

§ 4 Namba'-forcing

\mathbb{N}' is a variation of the Namba-conditions \mathbb{N} which is discussed at length in [PIF] and by us in [LF] § 6 and [SPSC]. We define \mathbb{N} as the set of all subtrees T of $(\omega_2)^{<\omega}$ = the set of monotone $f: m \rightarrow \omega_2$ ($m < \omega$) s.t.

whenever $\alpha \in T$, then $\{t \mid \alpha \leq t \text{ in } T\}$ has size ω_2 .

\mathbb{N}' is the set of $T \in \mathbb{N}$ s.t. for some $\alpha \in T$, $T = \{t \mid t \leq_T \alpha \vee \alpha \leq_T t\}$

and if $t \geq_T \alpha$ in T , then t has ω_2 many immediate successors.

→ In both cases the partial ordering of the conditions is inclusion.

In [PIF] it was shown that, assuming CH, we have:

- \mathbb{N}' adds no reals
- \mathbb{N}' is different from \mathbb{N} in the

α is called the stem of T
 $(\alpha = \text{stem}(T))$.

sense that no \mathbb{N}' -generic contain
an \mathbb{N} -generic sequence and conversely,

(Note We say that $c = \langle x_i \mid i < \omega \rangle$
is \mathbb{N} -generic iff $c = c_G = \bigcup G$,

where G is \mathbb{N} -generic. G is then
recoverable from c as the set G_c
of $T \in \mathbb{N}$ in which c is a branch.
Similarly for \mathbb{N}' .)

In [SPSC] we showed that,
assuming $CH + 2^{\omega_1} = \omega_2$, \mathbb{N}'
is subproper, (hence can be
iterated without collapsing ω_2),
Here we assume only CH and
show that \mathbb{N}' is ω_1 -subproper
and Dee -subproper. Hence it
can be iterated without adding
reals.

From now on assume CH .

\mathbb{N}' satisfies a weak amalgamation lemma:

Def $\text{stem}(T)$ = the stem of T = the maximal $s \in T$ s.t. $\wedge t \in T (s \leq t \vee t \leq s)$.

Lemma 1 Let $T \in \mathbb{N}'$, $s = \text{stem}(T)$. Let $\langle T_u \mid u \in (\omega_2)^{<\omega} \rangle$ be s.t.

• $T_u \in \mathbb{N}'$ and $T_\emptyset = T$

• $|s_u| = |s| + |u|$ where $s_u = \text{stem}(T_u)$

• $s_u \wedge \langle v \rangle \neq s_u \wedge \langle \bar{v} \rangle$ for $v \neq \bar{v} < \omega_2$

Then $T' \in \mathbb{N}'$, where

$$T' = \bigcap_{n < \omega} \bigcup_{|u|=n} T_u = \bigcup_{f: \omega \rightarrow \omega_2} \bigcap_{n < \omega} T_{f \upharpoonright n}$$

(Note: $\text{stem}(T') = \text{stem}(T)$.)

The refinement lemma reads:

Lemma 2 Let $T \in \mathbb{N}'$, $f: T \rightarrow \omega_1$.

There exist $T' \leq T$, $g: \omega \rightarrow \omega_1$ s.t.

• $\text{stem}(T') = \text{stem}(T)$

• $f(t) = g(|t|)$ for $t \in T'$.

This is proven with a game theoretic argument due to Shelah. A proof is given in [LF] §6. Using these two lemmas one can show:

Lemma 3 \mathbb{N}' adds no reals.

A proof of this is also given in [LF] §6.

We now prove:

Thm 4 \mathbb{N}' is Dec-subproper.

The proof stretches over many sublemmas. We first define the completeness system which will verify Dec-subproperness. Let $\bar{N} = L^A_{\bar{I}}$ be a countable ZFC-model and let $\bar{N}' \in \bar{N}$ satisfy the definition of \mathbb{N}' in \bar{N} . Let $\bar{c} = \langle \bar{\delta}_i \mid i < \omega \rangle$ be monotone and cofinal in $\bar{\omega}_2 = \omega_2^{\bar{N}}$.

Set:

$A_{\bar{c}} = \{ \text{the set of } G \text{ s.t.} \}$

• G is \bar{N}' -generic over \bar{N}

• Let $c = \langle \delta_i \mid i < \omega \rangle = c_G = \cup G$. Then

there is $m < \omega$ s.t. for all $i \geq m$

there is $j < \omega$ s.t.

If $u \in H_{\omega_1}$ is not such a \bar{c} , we simply set:

$A_u = \{ \text{the set of } G \text{ s.t. } G \text{ is } \bar{N}'\text{-generic over } \bar{N}. \}$

Set:

$$\mathbb{D}(\bar{N}, \bar{N}') = \{A_u \mid u \in H_{\omega_1}\}.$$

Lemma 4.1 \mathbb{D} is a completeness system.

prf.

