{df}}$ the set of $r \leq q$ s.t. for some $\langle \bar{a}, a \rangle \in F^p$, some $z \in \operatorname{dom}(\pi^r_{|p|})$ and some ψ we have:

$$\langle M^p_{|p|}, \bar{a} \rangle \models \psi(z) \not\leftrightarrow \langle M, a \rangle \models \psi(\pi^r_{|p|}(z)),$$

(where, as before, $\pi^r_i = F^r \circ \pi_{i,|r|}^r$).

<u>Claim</u> Δ is dense in $\{q' \mid q' \leq q\}$.

prf. Let $q' \leq q$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(q')$. Let $\langle a, \bar{a} \rangle \in F^p$, $z \in M^p_{|p|} = \dot{M}^{\mathfrak{A}}_{|p|}$ s.t.

(1) $\langle M^p_{|p|}, \bar{a} \rangle \models \psi(z) \not\leftrightarrow \langle M, a \rangle \models \psi(\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}(z))$.

Pick $\alpha > |q'|$ s.t. for all $a \in R = R^{q'} \cup R^p \cup \{\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}(z)\}$ there is a (necessarily unique) a^* s.t.

(2) $\dot{\pi}^{\mho}_{\alpha,\omega_1} : \langle \dot{M}^{\mho}_{\alpha}, a^* \rangle \prec \langle M, a \rangle$.

(Hence, letting $\langle a_i \mid i<\omega \rangle$ ennumerate $R^{q'} \cup R^p$ in V, we have $\langle a_i^* \mid i<\omega \rangle \in V$.)

Define $\tilde{r} \in \tilde{\mathbb{P}}$ by:

$M^r = \langle \dot{M}_i^{\mho} \mid i \leq \alpha \rangle$, $\pi^r = \langle \dot{\pi}_{ij}^{\mho} \mid i \leq j \leq \alpha \rangle$,

$b^r = (\dot{\pi}^{\mho}_{\alpha,\omega_1})^{-1} {}'' \dot{B}^{\mho}$, $F^r = \{ \langle a, a^* \rangle \mid a \in R \}$.

Then clearly, $\mho \models \varphi_r$. Hence $r \in \mathbb{P}$. Hence $r \leq q'$.

<u>Claim</u> $r \in \Delta$.

Let $x = \dot{\pi}^{\mho}_{|p|,\omega_1}(z)$, $x^* = (\pi^{\mho}_{|r|,\omega_1})^{-1}(x)$.

Then $x^* = \dot{\pi}^{\mho}_{|p|,|r|}(z) = \pi^r_{|p|,|r|}(z)$. Hence

$x = F^r \pi^r_{|p|,|r|}(z) = \pi^r_{|p|}(z)$. Then

$\langle M^p_{|p|}, \bar{a} \rangle \models \psi(z) \leftrightarrow \langle M, a \rangle \models \psi(\pi^r_{|p|}(z))$

QED (Claim)

Now let $G \ni q$ be $\mathbb{P}$-generic. Since $G \cap \Delta \neq \emptyset$ there is $\langle a, \bar{a} \rangle \in F^p$ s.t.

$\langle M^p_{|p|}, \bar{a} \rangle \models \psi(z) \not\leftrightarrow \langle M, a \rangle \models \psi(\pi^G_{|p|,\omega_1}(z))$

for some $z \in M^p_{|p|}$, since $\pi^G_{|p|,\omega_1} \supset \pi^r_{|p|}$, where $r \in G \cap \Delta$. Hence $N^*[G] \models \neg\varphi_p$. But $N^*[G] \models \varphi_p$, since $q \in [\![\varphi_p]\!]$. Contr!

QED (Lemma 2.3)

<u>Note</u> Given the set of conditions $\mathbb{P}$, the definition of the partial order $\leq_{\mathbb{P}}$ was purely structural, so we have indeed shown that $\mathcal{L}$ and $\mathcal{L}^*$ yield the same partially ordered set $\mathbb{P} = \mathbb{P}_{\mathcal{L}} = \mathbb{P}_{\mathcal{L}^*}$.

