§ 3 Adding no reals

All of the forcings we have hitherto discussed add new reals. In the first application this was, of course, necessitated by the problem, since a countable $\bar{M}$ which iterates to M in ω_1 steps must be new. In the last application, however, the problem was simply to collapse ω_2 to ω while giving it cofinality ω. The emergence of new reals in the generic extension resulted only from the fact that our conditions say too little about the extension. In this chapter we define a class of analogous forcings which, however, do not add new reals. This will enable us fr. ins. to give a positive answer to the following question: Assume GCH. Let κ be a measurable cardinal and $\beta > \kappa$ be a cardinal. Let U be a normal measure on κ. Let $M = \langle L^A_\beta, U \rangle$, where $L^A_\beta = H_\beta$. Is there a forcing extension adding no reals in which an $\bar{M}$ iterates in ω_1 many steps to M? ($\bar{M}$ will obviously have to be uncountable.)

In the following ω_1 will play the role of κ in §1. We assume $2^\omega = \omega_1$. Let $\beta > \omega_1$ be a cardinal s.t. $2^\beta = \beta$.

Let $M = L_\beta^A = \langle L_\beta[A], A\rangle$, where $L_\beta[A] = H_\beta$.

Set $N = \langle H_{\beta^+}, M, <, m\rangle$, where $<$ well orders N.

Let $\mathcal{L}$ be any infinitary language on the ZFC^--model N with:

<u>Predicate</u> $\dot\in$

<u>Constants</u> $\underline{x}$ $(x \in N)$, $\dot M$, $\dot\pi$, $\dot B$

<u>Axioms</u>: ZFC^-, $\bigwedge v \, (v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$ (for all $x \in N$) and

- $\dot M = \langle \dot M_i \mid i \leq \omega_1\rangle$, $\dot\pi = \langle \dot\pi_{ij} \mid i \leq j \leq \omega_1\rangle$
- π is a continuous commutative sequence of elementary embeddings $\dot\pi_{ij} : \dot M_i \prec \dot M_j$
- $\dot M_{\omega_1} = \underline{M}$; $\dot M_i$ is countable for $i < \underline{\omega}_1$
- $\dot\pi_{ij} \upharpoonright \dot\alpha_i = id$, $\dot\pi_{ij}(\dot\alpha_i) = \dot\alpha_j$ where $\dot\alpha_i =_{\text{df}} \omega_1^{\dot M_i}$
- $\dot\beta_i < \dot\alpha_{i+1}$ for $i < \underline{\omega}_1$, where $\dot\beta_i =_{\text{df}} On \cap \dot M_i$
- $\dot B \subset \underline{M}$
- $H_{\omega_1} = \underline{H}_{\omega_1}$

We call these the <u>basic</u> <u>axioms</u>. Note that by the last axiom we have:

$\langle \dot{M}_i \mid i \le \underline{\xi} \rangle, \langle \dot{\pi}_{ij} \mid i \le j \le \underline{\xi} \rangle \in \underline{H}_{\omega_1}$ for $\xi < \omega_1$.

Moreover, setting $\dot{M}_i^* =_{\mathrm{Df}} J_{\beta_i}^{\dot{A}_i, \dot{B}_i}$ where

$\dot{B}_i = \dot{\pi}_{i\underline{\omega}_1}^{-1} {}'' \dot{B}$, $\dot{A}_i = \dot{\pi}_{i\underline{\omega}_1}^{-1} {}'' \underline{A}$,

we have $\langle \dot{M}_i^* \mid i \le \underline{\xi} \rangle \in \underline{H}_{\omega_1}$ for $\xi < \omega_1$.

We also set: $M^* = \dot{M}^*_{\underline{\omega}_1} = J_{\underline{\beta}}^{\underline{A}, B'}$.

<u>Note</u> The basic axioms do <u>not</u> say that $\dot{\pi}_{ij}$ takes $\dot{M}_i^*$ to M_j^*, though $\dot{\pi}_{ij}$ will, of course, be a structurally preserving embedding of $\langle \dot{M}_i, \dot{B}_i \rangle$ to $\langle \dot{M}_j, \dot{B}_j \rangle$

We now associate to each $\mathcal{L}$ satisfying the basic axioms a set of conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$.