We verify (a) - (d) in the definition of "completeness system". (a), (d) are trivial. We verify (b), (c).

(b) Let $T \in \bar{N}'$, $\kappa = \text{ctm}(T)$, $n = |\kappa|$.

Claim There is $G \in T$ s.t. $G \in \mathbb{D}(\bar{N}, \bar{N}')$

prf.

Let $\bar{c} = \langle \bar{\delta}_i \mid i < \omega \rangle$ be monotone and cofinal in $\bar{\omega}_2 = \omega_2^{\bar{N}}$. We show:

Claim 1 There is $G \in T$ s.t.

• G is \bar{N}' -generic over \bar{N}

• $\forall i \geq n \forall j (\delta_i \leq \bar{\delta}_j \leq \bar{\delta}_{j+1} \leq \delta_{i+1}')$

where $c = \langle \delta_i \mid i < \omega \rangle = c_G$.

This follows easily by:

Claim 2 Let $T \in \bar{N}'$, $\alpha = \text{atm}(T)$. Let $\Delta \in \bar{N}$ be strongly dense in \bar{N}' . There is $T' \leq T$ in \bar{N}' s.t. $T' \in \Delta$ and, letting $\alpha' = \text{atm}(T')$:

(*) If $\alpha \leq i < i+1 \leq \alpha'$, then

$$\forall j \quad \alpha'_j \leq \bar{\delta}_j < \bar{\delta}_{j+1} \leq \alpha'_{i+1}$$

proof

For $t \in T$ set:

$$m(t) = \begin{cases} 1 & \text{if there is } T' \leq T \text{ s.t.} \\ & \text{atm}(T') = t \text{ and } T' \in \Delta \\ 0 & \text{if not} \end{cases}$$

By the refinement lemma there is $T^* \leq T$ s.t. $\text{atm}(T^*) = \text{atm}(T)$

and $m(t) = g(|t|)$ for $t \in T^*$.

But then $g(m) = 1$ for some m , since there is $T' \leq T^*$ s.t. $T' \in \Delta$,

let $\alpha' = \text{atm}(T')$. Clearly $\alpha \leq m$

where $\alpha = \text{atm}(T)$. But then

there is $\alpha' \in T^*$ s.t. $|\alpha'| = m$

and (*) holds for α' .

QED (Claim 2)

This proves (b). We prove (c).

Let $\bar{c}_i = \langle \bar{\gamma}_j^i \mid j < \omega \rangle$ be monotone and cofinal in $\bar{\omega}_2$ for $i < \omega$. Set $\gamma_i^* = \sup \{ \bar{\gamma}_j^h \mid h, j \leq i \}$ for $i < \omega$.

Then $C^* = \langle \gamma_i^* \mid i < \omega \rangle$ is monotone and cofinal in $\bar{\omega}_2$. Clearly, however, $A_{C^*} \subset \bigcap_i A_{\bar{c}_i}$.

QED (Lemma 4.1)

Now let $\theta > 2^{2^{\omega_2}}$. We show:

Main Claim $\langle \theta, \mathbb{D} \rangle$ witnesses the Deceit-subproperness of \mathbb{N}' .

Let $N = L_{\bar{\tau}}^A$ be a ZFC-model s.t.

$H_\theta \subset N$ and $\theta < \bar{\tau}$. Let $\pi: \bar{N} \prec N$ s.t.

\bar{N} is countable and full, where $\bar{\tau} \in N$.

Let $\pi(\bar{\theta}, \bar{N}') = \theta, N'$. We claim

that there is a $\bar{c} = \langle \bar{\gamma}_i \mid i < \omega \rangle$ monotone and cofinal in $\bar{\omega}_1 = \omega_1^{\bar{N}}$ s.t. whenever $\bar{c} \in A_{\bar{c}}$, $\bar{\alpha} \in \bar{N}$, $\alpha = \pi(\bar{\alpha})$,

Then there is $T \in \mathbb{N}'$ forcing that
whenever $G \in T$ is \mathbb{N}' -generic, then
there is $\sigma \in V[G]$ s.t.

(a) $\sigma: \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{B}, \bar{\pi}) = \theta, B, \pi$

(c) $C_{\omega_2}^N(\text{rng } \sigma) = C_{\omega_2}^N(\text{rng } \pi)$

(d) $\sigma'' \bar{G} \subset G$.

In order to find such a \bar{c} we work
in $V[G^*]$ where G^* is \mathbb{N}' -generic.

Note that $H_{\omega_1}^V = H_{\omega_1}^{V[G^*]}$.

Def Let $c = \langle \delta_i : i < \omega \rangle$ be monotone
and cofinal in ω_2 . c is good iff
whenever $F \in V$, $F: \omega_2 \rightarrow \omega_2$, then

$$\forall n \wedge i \geq n \quad \delta_i > \sup_{h < i} F(\delta_h).$$

Lemma 4.2 There is a good $c \in V[G^*]$

Proof.

