§3 Variations

We can, of course, formulate the definition of subcomplete directly for a set of conditions $\mathbb{P}$ rather than for a complete BA $\mathbb{B}$. This condition appears, however to be stronger than saying that $\mathbb{B} = BA(\mathbb{P})$ is subcomplete, since our necessary condition for selecting the sequence $\lambda_1, \ldots, \lambda_n$ is now $\overline{\overline{\mathbb{P}}} < \lambda_i$, which in general is weaker than $\overline{\overline{\mathbb{B}}} < \lambda_i$. This suggests the following modification of the notion of subcompleteness:

Def Let $\mathbb{B}$ be a complete Boolean algebra. $d(\mathbb{B})$ = the smallest cardinality of a dense subset of $\mathbb{B}$.

Def $\mathbb{B}$ is very subcomplete iff $\mathbb{B}$ satisfies the definition of subcompleteness with '$\overline{\overline{\mathbb{B}}} < \lambda_i$' weakened to '$d(\mathbb{B}) < \lambda_i$'.

Note We have thus far found no example of a $\mathbb{B}$ which is subcomplete without being very subcomplete.

With a very slight modification of the previous proof we get:

Thm 2 Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS-iteration. Set: $d_i = d(\mathbb{B}_i)$. Assume that for all $i+1 < \alpha$:

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1}/\dot{G}$ is very subcomplete$)$

(c) $\Vdash_{i+1} (\check{d}_i$ has cardinality $\leq \omega_1)$

Then every $\mathbb{B}_i$ is very subcomplete.

proof (sketch)

We need only a very slight modification of the previous proof. We first note:

(1) $d_i \leq d_j$ for $i \leq j < \alpha$,

since if X is dense in $\mathbb{B}_j$, then $\{h_i(a) \mid a \in X\}$ is dense in $\mathbb{B}_i$.

(2) $\bar{\bar{\nu}} \leq d_\nu$ for $\nu < \alpha$.

Suppose not. Let ν be the least counterexample. Then $\nu > 0$ is a cardinal. If $\nu < \omega$, then $d_\nu < \nu < \omega$ and hence $\mathbb{B}_\nu$ is atomic and d_ν is the number of atoms. But then $\nu = \eta + 1$ for some η and $d_\eta < d_\nu$ by (a). Thus $\nu \geq \omega$ is a cardinal. If ν is a limit cardinal, then $d_\nu \geq \sup_{i<\nu} d_\nu \geq \nu$. Contradiction! Thus ν is a successor cardinal. Let $X \subset \mathbb{B}_\nu$ be dense in $\mathbb{B}_\nu$ with $\bar{\bar{X}} = d_\nu < \nu$. Then $X \subset \mathbb{B}_\eta$ for an $\eta < \nu$. Hence X is dense in $\mathbb{B}_i = \mathbb{B}_\nu$ for $\eta \leq i \leq \nu$, contradicting (a).

QED (2)

We again prove by induction on i:

<u>Claim</u> Let $h \leq i$. Let G be $\mathbb{B}_h$ – generic. Then $\mathbb{B}_i / G$ is very subcomplete in $V[G]$.

The cases $h = i$ and $i = j+1$ are as before (since the two step iteration theorem easily carries over to very subcomplete algebras).

The case: $i = \lambda$ with λ a limit ordinal again splits into two cases:

<u>Case 1</u> $\mathrm{cf}(\lambda) \leq d_i$ for an $i < \lambda$.

The proof is exactly as before with slight notational changes (replacing $\bar{\bar{\mathbb{B}}}_i$ by d_i).

<u>Case 2</u> Case 1 fails.

Then λ is regular and $\lambda > d_i$ for $i < \lambda$. This again enables us to repeat the previous proof. QED (Thm 2)

Another variation on subcompleteness was mentioned in [SPSC].