The conditions in § 1 gave - essentially - finite information about $\dot{M}, \dot{\pi}$. The new conditions give countable information.

We first define a set $\tilde{\mathbb{P}}$ of preconditions:

Def $\tilde{\mathbb{P}}$ = the set of pairs $\langle p_0, p_1 \rangle$ s.t. for some γ:

(a) $p_0 = \langle M^p, \pi^p, b^p \rangle$, $p_0 \in H_{\omega_1}$

- $M^p = \langle M_i^p \mid i \leq \gamma \rangle$, $\pi^p = \langle \pi_{ij}^p \mid i \leq j \leq \gamma \rangle$
- $M_i^p = J_{\alpha_i}^{A_i}$ where $\alpha_i < \omega_1$
- $b^p \subset M_\gamma^p$
- π^p is a continuous, commutative system of elementary embeddings $\pi_{ij}^p : M_i^p \prec M_j^p$ $(i \leq j)$

(b) p_1 is a countable set of pairs $\langle a, \bar{a} \rangle$ s.t. $a \subset M$, $\bar{a} \subset M_\gamma^p$.

We set: $|p| =_{\mathrm{df}} \gamma$, where γ is as above, for $p \in \tilde{\mathbb{P}}$.

Def Let $p \in \tilde{\mathbb{P}}$. φ_p is the following $\mathcal{L}$-fmla:

$$\underline{\dot{\pi}^p} = \langle \dot{\pi}_{ij} \mid i \leq j \leq \underline{|p|} \rangle \wedge \underline{M^p} = \langle \dot{M}_i \mid i \leq \underline{|p|} \rangle \wedge$$

$$\wedge \bigwedge_{\langle a, \bar{a} \rangle \in p_1} \dot{\pi}_{\underline{|p|}, \underline{\omega_1}} : \underline{\langle M_{|p|}^p, \bar{a} \rangle} \prec \underline{\langle M, a \rangle} \wedge$$

$$\wedge\ \underline{b^p} = \dot{\pi}^{-1}_{\underline{|p|}, \underline{\omega_1}} {}'' \dot{B}.$$

Set $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. We define $\mathbb{P} = \mathbb{P}_{\mathcal{L}} = \langle \mathbb{P}, \leq \rangle$ by:

Def $\mathbb{P} = \{p \in \tilde{\mathbb{P}} \mid \mathcal{L}(p)$ is consistent$\}$

$$p \leq q \leftrightarrow (M^q = M^p \restriction |q|+1 \wedge \pi^q = \pi^p \restriction (|q|+1)^2 \wedge$$
$$\wedge \bigwedge \langle \bar{a}, a \rangle \in q_1 \; \bigvee a' (\langle a, a' \rangle \in p_1 \wedge$$
$$\wedge \pi^p_{|q|,|p|} : \langle M^q_{|q|}, \bar{a} \rangle \prec \langle M^p_{|p|}, a' \rangle \wedge$$
$$\wedge b^q = \pi^p_{|q|,|p|}{}^{-1} \text{''} b^p)$$

It is easily seen that $\leq$ is a partial ordering of $\mathbb{P}$.

<u>Lemma 0.1</u> p_1^{-1} is a function

proof. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$ (in some forcing collapse of β^+ to ω). Then $\langle a, \bar{a} \rangle \in p_1 \rightarrow \bar{a} = \dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}{}^{-1} \text{''} a$.

Def Let $p \in \mathbb{P}$. We set:

$F^p = p_1$, $R^p = \text{rng}(p_1)$, $D^p = \text{dom}(p_1)$.

For $a \in R^p$ set: $\bar{a}^p = p_1^{-1}(a)$.

