§2 The Complete Forcing Axiom (CFA)

Complete forcing was defined by Shelah in [S]. An equivalent definition is:

Def Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is a complete forcing iff there is θ s.t. $\mathbb{B} \in H_\theta$ and whenever $N = L_\tau^A$ is a ZFC$^-$ model with $H_\theta \subset N$ and $\theta < \tau$, then the following holds: Let $\pi : \bar{N} \prec N$ where $\bar{N}$ is transitive and countable. Let $\pi(\bar{\theta}, \bar{\mathbb{B}}) = \theta, \mathbb{B}$. Let $\bar{G}$ be $\bar{\mathbb{B}}$-generic over $\bar{N}$. There is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, then $\pi `` \bar{G} \subset G$. (Hence π extends uniquely to a $\pi^* : \bar{N}[\bar{G}] \prec N[G]$ s.t. $\pi^*(\bar{G}) = G$.)

(Note L_τ^A is an abbreviation for $\langle L_\tau[A], \in, A \cap L_\tau[A] \rangle$. Similarly for $L_\tau^{A_1, \ldots, A_n}$.)

We say that θ witnesses the completeness of $\mathbb{B}$, if it is as above. We say that θ verifies the completeness of $\mathbb{B}$ if every $\theta' \geq \theta$ witnesses the completeness of $\mathbb{B}$.

It is easily seen that, if the completeness of $\mathbb{B}$ is witnessed by θ, then it is verified by $(2^\theta)^+$. (Let θ be the smallest witness. Then θ is $H_{\theta'}$ - definable for $\theta' > 2^\theta$.)

It is easily seen that complete forcing adds no new countable sets of ordinals. If $\mathbb{B} = BA(\mathbb{P})$ and $\mathbb{P}$ is an ω-closed set of conditions, then $\mathbb{B}$ is complete.

(Here $BA(\mathbb{P})$ is the canonical BA over $\mathbb{P}$, defined as the set of $X \subset \mathbb{P}$ s.t. $X = \neg\neg X$, where:

$\neg X = \{q \mid q$ is incompatible with every $p \in X\}$. $\neg$ is then the complement function and the intersection $\bigcap^*$ and union $\bigcup^*$ are defined by:

$$\bigcap^* Z = \bigcap Z \ , \ \bigcup^* Z = \neg\neg \bigcup Z$$

for $Z \subset BA(\mathbb{P})$.)

However, the converse also holds:

Lemma 1 $\mathbb{B}$ is a complete forcing iff $\mathbb{B} \simeq BA(\mathbb{P})$ for a set of conditions $\mathbb{P}$ which is ω-closed.

proof.

$(\leftarrow)$ is left to the reader

$(\rightarrow)$ Let θ verify the completeness of $\mathbb{B}$. Fix $N \supset H_\theta$ as above. Rather than working with embeddings $\sigma: \bar{N} \prec N$, we now follow Shelah in working with $X \prec N$ s.t. $\mathbb{B}, \theta \in X$. Again following Shelah, we call $G \subset X \cap \mathbb{P}$ $\mathbb{P}$-<u>generic</u> <u>over</u> X if $G \cap \Delta \neq \emptyset$ for all dense $\Delta \in X$. Set:

$\mathbb{P}$ = the set of $p = \langle X_p, G_p \rangle$ s.t. $\mathbb{B}, \theta \in X_p \prec N$, X_p is countable, and G_p is $\mathbb{P}$-generic over X_p.

For $p, q \in \mathbb{P}$ set:

$p \leq q \longleftrightarrow (X_p \supset X_q \wedge G_q = G_p \cap X_q)$.

Clearly $\mathbb{P}$ is ω-closed, since if $p_{i+1} \leq p_i$, $p_i = \langle X_i, G_i \rangle$ for $i < \omega$, then $q \in \mathbb{P}$ and $q \leq p_i$ $(i < \omega)$, where $q = \langle \bigcup_i X_i, \bigcup_i G_i \rangle$. We claim:

<u>Claim</u> $\mathbb{B} \simeq BA(\mathbb{P})$.

proof.