We can then repeat our earlier proofs to get:

<u>Cor 2.4</u> $p \| q \longleftrightarrow \mathcal{L}^*(p) \cup \mathcal{L}^*(q)$ is consistent

<u>Cor 2.5</u> $[p] \subset [q] \longleftrightarrow \mathcal{L}(p^*) \vdash \mathcal{L}^*(q)$

$\longleftrightarrow \mathcal{L}^* \vdash (\varphi_p \to \varphi_q)$.

But then:

<u>Cor 2.6</u> $[p] = [\![\varphi_p]\!]$ for $p \in \mathbb{P}$

proof.

$(\subset)$ since $p \Vdash \varphi_p$

$(\supset)$ If not, there is $q \in [\![\varphi_p]\!]$ s.t. $q \perp p$. Hence $\mathcal{L}^*(p) \cup \mathcal{L}^*(q)$ is inconsistent. But $q \Vdash \varphi_p \wedge \varphi_q$. Hence $\mathcal{L}^*(p) \cup \mathcal{L}^*(q)$ is consistent

Contr! QED (2.6)

<u>Cor 2.7</u> $p \Vdash \psi \longleftrightarrow \mathcal{L}^*(p) \vdash \psi$

prf. $\mathcal{L}^*(p) \vdash \psi \longleftrightarrow \mathcal{L}^* \vdash (\varphi_p \to \psi) \longleftrightarrow$

$\longleftrightarrow [\![\varphi_p]\!] \subset [\![\psi]\!]$ (by 2.2) $\longleftrightarrow$

$\longleftrightarrow [p] \subset [\![\psi]\!] \longleftrightarrow p \Vdash \psi$.

QED (2.7)

Cor 2.8 Let G be $\mathbb{P}$-generic over V. Then G is definable from $\langle M^G, \pi^G, B^G \rangle$ since for all $p \in \mathbb{P}$ we have:

$$p \in G \longleftrightarrow (M^p = M^G \upharpoonright (1+|p|) \wedge \pi^p = \pi^G \upharpoonright (1+|p|)^2 \wedge$$
$$\wedge\ b^p = (\pi^G_{|p|,\omega_1})^{-1} {}'' B^G \wedge$$
$$\wedge \bigwedge \langle a, \bar{a} \rangle \in F^p\ \pi^G_{|p|,\omega_1} : \langle M^p_{|p|}, \bar{a} \rangle \prec \langle M, a \rangle)$$

prf.

$p \in G \longleftrightarrow [p] \cap G \neq \emptyset \longleftrightarrow [\![\varphi_p]\!] \cap G \neq \emptyset \longleftrightarrow$

$\longleftrightarrow N^*[G] \models \varphi_p$. QED (2.8)

This theorem is frequently useful.

Note: all of the previous theorems in this section hold if $\mathcal{L}$ satisfies the basic axioms and $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ adds no new reals. (Lemmas 1 and 2.2 – 2.6 in fact require only that $\mathcal{L}$ satisfy the basic axioms.)

Lemma 3 Let $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ where $\mathcal{L}$ satisfies the basic axioms and $\mathbb{P}$ adds no new reals. and let $cf(\beta) = \omega_1$. Let G be $\mathbb{P}$-generic and let β be a cardinal in $V[G]$. Then β^+ is a cardinal in $V[G]$.

proof.
We can assume w.l.o.g. that $\Vdash cf(\check{\beta}) = \check{\omega}_1$.
(Otherwise replace $\mathbb{P}$, $\mathcal{L}$ by $\mathbb{P}' = \mathbb{P}_{\mathcal{L}'}$, where $\mathcal{L}' = \mathcal{L}(p)$ and $p \in G$ with $p \Vdash cf(\check{\beta}) = \omega_1$. Then G is $\mathbb{P}'$-generic and $\Vdash_{\mathbb{P}'} cf(\check{\beta}) = \check{\omega}_1$.)
Thus it suffices to show:

Lemma 3.1 Let $\mathbb{P}$ be as above where $\Vdash cf(\check{\beta}) = \check{\omega}_1$. Let $\mathbb{B} = BA(\mathbb{P})$. Then $\mathbb{B} \setminus \{\emptyset\}$ has a dense subset of size β.

proof.
It suffices to show:
Claim Let $p \in \mathbb{P}$. There is an $\mathcal{L}$-statement $\varphi \in M$ s.t. $\emptyset \neq [\![\varphi]\!] \subset [p]$

It suffices to prove this for a dense set of $p \in \mathbb{P}$, so assume w.l.o.g. that p conforms to $N^* = \langle H_{\beta^{++}}, N, < \rangle$. Let G be $\mathbb{P}$-generic s.t. $p \in G$. Let $\alpha = |p|$. Then $M^p = \langle M_i^G \mid i \leq \alpha \rangle$, $\pi^p = \langle \pi_{ij}^G \mid i \leq j \leq \alpha \rangle$, and $b^p = (\pi_{\alpha,\omega_1}^G)^{-1} {}'' B^G$. Let $\tilde{\beta} = \sup \pi_{\alpha,\omega_1}^G {}'' \beta_\alpha^p$. Then $\tilde{\beta} < \beta$. Set $\tilde{M} = J_{\tilde{\beta}}^A$ (where $M = J_\beta^A$). For $a \in R^p$ set $\tilde{a} = a \cap \tilde{M}$.

X Then $\pi_{\alpha\omega_1}^G : \langle \bar{M}, \bar{a} \rangle \to \langle \tilde{M}, \tilde{a} \rangle$ is cofinal and Σ_0-preserving. But then

$$(1) \quad \tilde{a} = \bigcup_{z \in M_\alpha^p} \pi_{\alpha\omega_1}^G (z \cap \bar{a}).$$

Let $\langle a_i \mid i < \omega \rangle$ enumerate R^p (in V). Then $\langle \bar{a}_i \mid i < \omega \rangle \in H_{\omega_1}$. Moreover $\langle \tilde{a}_i \mid i < \omega \rangle \in M$, since $\tilde{a}_i \in M = J_\beta^A$ for $i < \omega$ and $cf(\beta)^V \geq cf(\beta)^{V[G]} = \omega_1$.

Let Ψ be the statement:

$$\dot{M} | \underline{\alpha+1} = \underline{M^p} \wedge \dot{\pi} \upharpoonright \underline{(\alpha+1)^2} = \underline{\pi}^p \wedge$$

$$\wedge\ \underline{\tilde{\beta}} = \sup \dot{\pi}_{\underline{\alpha}\,\omega_1} {}'' \underline{\beta}^p \wedge$$

$$\wedge \bigwedge_{i<\omega} \underline{\tilde{a}_i} = \bigcup_{z \in \underline{M_\alpha^p}} \dot{\pi}_{\underline{\alpha}\,\omega_1} (z \cap \underline{\bar{a}_i}).$$

Then $\Psi \in M$. But

(2) $N^*[G] \models \psi$. Hence $[\![\psi]\!] \neq \emptyset$.

Hence it suffices to show:

(3) $[\![\psi]\!] \subset [p]$.

Since $[p] = [\![\varphi_p]\!]$, this is equivalent to

(4) $\Vdash_{\mathbb{P}}^{N^*} (\psi \to \varphi_p)$

Let G be $\mathbb{P}$-generic over N^*. We must show:

$N^*[G] \models (\psi \to \varphi_p)$. Let $N^*[G] \models \psi$.

It suffices to show that:

Claim $\pi^G_{\alpha,\omega} : \langle M^p_\alpha, \bar{a} \rangle \prec \langle M, a \rangle$ for $\langle a, \bar{a} \rangle \in F^p$.