For $i \leq |p|$ set:

$$F^p_i = \{\langle a, \bar{a} \rangle \mid a \in R^p \wedge \bar{a} = \pi^p_{i,|p|}{}^{-1} \text{''} \bar{a}^p \wedge \pi^p_{i,|p|} : \langle M^p_i, \bar{a} \rangle \prec \langle M^p_{|p|}, \bar{a}^p \rangle\}$$

(Hence $F^p = F^p_{|p|}$)

$R^p_i = \text{rng}(F^p_i)$, $D^p_i = \text{dom}(F^p_i)$

<u>Lemma 0.2</u> Let R^p be closed under set difference. Then F^p is a bijection of D^p onto R^p.

proof.

Let $\langle a, \bar{a}\rangle, \langle b, \bar{b}\rangle \in F^p$. It suffices to show:

<u>Claim</u> $\bar{a} \subset \bar{b} \to a \subset b$.

Set $\bar{c} = \bar{b} \setminus \bar{a}$, $c = b \setminus a$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Let $\pi = \pi^{\mathfrak{A}}_{|p|, \omega_1}$. Then

$$F^{-1}(c) = \pi^{-1}``(b \setminus a) = \pi^{-1}``b \setminus \pi^{-1}``c = \bar{b} \setminus \bar{a} = \bar{c}.$$

Hence $\bar{b} \subset \bar{a} \to \bar{c} = \emptyset \to c = \emptyset \to b \subset a$,

since $\pi : \langle M^p_{|p|}, \bar{c}\rangle \prec \langle M, c\rangle$. QED (0.2)

<u>Note</u> p_1 plays largely the role of p_2 in §2. The component $p_1 = \langle \pi^p_i \mid i \in D(p)\rangle$ of §2 is superfluous here, since the information it contains can be subsumed in our $p_1 = F^p$:

<u>Def</u> $\pi^p_i = \pi^p_{i, \omega_1} =_{Df} F^p \circ \pi^p_{i, |p|}$ for $i \leq |p|$

<u>Lemma 0.3</u> π^p_i is a partial injection of M^p_i into M

proof.

It suffices to prove this for $i = |p|$. Let $\langle x, \bar{x}\rangle \in F^p$ where $\bar{x} \in M^p_{|p|}$. ~~Let~~ $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$.

Then $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1} : \langle M^p_{|p|}, \bar{x} \rangle \prec \langle M, x \rangle$. Hence $x \in M$ and $x = \dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1}(\bar{x})$. QED (0.3)

Note It follows that if $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then $\pi^p_i \subset \dot{\pi}^{\mathfrak{A}}_{i,\omega_1}$ for all $i \leq |p|$.

Note It is easily seen that

$$p \leq q \rightarrow F^q_i \subset F^p_i \quad \text{for } i \leq |q|$$
$$\rightarrow \pi^q_i \subset \pi^p_i \quad \text{''} \quad .$$

We define:

Def Let $p, q \in \mathbb{P}$.

$p \| q \longleftrightarrow_{Df}$ p, q are compatible in $\mathbb{P}$ (i.e. $\bigvee r \; r \leq p, q$)

$p \perp q \longleftrightarrow_{Df} \neg (p \| q)$.

Lemma 1.1 $p \| q \longleftrightarrow \mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent.

prf.

($\rightarrow$) Let $r \leq p, q$. Then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$

($\leftarrow$) Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p) \cup \mathcal{L}(q)$. Since ω_1 is regular in $\mathfrak{A}$ and $R^p \cup R^q$ is countable in $\mathfrak{A}$, there is an $\alpha < \omega_1$ s.t. $\alpha \geq |p|, |q|$ and for all $a \in R^p \cup R^q$ we have: $\dot{\pi}^{\mathfrak{A}}_{\alpha,\omega_1} : \langle M^{\mathfrak{A}}_{\alpha}, a^* \rangle \prec \langle M, a \rangle$, where $a^* = \dot{\pi}^{\mathfrak{A}\,-1}_{\alpha,\omega_1} {}'' a$.

Define $r \in \widetilde{\mathbb{P}}$ by:

$M^r = \langle M_i^{\mathfrak{A}} \mid i \le \alpha \rangle$, $\pi^r = \langle \pi_{ij}^{\mathfrak{A}} \mid i \le j \le \alpha \rangle$

$b^r = \dot{\pi}^{\mathfrak{A}\,-1}_{\alpha,\omega_1} {}'' \dot{B}^{\mathfrak{A}}$, $F^r = \{\langle a, a^* \rangle \mid a \in R^p \cup R^q\}$.