Set $\mathbb{B}' = BA(\mathbb{P})$. For $p \in \mathbb{P}$ set:

$b_p = \bigcap G_p$ in $\mathbb{B}$. Then $b_p \neq 0$ in $\mathbb{B}$, since by completeness there is $b \neq 0$ s.t. $G_p \subset G$ whenever $G \ni b$ is $\mathbb{B}$-generic. Hence $b \subset \bigcap G_p$. Moreover,

(1) $p \leq q \rightarrow b_p \subset b_q$

However:

(2) $p \| q \leftrightarrow b_p \cap b_q \neq 0$ (where $p \| q$ means "p is compatible with q in $\mathbb{P}$")

proof.

($\rightarrow$) Let $r \leq p, q$. Then $b_r \subset b_p \cap b_q$

($\leftarrow$) Let $b_p \cap b_q \neq 0$, Let $X \prec H_\theta$ s.t X is countable, $X_p \cup X_q \subset X$, and $b_p \cap b_q \in X$. Let G be $\mathbb{B}$-generic over X s.t. $b_p \cap b_q \in G$. Then $r = \langle X, G \rangle \leq p, q$. QED (2)

$\{[p] \mid p \in \mathbb{B}\}$ is dense in $\mathbb{B}' = BA(\mathbb{P})$ (where $[p] = \neg\neg\{p\} \doteq$ the smallest $X \in \mathbb{B}'$ s.t. $p \in X$). Moreover, $\{b_p \mid p \in \mathbb{P}\}$ is dense in $\mathbb{B}$. But then:

$$[p] \subset [q] \leftrightarrow \bigwedge r \in \mathbb{P}\ (r \| p \to r \| q)$$

$$\leftrightarrow \quad '' \quad (b_r \wedge b_p \neq 0 \to b_r \wedge b_q \neq 0)$$

$$\leftrightarrow \quad b_p \subset b_q \text{ in } \mathbb{B}.$$

Hence there is an isomorphism

$$\sigma : \langle \{[p] \mid p \in \mathbb{P}\}, \subset \rangle \overset{\sim}{\leftrightarrow} \langle \{b_p \mid p \in \mathbb{P}\}, \subset \rangle$$

defined by $\sigma([p]) = b_p$. Hence σ extends uniquely to an automorphism $\sigma' : \mathbb{B}' \overset{\sim}{\leftrightarrow} \mathbb{B}$.

QED (Lemma 1)

<u>Note</u> Lemma 1 does <u>not</u> say that there is an ω-closed dense subset of $\mathbb{B}$ if $\mathbb{B}$ is a complete forcing. We don't know whether that is true.

Lemma 1 means that complete forcings had, in effect, been exhaustively studied before Shelah defined them. We shall continue, however, to use Shelah's original definition here, since some arguments will be a template for later applications.

We remark that the concept of complete forcing is "locally based" in the following sense: In order to decide whether θ witnesses completeness, we need only consider $N \supset H_\theta$ which have the same cardinality as H_θ, by a Löwenheim – Skolem argument. Hence we need only to know $\mathcal{P}(H_\theta)$, regardless of what might exist further out in the universe. This means, in particular, that if W is an inner model with $\mathcal{P}(H_\theta) \subset W$, then θ witnesses the completeness of $\mathbb{B}$ in W iff in V.

The iteration theorem for complete forcing reads:

Theorem 2 Let $\langle \mathbb{B}_i \mid i < \alpha \rangle$ be a countable support iteration s.t.

(a) $\mathbb{B}_0 = 2$

(b) $\Vdash_i \mathbb{B}_{i+1}/\dot{G}_i$ is a complete forcing for $i+1 < \alpha$.