Let $c = c_{G^*} = \bigwedge UG^*$. Then c is
good. To see this, note that
if $F \in V$, $F: \omega_2 \rightarrow \omega_2$ and

$T \in N'$, then $T' \in N'$ where
 T' is the set of $t \in T$ s.t.,
 letting $m = \text{stm}(T)$,

$$\forall i \in [m, |T|) \quad t_i > \sup_{h < i} F(t_h).$$

Then $T' \leq T$ in N' and

$$T' \Vdash \forall i \geq m \quad \delta'_i > \sup_{h < i} F(\delta'_h).$$

QED (Lemma 4.2)

(Note This shows that every N' -generic sequence is good. However, no N -generic sequence is good.)

Until further notice let c be a fixed good sequence $\in V[G^*]$. We note that if $c' = \langle \delta'_i \mid i < \omega \rangle$ is monotone and cofinal in ω_1 and $\forall m \forall i \geq m \forall j \quad \delta'_i \leq \delta_i < \delta_{i+1} \leq \delta'_{i+1}$, then c' is trivially good.

Def $N' = \langle N, \theta, N', \tau, \pi \rangle$

$\sigma^* : \tilde{N}' \xrightarrow{\sim} \mathbb{Z}$, where \mathbb{Z} = the smallest

$\mathbb{Z} < N'$ s.t. $C \subset \mathbb{Z}$.

Let $\tilde{N}' = \langle \tilde{N}, \tilde{\theta}, \tilde{N}', \tilde{\tau}, \tilde{\pi} \rangle$.

Then $\sigma^* : \tilde{N}' \prec N'$.

Def $D =$ the set of \tilde{N}' -definable $X \subset \omega_2$

$$\tilde{D} = (\sigma^*)^{-1} \cap D.$$

$$F = \sigma^* \cap \tilde{D}.$$

Then F bijects \tilde{D} onto D .

Lemma 4.3 $F \in V$

prf.

Clearly $D \in V$ is countable. Let $\langle x_i \mid i < \omega \rangle$

enumerate D . Set $\tilde{x}_i = (\sigma^*)^{-1}(x_i)$.

Then $\langle \tilde{x}_i \mid i < \omega \rangle \in H_{\omega_1} \subset V$. Hence

$$F = \{ \langle x_i, \tilde{x}_i \rangle \mid i < \omega \} \in V, \text{ QED (4.3)}$$

Def Let $f : \tilde{\omega}_2 \rightarrow \omega_2$ cofinally, where

$\tilde{\omega}_2 =_{\text{nt}} \omega_2^{\tilde{N}}$, f is smooth iff for

each $X \in \tilde{D}$ and all $\nu_1, \dots, \nu_n < \tilde{\omega}_2$;

$$\langle \nu_1, \dots, \nu_n \rangle \in X \iff \langle f(\nu_1), \dots, f(\nu_n) \rangle \in F(X),$$

($\langle \nu \rangle$ being Gödel's tuple function on ordinals).

Lemma 4.4 Let f be smooth. There

is a unique extension $f^* : \tilde{N}' \prec N'$

prf. of Lemma 4.4

$$(1) \tilde{N}' \models \varphi[\vec{v}] \leftrightarrow N' \models \varphi[f(\vec{v})]$$

for $\nu_1, \dots, \nu_m < \tilde{\omega}_2$.

prf.

There is $Y \in D$ s.t.

$$\langle \vec{v} \rangle \in Y \leftrightarrow N' \models \varphi[\vec{v}] \text{ for } \nu_1, \dots, \nu_m < \omega_2.$$

Set $\tilde{Y} = (\sigma^*)^{-1}(Y)$. Then

$$\langle \vec{v} \rangle \in \tilde{Y} \leftrightarrow \tilde{N}' \models \varphi[\vec{v}] \text{ for } \nu_1, \dots, \nu_m < \tilde{\omega}_2.$$

The conclusion is immediate. $\square \text{ED}(1)$

But each $x \in \tilde{N}'$ is \tilde{N}' -definable from ordinals $\nu_1, \dots, \nu_m < \tilde{\omega}_2$. If $x = \tilde{g}(\vec{v})$, \tilde{g} being a definable function, set:

$$f^*(x) = g(f(\vec{v})),$$

where g is N' -definable by the same formula. It is apparent from (1) that this definition is independent of the choice of \vec{v} and the defining formula of \tilde{g} . Now let $x_1, \dots, x_m \in \tilde{N}'$, with $x_i = \tilde{g}_i(\vec{v})$ for $i = 1, \dots, m$. By (1) we

have:

$$\tilde{N}' \models \varphi[\vec{x}] \leftrightarrow \tilde{N}' \models \varphi[\tilde{g}_1(\vec{v}), \dots, \tilde{g}_m(\vec{v})] \leftrightarrow$$

$$\leftrightarrow N' \models \varphi[g_1(f(\vec{v})), \dots, g_m(f(\vec{v}))] \leftrightarrow$$

$$\leftrightarrow N' \models \varphi[f^*(x_1), \dots, f^*(x_m)].$$

This proves existence. To see uniqueness, let f' have the same property. Then $f'(\tilde{q}(\vec{v})) = g(f(\vec{v})) = f^*(\tilde{q}(\vec{v}))$,

QED (Lemma 4.4)

Lemma 4.5 Let f^* be as above. Set: $C = C_{\omega_2}^N(\text{rng } \pi)$, $\tilde{C} = C_{\omega_2}^{\tilde{N}}(\text{rng } \tilde{\pi})$. Then $\tilde{C} \in \tilde{N}$ and $f^*(\tilde{C}) = C$.

proof.