Def $\mathbb{B}$ is μ-subcomplete iff for sufficiently large cardinals θ we have:

Let $\mu, \mathbb{B} \in H_\theta$. Let $\tau > \theta$ be regular s.t. $H_\theta \subset W = L^A_\tau$. Let $\sigma : \overline{W} \prec W$ where $\overline{W}$ is countable, transitive, and full. Let $\sigma(\bar{\theta}, \bar{\mu}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_1, \ldots, \bar{\lambda}_n) = \theta, \mu, \mathbb{B}, s, \lambda_1, \ldots, \lambda_n$, where λ_i is regular s.t. $\overline{\overline{\mathbb{B}}} < \lambda_i$ $(i = 1, \ldots, n)$. Let $\bar{G}$ be $\bar{\mathbb{B}}$-generic over $\overline{W}$. There is $a \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni a$ is $\mathbb{B}$-generic, then there is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0 : \overline{W} \prec W$ and $\sigma_0 \upharpoonright \mu = \sigma \upharpoonright \mu$

(b) $\sigma_0(\bar{\theta}, \bar{\mu}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mu, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \sigma_0 '' \bar{\lambda}_i = \sup \sigma '' \bar{\lambda}_i$ $(i = 0, \ldots, n)$
where $\lambda_i = On \cap \overline{W}$.

(d) $\sigma_0 '' \bar{G} \subset G$.

<u>Note</u> Every subcomplete algebra is ω_1-subcomplete.

<u>Note</u> By (a) we have $\sigma_0 \upharpoonright V_{\bar{\mu}}^{\overline{W}} = \sigma \upharpoonright V_{\bar{\mu}}^{\overline{W}}$.

Weak μ-subcompleteness again implies μ-subcompleteness. A repetition of the proof of the two step thm gives:

If $\mathbb{B}$ is μ subcomplete and $\Vdash_{\mathbb{B}} (\mathring{\mathbb{C}}$ is $\check{\mu}'$-subcomplete$)$ and $\mu \le \mu'$, then $\mathbb{B}$ is μ-subcomplete.

We then get:

Thm 3 Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS – iteration, let $\langle \mu_i \mid i+1 < \alpha \rangle$ be weakly monotone s.t. for all $i+1 < \alpha$:

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i$ ($\check{\mathbb{B}}_{i+1}/\dot{G}$ is $\check{\mu}_i$ – subcomplete)

(c) $\Vdash_{i+1}$ ($\check{\mathbb{B}}_i$ has cardinality $\leq \check{\mu}_i$)

Then every $\mathbb{B}_i$ is μ_0 – subcomplete.

proof (sketch)

Again only a slight modification is needed. By induction on i we prove:

Claim Let $h \leq i$. Let G be $\mathbb{B}_h$ – generic. Then $\mathbb{B}_i/G$ is μ_h – subcomplete in $V[G]$.

The cases $h = i$ and $i = j+1$ are again easy. If $i = \lambda$ is a limit ordinal we again have the same two cases:

Case 1 $cf(\lambda) \leq \overline{\overline{\mathbb{B}}}_i$ for an $i < \lambda$.

Case 2 Case 1 fails.

In Case 1 it again suffices to prove the claim for sufficiently large $h < \lambda$, so we may assume $cf(\lambda) \leq \overline{\overline{\mathbb{B}}}_i$ for an $i < h$. But then $cf(\lambda) \leq \mu_h$

in $V[G]$. Hence we carry out our proof for the case; $h=0$, $cf(\lambda) \leq \mu_0$, since the same proof will work in $V[G]$ for $\langle \mathbb{B}_{h+j}/G \mid j<\alpha-h \rangle$, $\langle \mu_{h+j} \mid j+1<\alpha-h \rangle$.