Then each $\mathbb{B}_i$ is a complete forcing.

($\dot{G}_i$ being the canonical generic name.)

(It follows, of course, that if $\mathbb{A}$ is a complete forcing and

$$\Vdash_{\mathbb{A}} \mathring{\mathbb{B}} \text{ is a complete forcing,}$$

then $\mathbb{A} * \mathring{\mathbb{B}}$ is a complete forcing.)

We take this as given. It can either be proven directly, as Shelah did, or derived from Lemma 1.

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The complete forcing axiom (CFA) says that MA($\mathbb{B}$) holds for every complete forcing $\mathbb{B}$.
CFA$^+$ says that MA$^+$($\mathbb{B}$) holds.

Thm 3 CFA$^+$ + CH is consistent relative to the existence of a supercompact cardinal.

The argument of this proof is a paradigm for further arguments, which we will therefore not need to repeat. It was first used by Baumgartner to prove the consistency of PFA$^+$ relative to a supercompact.

proof.
Let κ be supercompact and let f be a Laver function for κ (i.e. $f : \kappa \to V_\kappa$ s.t. for each $\langle x, \beta \rangle$ there is a supercompact embedding $\pi : V \prec W$ with $x = \pi(f)(\kappa)$ and ${}^{\beta}W \subset W$). Define a CS iteration $\langle \mathbb{B}_i \mid i \leq \kappa \rangle$ by:

- $\mathbb{B}_0 = 2$
- If $\Vdash_i f(i)$ is a complete forcing, then $\Vdash_i \check{\mathbb{B}}_{i+1} / \dot{G} \simeq f(i) * \mathrm{coll}(\omega_1, \overline{\overline{f(i)}})$
- If $\not\Vdash_i f(i)$ is a complete forcing, then $\Vdash_i \check{\mathbb{B}}_{i+1} / \dot{G} = \mathrm{coll}(\omega_1, \omega_2)$.

Then $\mathbb{B}_\kappa$ is a complete forcing by the iteration Thm for complete forcings.

<u>Claim</u> Let G be $\mathbb{B}_\kappa$-generic. Then $V[G] \models \mathrm{CFA}^+ + \mathrm{CH}$.

proof.
The collapsing will, at some point in the iteration make CH true. But then it remains true, since complete forcings do not add reals.

We show now that CFA^+ holds in $V[G]$. Let $\mathbb{A} \in V[G]$ be a complete forcing as verified by θ. We can assume w.l.o.g. that $\mathbb{A} = \mathring{\mathbb{A}}^G$ and that:

(1) $\Vdash_\kappa \mathring{\mathbb{A}}$ is complete as verified by $\check{\theta}$

(2) $[\![x \in \mathring{\mathbb{A}}]\!] \subset \bigcup_{z \in U} [\![z = x]\!]$ for all $x \in V^{\mathbb{B}_\kappa}$,

where $U \subset V^{\mathbb{B}_\kappa}$, $U \in V$.

Let $\langle \Delta_i \mid i < \omega_1 \rangle \in V[G]$ be s.t. Δ_i is dense in $\mathbb{A}$ for $i < \omega_1$. Let $\mathring{\Delta}^G = \langle \Delta_i \mid i < \omega_1 \rangle$. Let $\dot{a} \in V[G]^{\mathbb{A}}$ s.t. $\Vdash \dot{a}$ is stationary in ω_1. We assume: $\dot{a} = \ddot{a}^G$.

Now let $\beta = \overline{\overline{V_\beta}}$ s.t. $\mathring{\mathbb{A}}, U, \mathring{\Delta}, \mathring{\ddot{a}} \in V_\beta$.

Let $\pi : V \to W$ be a supercompact embedding s.t. $W^\beta \subset W$. (Hence $V_{\beta+1} \subset W$.) Now let:

(3) $\pi(\langle \mathbb{B}_i \mid i \leq \kappa \rangle) = \langle \mathbb{B}_i' \mid i \leq \kappa' \rangle$.