Then $r \in \mathbb{P}$, since $\mathfrak{A} \models \mathcal{L}(r)$. It follows easily that $r \le p, q$. QED (1.1)

We also obtain the following lemmas on extending conditions:

<u>Lemma 1.2</u> Let $p \in \mathbb{P}$. Let $u \subset H(M)$ be at most countable. There is $q \le p$ s.t. $u \subset R^q$.

prf. Like 1.1

Hence:

<u>Cor 1.3</u> Let $p \in \mathbb{P}$, $u \subset M$, $\bar{\bar{u}} \le \omega$. There is $q \le p$ s.t. $u \subset \mathrm{rng}(\pi^q_{|q|})$

<u>Lemma 1.4</u> Let $p \in \mathbb{P}$ + let $u \subset M_i^p$ be finite. There is $q \le p$ s.t. $u \subset \mathrm{dom}(\pi_i^q)$.

prf.

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$.

Define q by: $M^q = M^p$, $\pi^q = \pi^p$,

$R^q = R^p \cup R$, where

$R = \{\langle \dot{\pi}^{\mathfrak{A}}_{|p|\,\omega_1}(x), \pi^p_{i,|p|}(x) \rangle \mid x \in u\}$.

Then $\mathfrak{A} \models \mathcal{L}(q)$. Hence $q \le p$.

QED (1.4)

Lemma 1.5 Let $p \in \mathbb{P}$, $|p| \leq \alpha < \omega_1$. There is $q \leq p$ s.t. $|q| \geq \alpha$.

prf. Like 1.1

Using the extension lemma we easily get:

Lemma 2 Let G be $\mathbb{P}$-generic over V. Define $M^G = \langle M_i^G \mid i \leq \omega_1 \rangle$, $\pi^G = \langle \pi_{ij}^G \mid i \leq j \leq \omega_1 \rangle$, $B^G \subset M$ by:

$M^G \upharpoonright \omega_1 = \bigcup_{p \in G} M^p$, $\pi^G \upharpoonright \omega_1^2 = \bigcup_{p \in G} \pi^p$,

$M_{\omega_1}^G = M$, $\pi_{i,\omega_1}^G = \bigcup_{p \in G} \pi_i^p$,

$B^G = \bigcup_{p \in G} \pi_i^p{}''b^p$.

Then the following hold:

(A) M^G, π^G is a continuous commutative sequence of elementary embeddings.

(B) $\omega_1 \in \operatorname{rng}(\pi_{i,\omega_1}^G)$. Moreover, $\alpha_i^G = \operatorname{crit}(\pi_{i,\omega_1}^G)$, where $\alpha_i^G =_{\text{df}} \pi_{i,\omega_1}^{G\,-1}(\omega_1)$.

(C) $\beta_i^G < \alpha_{i+1}^G$, where $\beta_i^G = On_{M_i^G}$

(D) $b_i^p = \pi_{i\omega_1}^{G\,-1}{}''B^G$ for all $p \in G$, $i \leq |p|$.

(E) If $p \in G$, $\alpha = \omega_1^V$, and $\langle a, \bar{a} \rangle \in F^p$, then $\pi_{|p|,\alpha} : \langle M_{|p|}^p, \bar{a} \rangle \prec \langle M, a \rangle$.

We now formulate a condition called <u>revisability</u> which will guarantee that $\mathbb{P}$ adds no new reals.

We first define:

<u>Def</u> Let $N^* = \langle H_\delta, M, <, \dots \rangle$ be a model of countable or finite type, where $\delta \geq \beta^+$ is a cardinal and $<$ well orders H_δ. Let $p \in \mathbb{P}$. p <u>conforms</u> to N^* iff whenever $a_1, \dots, a_n \in \mathbb{R}^p$ and $b \subset M$ is N^*-definable in $a_1, \dots, a_n$ $(n \geq 0)$, then $b \in \mathbb{R}^p$.