There is then an $f: \mu_0 \to \lambda$ s.t. $\sup f''\mu_0 = \lambda$ and $\sigma(\bar{f}) = f$. As before, pick $\langle \nu_i \mid i<\omega \rangle$ in $\bar{\mu}_0$ s.t. the function $\bar{\xi}_i = \bar{f}(\nu_i)$ is monotone and cofinal in $\bar{\lambda}$, where $\sigma(\bar{\lambda}) = \lambda$. Set $\xi_i = \sigma(\bar{\xi}_i) = f(\sigma(\nu_i))$. Then $\langle \xi_i \mid i<\omega \rangle$ is monotone and cofinal in $\tilde{\lambda} = \sup \sigma''\lambda$ and $\sigma'(\bar{\xi}_i) = \xi_i$ $(i<\omega)$ whenever $\sigma': \bar{W} \prec W$ s.t. $\sigma'(\bar{f}) = f$ and $\sigma' \upharpoonright \bar{\mu}_0 = \sigma \upharpoonright \bar{\mu}_0$.

We then closely imitate our previous proof, constructing a thread $\langle c_i \mid i<\omega \rangle$ in $\langle \mathbb{B}_{\xi_i} \mid i<\omega \rangle$ s.t. c_i forces the existence of $\sigma_i: \bar{W} \prec W$ with certain properties. It follows that if letting $c = \bigcap_i c_i$, then the $G \ni c$ is $\mathbb{B}_\lambda$-generic, then the desired map $\sigma': \bar{W} \prec W$ can be defined from $\langle \sigma_i \mid i<\omega \rangle$ as before. We must, however, ensure that $\sigma_i(\bar{f}) = f$ and $\sigma_i \upharpoonright \mu_0 = \sigma \upharpoonright \mu_0$

for $i<\omega$. Given our assumptions this is straightforward. Virtually only minor notational changes are needed.

In Case 2 we proceed exactly as before, again ensuring that $\sigma_i \upharpoonright \mu_0 = \sigma \upharpoonright \mu_0$ for $i<\omega$.

QED (Thm 3)

[Note If we can arrange that $\mu_i > \overline{\overline{\mathbb{B}}}_i$ for $i+1<\alpha$, and $\mathbb{B}_{i+1}$ never collapses new cardinals, then ~~by~~ Thm 3 all $\mathbb{B}_i$ are subcomplete even though we did not collapse cardinals at successor stages. In [SPSC] we showed that the forcing for adding a Prikry sequence at a measurable cardinal κ is μ-subcomplete for every $\mu<\kappa$. Thus if $\langle \kappa_i \mid i<\alpha\rangle$ is a discrete sequence of measurables (i.e. $\sup_{h<i} \kappa_h < \kappa_i$ for all $i<\alpha$), we can use Thm 3 to successively add a Prikry sequence for an arbitrarily chosen normal measure U_i on κ_i without collapsing cardinals at successor stages. The value of this is questionable, since Thm 3 does not, in itself, prevent cardinals from being collapsed at limit stages. In fact, the application of Thm 3 to this situation seems rather pointless, since Magidor has shown that, if $\langle U_i \mid i<\alpha \rangle \in V$ is s.t. U_i is normal on κ_i for $i<\alpha$, then an Easton-like iteration will add a Prikry sequence for each U_i without adding reals or collapsing cardinals. There is an account of this in [F].]

. . . . .

In §1 ("Fact") we gave a necessary characterization of the RCS-iterations. If we omit the clause (a)(iii) we obtain a wider class of iterations which we can call <u>quasi-RCS-iterations</u>. In the paper [EN] we used a quasi-RCS-iteration to prove the main theorem. In [EN] §1 we proved an iteration theorem for certain quasi-RCS-iterations which we now reprove in a slightly generalized form. We first define:

<u>Def</u> An iteration $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ is <u>nicely</u> <u>subcomplete</u> iff the following hold:

(a) For all $i+1 < \alpha$:

(i) $\Vdash_i \check{\mathbb{B}}_{i+1}/\dot{G}$ is subcomplete

(ii) $\Vdash_{i+1} \check{\mathbb{B}}_i$ has cardinality $\leq \omega_1$

(b) If $\lambda < \alpha$ and $\langle \xi_n \mid n < \omega \rangle$ is monotone and cofinal in λ, then:

(i) $\bigcap_n b_n \neq 0$ in $\mathbb{B}_\lambda$ whenever $b = \langle b_n \mid n < \omega \rangle$ is a Thread in $\langle \mathbb{B}_{\xi_n} \mid n < \omega \rangle$

(ii) $\mathbb{B}_\lambda$ is subcomplete if $\mathbb{B}_i$ is subcomplete for $i < \lambda$.