Let $\pi: \bar{N} \prec N^*$ cofinally. Then

$N^* = N \setminus \{\lambda \in N \text{ and } N^* \prec N\}$, where

$\lambda = \text{On} \cap N^*$. Hence $C = C_{\omega_2}^{N^*}(\text{rng } \pi)$.

Similarly $\tilde{C} = C_{\omega_2}^{\tilde{N}^*}(\text{rng } \tilde{\pi})$, where

$\tilde{\pi}: \bar{N} \prec \tilde{N}^*$ cofinally. But $f^*(\tilde{N}^*) = N^*$,

since $f^*(\tilde{\pi}) = \pi$. QED (4.5)

Lemma 4.6 There is $\tilde{\sigma}: \bar{N} \prec \tilde{N}$ s.t.

- $\tilde{\sigma}$ takes $\bar{\omega}_2 = \omega_2^{\bar{N}}$ cofinally to $\tilde{\omega}_2$

- $C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\sigma}) = C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\pi})$.

- $\tilde{\sigma}(\bar{\theta}, \bar{N}, \bar{\alpha}) = \tilde{\theta}, \tilde{N}, \tilde{\alpha}$

proof of Lemma 4.6

Let $k: \tilde{N}_1 \xrightarrow{\sim} C_{\tilde{\omega}_2}^{\tilde{N}}(\text{sing}(\tilde{\pi}))$. Then $k, \tilde{N}_1 \in \tilde{N}$

Let $\tilde{\pi}_1 = k^{-1} \tilde{\pi} : \tilde{N} \prec \tilde{N}_1$.

Let $\langle \tilde{N}_0, \tilde{\pi}_0 \rangle =$ the liftup of $\langle \tilde{N}, \tilde{\pi} \upharpoonright \tilde{\omega}_1 \rangle$.

$k_0: \tilde{N}_0 \prec \tilde{N}_1$ s.t. $k_0 \tilde{\pi}_0 = \tilde{\pi}_1$ and $k_0 \upharpoonright \tilde{\omega}_2^{\tilde{N}_0} = \text{id}$.

The maps $\tilde{\pi}_0, \tilde{\pi}_1$ are cofinal; hence so

is k_0 . \tilde{N}_0, \tilde{N} are almost full, since

\tilde{N} is full. Let \mathcal{L}_1 be the following

language on $L_{\delta_1}(\tilde{N}_1)$ ($\delta_1 = \delta_{\tilde{N}_1}!$)

Predicate \in ; Constants σ, x ($x \in L_{\delta_1}(\tilde{N}_1)$)

Axioms ZFC^- , $H_{\omega_1} = \underline{H}_{\omega_1}$, $\Lambda \sigma (v \in x \leftrightarrow \bigvee_{z \in x} v = z)$,

$\sigma: \tilde{N} \prec \tilde{N}_1$ $\tilde{\omega}_1$ -cofinally, \uparrow

Then \mathcal{L}_1 is consistent since the

corresponding language \mathcal{L}_0 on $L_{\delta_0}(\tilde{N}_0)$

($\delta_0 = \delta_{\tilde{N}_0}$) is modeled by:

$\langle H_{\omega_2}, \in, \tilde{N}_0, \tilde{\pi}_0 \rangle$.

Let \mathcal{M} be a solid model of \mathcal{L}_1 .

Then $\tilde{\sigma} = k_0 \circ \sigma^{\mathcal{M}}$ has the desired

properties. QED (Lemma 4.6)

$\sigma(\tilde{\omega}, \tilde{N}, \tilde{\omega}_1) = k^{-1}(\tilde{\omega}), k^{-1}(\tilde{N}'), k^{-1}(\tilde{\omega}_2)$

(Note N is called full iff N is a second order ZFC-model in $L_\mu(N)$, where $L_\mu(N)$ satisfies ZFC, $\delta_N =$ the smallest δ s.t. $L_\delta(N)$ is admissible. The concepts "liftup" and "almost full" are elucidated in [SPSC] §3.)

From now on let $\tilde{\sigma}$ be a fixed map $\tilde{\sigma}: \tilde{N} \rightarrow \tilde{N}$ satisfying Lemma 4.6.