Let $b = \{\vec{z} \in M \mid \langle M, a \rangle \models \chi(\vec{z})\}$. Then

$b \in R^p$ by the N^*-conformity of p.

Let $\langle b, \bar{b} \rangle \in F^p$. Then

$\bar{b} = \{\vec{z} \in M^p_\alpha \mid \langle M^p_\alpha, \bar{a} \rangle \models \chi(\vec{z})\}$

by the N^*-conformity of p. Hence:

$\langle M^p_\alpha, \bar{a} \rangle \models \chi(\vec{z}) \leftrightarrow \vec{z} \in \bar{b}$

$\leftrightarrow \pi^G_{\alpha\omega_1}(\vec{z}) \in \tilde{b} = b \cap \tilde{M}$

$\leftrightarrow \langle M, a \rangle \models \chi(\pi^G_{\alpha\omega_1}(\vec{z}))$

since $\tilde{b} = \bigcup_{u \in M^p_\alpha} \pi^G_{\alpha,\omega_1}(u \cap \bar{b})$

QED (Lemma 3.1)

If, on the other hand, $cf(\beta) = \omega$ in $V[G]$, then 2^β is not a cardinal in $V[G]$. This follows from:

<u>Lemma 4.1</u> Let β be a cardinal in an inner model W s.t. $2^{\beta} = \beta$ in W. Let $\gamma = 2^\beta$ in W. Suppose that in V we have:

$$\overline{\overline{\beta}} = \omega_1, \ cf(\beta) = \omega, \ 2^\omega = \omega_1.$$

Then $\overline{\overline{\gamma}} = \omega_1$ in V.

prf.

Let $\pi_{ij} : M_i \prec M_j$ $(i \leq j \leq \omega_1)$ be a commutative continuous sequence of embeddings s.t. for $i < \omega_1$ M_i is countable and transitive and $M_{\omega_1} = H^W_\beta$. (This clearly exists in V by our assumptions.) For each $a \in \mathcal{P}(\beta) \cap W$ there is a least i s.t.

$$\pi_{i,\omega_1} : \langle M_i, \overline{a} \rangle \prec \langle M, a \rangle \text{ for some } \overline{a}.$$

But then $\overline{a} \in H_{\omega_1}$ and $\langle M_i, \overline{a} \rangle$ is amenable.

Thus $a = \bigcup_{x \in M_i} \pi_{i,\omega_1}(x \cap \overline{a}) = F(\langle i, \overline{a} \rangle)$,

where F is a partial map on $\omega_1 \times H_{\omega_1}$.

Hence $\mathcal{P}(\beta) \cap W \subset F''(\omega_1 \times H_{\omega_1})$ and

$$\overline{\overline{\gamma}} = \overline{\overline{\mathcal{P}(\beta) \cap W}} \leq \omega_1. \quad \text{QED (4.1)}$$

<u>Cor 4.2</u> Let $\mathbb{P}$ be a set of conditions. Let G be $\mathbb{P}$-generic and ~~let~~ $2^{\omega} = \omega_1$, $\overline{\overline{\beta}} = \omega_1$, $cf(\beta) = \omega$ in $V[G]$, where ω_1 is absolute in $V[G]$ and $2^{\omega} = \omega_1$, $2^{\beta} = \beta$ in V. Then $\overline{\overline{\mathbb{P}}} \geq 2^{\beta}$. Moreover $cf(2^{\beta}) = \omega_1$ in $V[G]$ if $\overline{\overline{\mathbb{P}}} = 2^{\beta}$.

prf.
$card(2^{\beta}) = \omega_1$ in $V[G]$ by 4.1. But if $\overline{\overline{\mathbb{P}}} < 2^{\beta}$, then 2^{β} would be a cardinal in $V[G]$. Contr! At $\overline{\overline{\mathbb{P}}} = 2^{\beta}$ then $cf(2^{\beta}) \neq \omega$ in $V[G]$, since otherwise $card(2^{2^{\beta}}) = \omega_1$ in $V[G]$ by 4.1.