<u>Note</u> $\{p \mid p \text{ conforms to } N^*\}$ is dense in $\mathbb{P}$ by the extension lemmas.

<u>Note</u> If p conforms to N^*, then $\mathbb{R}^p \neq \emptyset$ and $F^p : D^p \longleftrightarrow F^p$ by Lemma 0.2.

Before defining revisability we must prove a theorem:

Lemma 3 Let p conform to N^*. There is a unique $\bar{N}^* = \bar{N}^*(p, N^*)$ s.t.

(i) $\bar{N}^*$ is transitive and of the same type as N^*

(ii) If $a_1, \dots, a_n \in \mathbb{R}^p$ ($n \geq 0$) and $b \subset M$ is N^*-definable in $a_1, \dots, a_n$, then $\bar{a}_1^p, \dots, \bar{a}_n^p \in \bar{N}^*$ and $\bar{b}^p$ is $\bar{N}^*$-definable in $\bar{a}_1^p, \dots, \bar{a}_n^p$ by the same definition.

(iii) Each $x \in \bar{N}^*$ is $\bar{N}^*$-definable from parameters in $M^p_{|p|} \cup D^p$

Moreover, If $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then $\dot{\pi}^{\mathfrak{A}}_{|p|,\omega_1} \cup F^p$ extends uniquely to a $\pi : \bar{N}^* \prec N^*$ s.t. $\pi(\bar{a}) = a$ whenever $\langle a, \bar{a} \rangle \in F^p$.

(Note that $M^p_{|p|} = \bar{M}^p \in \bar{N}^*$ by (b))

proof.
We first show the existence of $\bar{N}^*$ satisfying (i)–(iii). Exactly as in the proof of §2 Lemma 3 we get:

Fact For any $X \subset M$ the following are equivalent:

(a) $X \prec \langle M, a \rangle$ for all $a \in \mathbb{R}^p$

(b) Let Y = the smallest $Y \prec N^*$ s.t. $X \cup \mathbb{R}^p \subset Y$. Then $Y \cap M = X$

((b) → (a) is trivial. (a) → (b) follows from the fact that each $z \in Y$ is N^*-definable from parameters in $X \cup \mathbb{R}^p$.)

Now collapse β^{++} to ω and work in the resulting model $V[G]$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Then $\mathbb{R}^p \in N \subset \mathfrak{A}$, since $\mathbb{R}^p \subset N$ is countable. But $\dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1} : \langle M^p_{|p|}, \bar{a}^p \rangle \prec \langle M, a \rangle$ for all $a \in \mathbb{R}^p$, or in other words, $X \prec \langle M, a \rangle$ for all $a \in \mathbb{R}^p$, where $X = \operatorname{rng}(\dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1})$. Let Y be the smallest $Y \prec N^*$ s.t. $X \cup \mathbb{R}^p \subset Y$. Let $\pi : \bar{N}^* \stackrel{\sim}{\longleftrightarrow} N^* | Y$ be the transitivisation of $N^* | Y$.

<u>Claim 1</u> $\bar{N}^*$ satisfies (i) – (vii)

proof.
Since $X = Y \cap M$, we clearly have $\pi \upharpoonright M^p_{|p|} = \dot{\pi}^{\mathfrak{A}}_{|p|, \omega_1}$ and

$\pi^{-1}(a) = \pi^{-1''}(X \cap a) = \pi^{\mathcal{M}-1''}_{|p|,\omega_1}(X \cap a) =$

$= \bar{a}^{p}$ for $a \in R^{p}$.