Lemma 4.7 Let f be smooth and set $\sigma = f^* \tilde{\sigma}$. Then $C_{\omega_2}^N(\text{rng } \sigma) = C_{\omega_2}^N(\text{rng } \pi)$.
 p.f.

$$\begin{aligned} (\subset) \text{rng } \sigma &= f^* \text{rng } \tilde{\sigma} \subset f^* (C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\sigma})) = \\ &= f^* (C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\pi})) = C_{\omega_2}^N(\text{rng } \pi) \text{ by Lemma 4.6} \end{aligned}$$

$$\text{Hence } C_{\omega_2}^N(\text{rng } \sigma) \subset C_{\omega_2}^N(\text{rng } \pi).$$

$$(\supset) \text{ Let } x \in C_{\omega_2}^N(\text{rng } \pi), \text{ Then } x = \pi(g)(\xi)$$

where $\xi < \omega_2$ and $g \in \tilde{N}$. Hence

$$\tilde{\pi}(g) \in C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\pi}) = C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\sigma}). \text{ Let}$$

$$\tilde{\pi}(g) = \tilde{\sigma}(h)(\zeta), \zeta < \tilde{\omega}_2, h \in \tilde{N}. \text{ ; ;}$$

$$\text{Then } x = (\sigma(h)(f(\zeta)))(\xi) \text{ where}$$

$$f(\zeta), \zeta < \omega_2. \text{ Hence } x \in C_{\omega_2}^N(\text{rng } \sigma),$$

QED (4.7)

Remark $F = \sigma^* \uparrow \tilde{D}$ is smooth with $F^* = \sigma^*$. Hence $\sigma' = \sigma^* \circ \tilde{\sigma}$.

satisfies Lemma 4.7. We shall see that there are many other smooth functions in $V[G^*]$.

Lemma 4.8 Let σ be as above. There is $c' \in V[G^*]$ s.t. c' is good and $c' \subset \text{rng } \sigma$.

Prf.
Let $c = \langle \delta_i \mid i < \omega \rangle \in V[G^*]$ be good. Define $c' = \langle \delta'_i \mid i < \omega \rangle$ by: $\delta'_0 = 0$,
 $\delta'_{i+1} =$ the least $\delta \in \text{rng } \sigma$ s.t. $\forall j, \delta'_i \leq \delta_j < \delta'_{i+1} \leq \delta_j$.
QED(4.8)

Now let $\sigma' = \sigma^* \circ \tilde{\sigma}$. Assume that $c \in V[G^*]$ is not only good but that $c \subset \text{rng } \sigma'$. Set: $\bar{c} = (\sigma')^{-1} \circ c$.

We show:
Claim Let $\bar{G} \in A_{\bar{c}}$. There is $T \in \mathbb{N}$ satisfying the conclusion of the Main Claim.

Let $\bar{c}' = \langle \bar{\delta}'_i \mid i < \omega \rangle = c \bar{G} = \cap \cup \bar{G}$.
Set $\tilde{c}' = \sigma' \circ \bar{c}'$. Let \mathcal{L} be the following language over the admissible structure $\langle H_\theta, < \rangle$, where $<$ well orders H_θ .

Predicates: \in , Constants f, x ($x \in H_\theta$)

Axioms ZFC⁻, $H_{\omega_1} = \underline{H}_{\omega_1}$, $\forall v \in x \leftrightarrow \forall z \in x \exists!$

$f: \tilde{\omega}_2 \rightarrow \omega_2$ cofinally, " f " is smoothⁿ - ω .

$\forall x_1, \dots, x_n \in \tilde{\omega}_2 \wedge x \in \underline{D} \ (\langle \vec{x} \rangle \in x \leftrightarrow \langle f(\vec{x}) \rangle \in \underline{E}(x))$

for $n < \omega$.

Clearly this is consistent, since it is modeled by $\langle H_\theta, \in, F \rangle$.

We now define a subtree of $(\omega_2)^{<\omega}$ by:

Def $t \in T$ iff $\mathcal{L}_t \wedge \bigwedge_{i < |t|} t_i = f(\vec{x}'_i)$ is consistent

where $\vec{c}' = \langle \vec{x}'_i \mid i < \omega \rangle = \vec{\sigma} \text{ " } \vec{c}'$.

Then $c' = \langle x'_i \mid i < \omega \rangle$ is a branch through T .

Lemma 4.9 There is $T' \subset T$ s.t.

• $T' \in \mathcal{N}'$

• $s = \langle x'_i \mid i < |s| \rangle$ where $s = \text{stem}(T')$

proof.

For $m < \omega$ consider the following game

G_m : At the i -th move player I

picks an γ_i s.t. $\gamma_i > \gamma_j$ for $j < i$

Player II then picks - if possible - a ν_i s.t. $\langle \nu_h \mid h \leq i \rangle \in T$.

If $i < m$ he must pick $\nu_i = x'_i$.