(c) If $\lambda < \alpha$ and $\Vdash_i \mathrm{cf}(\check{\lambda}) > \omega$ for all $i < \lambda$, then $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$.

(d) If $i < \alpha$ and G is $\mathbb{B}_i$-generic, then (a)–(c) hold for $\langle \mathbb{B}_{i+j}/G \mid j < \alpha - i \rangle$ in $V[G]$.

We prove:

<u>Thm 4</u> Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be nicely sub-complete. Then every $\mathbb{B}_i$ is subcomplete.

proof. (sketch)
By induction on i we prove:

<u>Claim</u> Let $h \leq i$. Let G be $\mathbb{B}_h$-generic. Then $\mathbb{B}_i/G$ is subcomplete in $V[G]$.

The cases $h = i$ and $i = j+1$ are again trivial, so assume that $i = \lambda$ is a limit ordinal. We again have two cases:

<u>Case 1</u> $cf(\lambda) < \overline{\overline{\mathbb{B}}}_h$ for an $h < \lambda$.

<u>Case 2</u> Case 1 fails.

In Case 1 it again suffices to prove the Claim for sufficiently large $h < \lambda$, so we can assume that $cf(\lambda) \leq \omega_1$ in $V[G]$ whenever G is $\mathbb{B}_h$-generic. But then we can assume $cf(\lambda) \leq \omega_1$ in V since the same proof can be carried out for $\langle \mathbb{B}_{h+j}/G \mid j < \alpha - h \rangle$ in $V[G]$. This splits into two subcases:

<u>Case 1.1</u> $cf(\lambda) = \omega$.

Then $\mathbb{B}_\lambda$ is subcomplete by (b)(ii)

<u>Case 1.2</u> $cf(\lambda) = \omega$.

We literally repeat the argument in the proof of Thm 1. This gives us a $c \in \mathbb{B}_\lambda$ s.t. if $G \ni c$ is $\mathbb{B}_\lambda$-generic, there is $\sigma' \in V[G]$ s.t. $\sigma' : \overline{W} \prec W$ has the desired properties. In particular, we see that $\sigma'' \bigcup_{i<\bar\lambda} \bar G \cap \bar{\mathbb{B}}_i \subset G$. But since $cf(\bar\lambda) = \omega_1$ in $\bar W$, we know that $\bigcup_{i<\lambda} \bar G \cap \bar{\mathbb{B}}_i$ is dense in $\bar G$. Hence $\sigma'' \bar G \subset G$.

[Note If we instead assumed $cf(\lambda) = \omega$, then $cf(\bar\lambda) = \omega$ in $\bar W$ and the argument in the proof of Thm 1 would no longer work, since it used (a)(ii) in the "Fact" of §1 to establish that, if $\langle v_i \mid i<\omega\rangle \in \bar W$ is a monotone cofinal sequence in $\bar\lambda$, then the set of $\bigcap_{i<\omega} a_i$ s.t. $\langle a_i \mid i<\omega\rangle \in \bar W$ is a thread in $\langle \bar{\mathbb{B}}_{v_i} \mid i<\omega\rangle$ is dense in $\bar G$. We now no longer know this to be true.]

In Case 1.2 we again have: λ is regular and $\lambda > \overline{\overline{\mathbb{B}}}_i$ for $i<\lambda$. We then repeat the argument in the proof of Thm 1.

QED (Thm 4)