(ii), (iii) follow easily. QED(Claim 1)

But $\bar{N}^{*}$ was constructed in $V[G]$, so we must show:

<u>Claim 2</u> $\bar{N}^{*} \in V$

proof:

Let $\tilde{Y}$ = the smallest $\tilde{Y} \prec N^{*}$ s.t. $M \cup R^{p} \subset \tilde{Y}$. Then $\tilde{Y}$ has cardinality β in V. Hence, letting $\tilde{\pi} : \tilde{N}^{*} \overset{\sim}{\longleftrightarrow} N^{*}|\tilde{Y}$ be the transitivisation of $N^{*}|\tilde{Y}$, we have: $\tilde{N}^{*} \in N \subset \mathcal{M}$. Note that $\tilde{\pi} \upharpoonright M \cup R^{p} = \mathrm{id}$. Hence, setting $\hat{Y}$ = the smallest $\hat{Y} \prec \tilde{N}^{*}$ s.t. $X \cup R^{p} \subset \hat{Y}$, we have $\hat{Y} \in \mathcal{M}$. Hence $\hat{\pi}, \hat{N}^{*} \in \mathcal{M}$, where $\hat{\pi} : \hat{N}^{*} \overset{\sim}{\longleftrightarrow} \tilde{N}^{*}|\hat{Y}$ is the transitivisation of $\tilde{N}^{*}|\hat{Y}$. But obviously $\hat{N}^{*} = \bar{N}^{*}$; hence $\bar{N}^{*} \in H^{\mathcal{M}}_{\omega_1} = H^{V}_{\omega_1}$. QED(Claim 2)

Note We have shown that if $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then there is an $\bar{N}^* \in V$ satisfying (i) – (iii) with the property that $\pi^{i\,\mathfrak{A}}_{|p|,\omega_1}$ extends uniquely to a $\pi : \bar{N}^* \prec N^*$.
The last part of the theorem is, therefore, established as soon as we show:

Claim 3 There is at most one $\bar{N}^*$ satisfying (i) – (iii)

prf.
Let $\bar{N}_0^*, \bar{N}_1^*$ satisfy (a) – (c). Then:
Let D^p.
(1) Let $x_1, \ldots, x_n \in M^p_{|p|}$, $b_1, \ldots, b_m \in D^p$.
Then $\bar{N}_0^* \models \varphi(\vec{x}, \vec{b}) \longleftrightarrow \bar{N}_1^* \models \varphi(\vec{x}, \vec{b})$
for all N^*-fmlae φ.

proof.
Let $b_i = \bar{a}_i^p$, where $a_i \in B^p$ $(i = 1, \ldots, m)$
Set: $c = \{\langle \vec{x} \rangle \in M \mid N^* \models \varphi(\vec{x}, \vec{a})\}$.
Then $\bar{c}^p = \{\langle \vec{x} \rangle \in M^p_{|p|} \mid \bar{N}_h^* \models \varphi(\vec{x}, \vec{b})\}$
for $h = 0, 1$. QED (1)

But it then follows easily that $\mathrm{id} \restriction (M^p_{|p|} \cup D^p)$ extends to a $\sigma : \bar{N}_0^* \xleftrightarrow{\sim} \bar{N}_1^*$. Since $\bar{N}_0^*, \bar{N}_1^*$ are transitive, we conclude: $\bar{N}_0^* = \bar{N}_1^*$.

QED (Lemma 3)

As a corollary of the proof we get:

<u>Cor 3.1</u> Let G be $\mathbb{P}$-generic over V. Let $p \in G$ conform to N^*. Then $\pi^G_{|p|,\omega_1} \cup F^p$ extends uniquely to a $\pi : \bar{N}^* \prec N^*$.

proof.

By a proof similar to (1) in Claim 3:

(1) Let $x_1, \ldots, x_n \in M^p_{|p|}$, $b_1, \ldots, b_n \in D^p$. Then $\bar{N}^* \models \varphi(\vec{x}, \vec{b}) \longleftrightarrow N^* \models \varphi(\pi^G_{|p|,\omega_1}(\vec{x}), F^p(\vec{b}))$.

The result follows easily, since each element of $\bar{N}^*$ is definable in parameters from $M^p_{|p|} \cup D^p$. QED (3.1)

<u>Note</u> If we assume that $\Vdash \mathcal{P}(\check{\omega}) \subset \check{V}$, we obtain the following fact, whose proof we shall bring later.