If $i \geq m$ " " " $\nu_i > \gamma_i$.

If at some i II has no move, then I wins. Otherwise II wins.

Claim For arbitrarily large n , II has a winning strategy.

proof.

Suppose not. Then I has a winning strategy for all $n \geq n_0$. Let $D \subset \omega_2$ be the set of δ s.t. for all $n \geq n_0$ and all plays $\langle \nu_{0,m}, \nu_m \rangle$ by II s.t. $\nu_m < \delta$, we have $S_m(\nu_{0,m}, \nu_j) < \delta$ for $j \leq m$, where S_m is I's winning strategy. Then D is club in ω_2 . Set:

$F(\bar{\alpha}) =$ the least $\delta \in D$ s.t. $\bar{\alpha} < \delta$.

Since C' is good, there is $n \geq n_0$ s.t. $\bigwedge i \geq n \ \delta'_i > \sup_{h < i} F(\delta'_h)$. But then $\langle \delta'_i \mid i < \omega \rangle$ defeats the strategy S_m . Contr! QED (Claim).

Now let S be II's winning strategy for G_n . Let T' be the set of all $S(\gamma_{0,m}, \gamma_i)$ where $\langle \gamma_{0,m}, \gamma_i \rangle$ is any possible play by I. Then

T' has the desired properties.

QED (4.9)

Let T' be as in Lemma 4.9. We show that T' satisfies the conclusion of the Main Claim. Let $G \ni T'$ be N' -generic. We claim that there is $\sigma \in V[G]$ satisfying (a) - (d) of the Main Claim. Let $C'' = \langle \delta_i'' \mid i < \omega \rangle =$

$= C_G = \Delta \cup G$. For $i < \omega$, let:

$f_i'' =$ the N -least $f: \omega_1 \xrightarrow{\text{onto}} (\delta_i'' + 1)$

$f_i' =$ " \tilde{N} - " $f: \tilde{\omega}_1 \xrightarrow{\text{onto}} (\tilde{\delta}_i' + 1)$

Set $f_i = f_i'' \circ (f_i')^{-1}$. Then

$f_i: \tilde{\delta}_i' + 1 \rightarrow \delta_i'' + 1$. But if \mathcal{M} is a

solid model of $\mathcal{L} + \dot{f}(\underline{\tilde{\delta}_i'}) = \underline{\delta_i''}$,

and $f = \dot{f}^{\mathcal{M}}$, then f , being smooth,

extends to $f^*: \tilde{N}' \prec N'$. Hence

$f^*(\tilde{f}_i') = f_i''$ and $f_i = f \upharpoonright (\tilde{\delta}_i' + 1)$.

Hence $f_i \subset f_i'$ and $f_i(\tilde{\delta}_i') = \delta_i''$

for $i \leq i' < \omega$. Set $f = \bigcup_i f_i$

Then:

(1) $f: \tilde{\omega}_1 \rightarrow \omega_1$ cofinally and $f(\tilde{\delta}_i') = \delta_i''$.

Moreover:

(2) f is smooth

Proof.

Let $\nu_1, \dots, \nu_n < \tilde{\omega}_1$. Then $\vec{\nu} < \tilde{\delta}_i'$ for some i .

Let \mathcal{O} be a solid model of $\mathcal{L} + \dot{f}(\tilde{\delta}_i') = \delta_i''$.

Then $f^{\dot{\mathcal{O}}}$ is smooth and $f^{\dot{\mathcal{O}}} \upharpoonright \tilde{\delta}_i' = f \upharpoonright \tilde{\delta}_i'$

by the above argument. Hence, for

$X \in \tilde{A}$, we have:

$$\langle \vec{\nu} \rangle \in X \iff \langle f^{\dot{\mathcal{O}}}(\vec{\nu}) \rangle = \langle f(\vec{\nu}) \rangle \in F(X)$$

QED (2)

But then f extends to:

(3) $f^* : \tilde{N}' \rightarrow N'$,

and we set: $\sigma = f^* \circ \tilde{\sigma}$. We show that

(a)-(d) of the main Claim hold for the

σ . (a), (b) are immediate by (3) and

$\tilde{\sigma}(\bar{\theta}, \bar{N}', \bar{\alpha}) = \tilde{\theta}, \tilde{N}', \tilde{\alpha}$. (c) follows by

Lemma 4.7. Finally, (d) follows, since

$C'' = C_G$, $\bar{C}' = C_{\bar{G}}$, and $f \upharpoonright \bar{C}' = C''$.

QED (Theorem 4)

It remains only to show that N' is ω_1 -subproper. Before doing so, we attempt to distill some more information from the proofs just given. From now on let Θ be $(Z^{2^{\omega_2}})^+$.