QED (Cor 4.2)

Def $\mathbb{Q}$ is reshapable iff for some β there is a revisable set of conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ s.t. $BA(\mathbb{Q}) = BA(\mathbb{P})$.

It will turn out that many well known sets of forcing conditions are reshapable. The notion of "revisability" has something of the flavour of "properness", so it is not surprising that we get:

Lemma 5 Assume $2^{\omega} = \omega_1$ and $\beta = 2^{\beta}$, where $\beta > \omega_1$. Let $\mathbb{Q}$ be a proper set of conditions s.t. $\overline{\overline{\mathbb{Q}}} \leq \beta$ and $\Vdash_{\mathbb{Q}} (\overline{\overline{\beta}} = \omega_1 \wedge \mathcal{P}(\check{\omega}) = \mathcal{P}(\check{\omega}))$.

Then $\mathbb{Q}$ is reshapable.

prf.

Assume w.l.o.g. that $\mathbb{Q} \subset H_\beta$. Let $M_{\omega_1} =_{\text{Df}} L_\beta^A = H_\beta$. Let $\Vdash_{\mathbb{Q}} \mathring{f} : \check{\omega}_1 \xrightarrow{\text{onto}} \check{M}_{\omega_1}$. Let G be $\mathbb{Q}$-generic. Set: $f = \mathring{f}^G$ and $C = \{\alpha \leq \omega_1 \mid f''\alpha \prec M\}$. Let $\langle \alpha_i \mid i \leq \omega_1 \rangle$ be the monotone enumeration of C and set: $\pi_{i,\omega_1} : M_i \overset{\sim}{\longleftrightarrow} f''\alpha_i$, where M_i is transitive. Set: $\pi_{ij} = \pi_{j\omega_1}^{-1} \pi_{i\omega_1}$ for $i \leq j \leq \omega_1$.

Set: $N = \langle H_{\beta^+}, M, <, f, \mathbb{Q}, \ldots \rangle$ in V.

Let $\mathcal{L}$ be our standard infinitary language on N (in particular with the constants $\underset{\sim}{x}$ $(x \in N)$, $\dot{M}$, $\dot{\pi}$). Set: $M = \langle M_i \mid i \leq \omega_1 \rangle$, $\pi = \langle \pi_{ij} \mid i \leq j \leq \omega_1 \rangle$.

Let $\delta = \beta^{+V}$. (Hence $\delta = \omega_2^{V[G]}$.) Then $\langle H_\delta^{V[G]}, M, \pi, G \rangle$ models the basic axioms.

Moreover, there are $\mathbb{Q}$-names $\ddot{M}, \ddot{\pi}, \ddot{G}$ which are N-definable s.t. $M = \ddot{M}^G$, $\pi = \ddot{\pi}^G$, $G = \ddot{G}^G$ <u>whenever</u> G is $\mathbb{Q}$-generic over N.

Interpret $\mathcal{L}$ in $\ddot{N} = \langle N^{\mathbb{Q}}, \ddot{M}, \ddot{\pi}, \ddot{G} \rangle$, letting $\underset{\sim}{x}$ be interpreted by $\check{x}$ for $x \in N$ and $\dot{M}, \dot{\pi}, \dot{B}$ by $\ddot{M}, \ddot{\pi}, \ddot{G}$ respectively. This assigns to every $\mathcal{L}$-statement φ a truth value $[\![\varphi]\!] \in BA(\mathbb{Q})$. As usual, we write:

$\Vdash \varphi \underset{\text{df}}{\longleftrightarrow} [\![\varphi]\!] = 1\!\!1$, $p \Vdash \varphi \longleftrightarrow p \in [\![\varphi]\!]$

for $p \in \mathbb{Q}$ and $\mathcal{L}$-statements φ.

If G is $\mathbb{P}$-generic, set:

$\ddot{N}[G] = \langle H_\delta^{V[G]}, \ddot{M}^G, \ddot{\pi}^G, G \rangle$.