Set $\mathcal{L}^*$ = the set of φ which are forced by $\mathbb{P}$ to hold in $N[\mathring{G}]$, where $\dot{M}, \dot{\pi}, \dot{B}$ are interpreted by M^G, π^G, B^G. Then

(a) $\mathbb{P} = \mathbb{P}_{\mathcal{L}^*}$

(b) $p \Vdash^N_{\mathbb{P}} \varphi \longleftrightarrow \mathcal{L}^*(p) \vdash \varphi$

If $G \ni p$ is $\mathbb{P}$-generic over V, then $\langle N[G], M^G, \pi^G, B^G \rangle$ is a solid model of $\mathcal{L}^*(p)$. Thus Cor 3.1 becomes a special case of Lemma 3.

The condition of <u>revisability</u> then says that certain conditions p can be revised, leaving $M^p_{|p|}$ and F^p fixed, but changing $M^p \upharpoonright |p|$, $\pi^p \upharpoonright |p|^2$ and b^p.

<u>Def</u> $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is <u>revisable</u> iff for all sufficiently large cardinals $\delta > 2^{\beta}$ we have:

(*) Let $p \leq r$ in $\mathbb{P}$, where $\mathbb{P}$ conforms to $N^* = \langle H_\delta, M, <, \mathbb{P}, r, m \rangle$. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{M}, \bar{\mathbb{P}}, \bar{r}, \bar{m} \rangle$ be as Lemma 3. (Hence $\bar{r} \in \bar{\mathbb{P}}$.) Then there is $G \ni \bar{r}$ s.t. G is $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ and, letting M^G, π^G, B^G be defined over $\bar{N}^*[G]$ as in Lemma 2, we have: $q \in \mathbb{P}$, where $q = \langle\langle M^G, \pi^G, M^G \rangle, F^p \rangle$.

<u>Note</u> As defined here, revisability is a property of $\mathbb{P}$ rather than $\mathcal{L}$.

<u>Note</u> It follows easily that $q \leq r$.

<u>Note</u> In some applications we shall have a stronger form of revisability: For <u>every</u> $\bar{r} \in \bar{\mathbb{P}}$ there is $G \ni \bar{r}$ satisfying the above condition.

<u>Def</u> We say that μ <u>witnesses</u> the revisability of $\mathbb{P}$ iff (*) holds for all $\delta \geq \mu$.

We also define:

Def $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is <u>weakly</u> <u>revisable</u> iff there exists d s.t. for all sufficiently large cardinals $\delta > 2^{\beta}$ we have:
(**) Let $p \leq r$ in $\mathbb{P}$, where $\mathbb{P}$ conforms to $N^* = \langle H_\delta, M, <, d, \mathbb{P}, r, m \rangle$. Let $\bar{N}^* = \bar{N}^*(p, N^*) = \langle \bar{H}, \bar{M}, <, \bar{d}, \bar{\mathbb{P}}, \bar{r}, m \rangle$. Then there is $G \ni \bar{r}$ s.t. G is $\bar{\mathbb{P}}$-generic over $\bar{N}^*$ and: $q \in \mathbb{P}$, where $q = \langle\langle M^G, \pi^G, M^G \rangle, FP \rangle$.

Def $\langle d, \mu \rangle$ <u>witnesses</u> the weak revisability of $\mathbb{P}$ iff (**) holds for all $\delta \geq \mu$.

<u>Lemma 3.2</u> $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is revisable iff it is weakly revisable.

prf.
$(\rightarrow)$ is trivial. We prove $(\leftarrow)$.
Say that $\mathbb{P}$ is $\langle d, \delta \rangle$-<u>revisable</u> iff (**) holds.
For $\eta \leq \infty$ set:
$\mu(\eta) \simeq$ the least μ s.t. there is a $d \in H_\mu$ with: $\mathbb{P}$ is $\langle d, \delta \rangle$ revisable for all cardinals $\delta \in [\mu, \eta)$.
$X_\eta \simeq$ the set of such $d \in H_{\mu(\eta)}$
Then $\mu(\eta) < \eta$ and X_η are defined for sufficiently large cardinals η.
Moreover, if $\eta \leq \eta'$ and $\mu(\eta)$ is defined, then:
$\mu(\eta) \leq \mu(\eta') \leq \mu(\infty) \wedge X_\eta \supset X_{\eta'} \supset X_\infty$