It is enough, however, to take Θ as having this meaning in the sense of an N which is viable in the following sense:

Def N is viable iff

- $N = L_{\bar{\sigma}}^A$ is a model of $ZFC^- + CH$
- There are $\Theta, H \in N$ s.t.

$$L_{\bar{\sigma}}(N) \models (\Theta = (Z^{2^{\omega_2}})^+ \wedge H = H_{\Theta})^+$$

When dealing with a fixed viable N we shall write $\omega_1, \omega_2, \Theta, H_{\Theta}, N'$ etc. to denote the relativisation of the concept to N . All that we did in the proofs just given goes through for an arbitrary viable N .

where $\bar{\sigma} = \delta_N =$ the least δ s.t. $L_{\bar{\sigma}}(N)$ is admissible.

Assuming $N = L_{\omega_2}^A$ to model $ZFC^- + CH$,
 N will be viable in either of the cases:

- $2^{2^{\omega_2}} = \theta$ in V and $H_{\theta}^V \subset N$
- N is almost full.

For transitive $M, c \subset M$, set:

$$M[c] =_{df} \bigcup_{x \in M} L_{\text{On}_M} (TC(x \cup \{c\}))$$

We define:

Def Let N be viable and $c = \langle \gamma_i : i < \omega \rangle$
 monotone and cofinal in ω_2 .

c is tame iff the following hold:

- $\delta_N = \delta_{N[c]}$
- $H_{\omega_1}^N = H_{\omega_1}^{L_{\delta}(N[c])}$ ($\delta = \delta_N$)
- $\theta = (2^{2^{\omega_2}})^+$ in $L_{\delta}(N[c])$
- $H_{\theta}[c] = H_{\theta}^{L_{\delta}(N[c])}$.

c is good iff for all $F: \omega_2 \rightarrow \omega_2$ in N
 we have: $\forall n \wedge i \geq n \quad \gamma_i > \sup_{h < i} F(\gamma_h)$.

Clearly, any IV' -generic sequence
 over N is both good and tame.

Lemma 5.1 Let c be tame. Let $c' \in N[c]$ s.t. $c' = \langle \delta'_i : i < \omega \rangle$ is monotone and cofinal in ω_2 . Then $H_\theta[c] = H_\theta[c']$, (Hence c' is tame.)

prf.

$c' \in H_\theta[c]$, since c is tame.

Claim $c \in H_\theta[c']$

Let f map ω_1 onto ω_2 , $f \in H_\theta[c']$,

Set $\bar{c} = f^{-1} \circ c$. Then $\bar{c} \in H_{\omega_1}$ and

$c = f \circ \bar{c} \in H_\theta[c']$. QED (5.1)

Now let $\pi : \bar{N} \prec N$ where $\bar{N} \in N$ and $\bar{N} = L_{\bar{\tau}}^{\bar{A}}$ is countable and full in N ,

Let $\pi(\bar{\theta}, \bar{N}, \bar{\iota}) = (\theta, N, \iota)$ ($\bar{\iota} \in \bar{N}$ being arbitrary). We can literally repeat

our previous proof: Setting

$N' = \langle N, \theta, N', \iota, \pi \rangle$, let:

$Y =$ the smallest $Y \prec N'$ s.t. $c \subset Y$,

where c is tame for N' . Then

$\forall \delta \in L_\delta(N[c])$ where $\delta = \delta_N$. Hence

$\bar{N}', \sigma^* \in L_\delta(N[c])$ where

$\sigma^* : \tilde{N}' \xrightarrow{\sim} Y$ and $\tilde{N}' = \langle \tilde{N}, \tilde{\theta}, \tilde{N}', \tilde{\alpha}, \tilde{\pi} \rangle$ is transitive. Hence $\tilde{N}' \in H_{\omega_1}$. Define

$D, \tilde{D}, F = \sigma^* \upharpoonright \tilde{D}$ exactly as before. As before

we have $D, \tilde{D}, F \in H = H_{\omega_2}$. Define

the notion of a smooth $f : \tilde{\omega}_2 \rightarrow \omega_2$

exactly as before. A smooth f then

again extends uniquely to $f^* : \tilde{N}' \rightarrow N'$

s.t. $f^* \upharpoonright \tilde{\omega}_2 = f$. As before we get the

existence of $\tilde{\sigma} \in H_{\omega_1}$ s.t. $\tilde{\sigma} : \tilde{N} \rightarrow \tilde{N}'$,

$\tilde{\sigma}(\tilde{\theta}, \tilde{N}', \tilde{\alpha}) = \tilde{\theta}, \tilde{N}, \tilde{\alpha}$, and

$C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\sigma}) = C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\pi})$. Then

setting $\sigma = f^* \circ \tilde{\sigma}$, where $f \in N[c]$ is

smooth, we get:

$$C_{\omega_2}^N(\text{rng } \sigma) = C_{\omega_2}^N(\text{rng } \pi),$$

But then:

Lemma 5.2 $\sigma \in N[c]$.

Proof.

Clearly $\sigma \in L_\sigma[N[c]]$. Let $k : N^* \xrightarrow{\sim} C_{\omega_2}^N(\text{rng } \pi)$.