Then $\mathbb{B}_i = \mathbb{B}_i'$ for $i \leq \kappa$. Since G is $\mathbb{B}_\kappa$-generic, we can extend it to $G' \supset G$ which is $\mathbb{B}'_{n'}$ - generic over W. Then π extends uniquely to $\pi^* \supset \pi$ s.t.

(4) $\pi^*: V[G] \prec W[G']$, $\pi^*(G) = G'$.

$\mathbb{A}$ is complete in $W[G']$ since θ witnesses completeness and $\theta < \beta$. (This is where we use that completeness is "locally based".) Hence:

(5) $\mathbb{B}'_{n+1}/G = \mathbb{A} * \text{coll}(\omega_1, \bar{\mathbb{A}})$ in $W[G]$.

Now let $\sigma = \pi^* \upharpoonright \mathbb{A}$. Then σ is a homomorphism of $\mathbb{A}$ into $\mathbb{A}' = \pi^*(\mathbb{A})$. But $\sigma \in W[G']$, since it is definable from $\pi \upharpoonright U$, $\mathbb{A}$, G, G' by:

$$\sigma(t^G) = \pi(t)^{G'} \text{ if } t^G \in \mathbb{A},\ t \in U.$$

By (5) there is $A \in W[G']$ which is $\mathbb{A}$-generic over $W[G]$ (hence over $V[G]$). Let $\tilde{A}$ be the filter on $\mathbb{A}'$ generated by $\pi^{*}{}''A$. Let

(6) $\pi^*(\langle \Delta_i \mid i < \omega_1 \rangle) = \langle \Delta'_i \mid i < \omega_1 \rangle$

(7) $\pi^*(\dot{a}) = \dot{a}'$.

Then:

Δ_i' is dense in $\mathbb{A}'$ for $i<\omega$, and $\Vdash_{\mathbb{A}'}$ ($\mathring{a}'$ is stationary in ω_1) holds in $W[G']$. Clearly $\tilde{A} \cap \Delta_i' \neq \emptyset$ for $i<\omega_1$. Moreover, $\mathring{a}'^{\tilde{A}} = \mathring{a}^A$ is stationary in ω_1 in $W[G']$.

Since $\pi^*: V[G] \prec W[G']$, there is $\tilde{A} \in V[G]$ s.t. $\tilde{A}$ is a filter on $\mathbb{A}$, $\tilde{A} \cap \Delta_i \neq \emptyset$ for $i<\omega$, and $\mathring{a}^{\tilde{A}}$ is stationary in ω_1.

QED (Theorem.)

We refer to $V[G]$ as the "natural model" of CFA. We could, of course, have done a prior application of Silver forcing to make GCH hold in V, in which case GCH will also hold in $V[G]$.

The natural model also satisfies $\diamondsuit$, since $coll(\omega_1, \beta)$ makes $\diamondsuit$ true and no later stage can make it false. We can conclude that by Lemma 1, since it is well known that every $\diamondsuit$-sequence remains a $\diamondsuit$-sequence under ω-closed forcings. Hence:

<u>Corollary 2.1</u> $CFA^+ + GCH + \diamondsuit$ is consistent relative to a supercompact cardinal.

We note that in the course of the iteration we repeated create new $\diamondsuit$-sequences and hence new Souslin trees. But a Souslin tree can also not be destroyed by ω-closed forcing. Hence the natural model is particularly rich in Souslin trees.

The most striking consequence of CFA^+ is

<u>Lemma 3</u> Let CFA^+ hold. Then every forcing which preserves stationary subsets of ω_1 is semiproper.

This is proven in [FMS]. Since Shelah

showed that if Namba forcing is semiproper, then a strong form of Chang's conjecture holds. Hence:

Corollary 4 $CFA^+ \rightarrow$ The strong Chang's conjecture.