Since $\text{rng } \sigma \subset C_{\omega_2}^N(\text{rng } \pi)$, we can

set $u = k^{-1} \upharpoonright \text{rng } \sigma$. Then $u \subset N^*$ is

countable, where $N^* \in H_\theta^N$. Hence

$u \in H_{\theta}[c] = H_{\theta}^N[c]$, Hence.

$\text{rng } \sigma = k''u \in N[c]$. QED (5.2)

The main thing we now distill from this is:

Lemma 5.3 Let c be tame. There is $\sigma \in N[c]$ s.t.

- $\sigma: \bar{N} \prec N$
- $\sigma(\bar{\theta}, \bar{N}, \bar{a}) = \theta, N', a$
- $C_{\omega_2}^N(\text{rng } \sigma) = C_{\omega_2}^N(\text{rng } \tau_0)$.

But by the proof of Lemma 4.8:

Lemma 5.4 Let σ, c be as above, where c is both tame and good. There is $c' \in N[c]$ s.t. $c' \subset \text{rng } \sigma$ and c' is tame and good (hence $N[c] = N[c']$).

Note It might appear that our formulation of Lemma 5.3 throws out valuable information since, in order to get σ , we first defined $\sigma^*, \bar{N}, \bar{\sigma}$.

Using Lemma 5.4, however, we can quickly restore the missing information:

Def $\langle c, \sigma \rangle$ is excellent for N, π iff

- c is tame and good
- $\sigma \in N[c]$
- $\sigma: \bar{N} \rightarrow N$
- $\sigma(\bar{\theta}, \bar{N}') = \theta, N'$
- $C_{\omega_2}^N(\text{rng } \sigma) = C_{\omega_2}^N(\text{rng } \pi)$
- $c \subset \text{rng } \sigma$.

Lemmas 5.3, 5.4 then say:

Cor 5.5 Let c' be tame and good, let $\pi(\bar{x}) = a$.

There is an excellent $\langle c, \sigma \rangle$ s.t. $\sigma(\bar{x}) = a$ and $N[c] = N[c']$.

From now on suppose that $\langle c, \sigma \rangle$ is excellent, $\bar{x} \in \bar{N}$, and $\sigma(\bar{x}) = a$.

(\bar{x} is not necessarily the same as our previous \bar{x} with $\pi(\bar{x}) = a$.) Set:

$N' = \langle N, \theta, N', \pi, a \rangle$. Since $\text{rng } \sigma \subset C_{\omega_2}^N(\text{rng } \pi)$, each $x \in \text{rng } \sigma$ is N' -definable from elements of $\omega_1 \cup c$. Since σ is countable in $L_\sigma(N[c])$, there is $\alpha < \omega_1$ s.t. each $x \in \text{rng } \sigma$ is N -definable in parameters from $d \cup c$.

Let $Y =$ the smallest $Y \subset N$ s.t. $d \cup c \subset Y$.
Then $Y \in L_\delta(N)$. Hence $\sigma^*, \tilde{N}' \in L_\delta(N)$,

where $\sigma^*; \tilde{N}' \leftrightarrow Y$ and $\tilde{N}' = \langle \tilde{N}, \tilde{\theta}, \tilde{N}', \tilde{\pi}, \tilde{z} \rangle$
is transitive. Hence $\tilde{N}' \in H_{\omega_1}$. Set:

$\tilde{\sigma} = (\sigma^*)^{-1} \sigma$. Letting $D =$ the set of
 N' -definable $X \subset \omega_1$ and $\tilde{D} = (\sigma^*)^{-1} \ulcorner D \urcorner$,
 $F = \sigma^* \upharpoonright \tilde{D}$, we again have that $D \in H_\theta$
is countable in N (arguing in $L_\delta(N)$).

It follows as before that \tilde{D}, F are
countable in N . We define the notion
of smooth function $f: \tilde{\omega}_1 \rightarrow \omega_1$ as
before and again conclude that each
smooth f extends uniquely to an

$f^*: \tilde{N}' \subset N'$. As before, we have

$f^*(\tilde{C}) = C$, where $C = C_{\omega_1}^N(\text{rng } \pi)$,

$\tilde{C} = C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\pi})$. Now set: $\tilde{\sigma} = (\sigma^*)^{-1} \sigma$.

Then $\tilde{\sigma}: \tilde{N} \subset \tilde{N}$. Moreover, $\tilde{C} = C_{\tilde{\omega}_2}^{\tilde{N}}(\text{rng } \tilde{\sigma})$,

(To see this note that $C_{\omega_2}^N(\text{rng } \sigma) = C$;

hence $C_{\omega_2}^{\tilde{N}}(\text{rng } \tilde{\sigma}) = \sigma^{*-1} \ulcorner C_{\omega_2}^N(\text{rng } \sigma) \urcorner =$

$= \sigma^{*-1} \ulcorner C \urcorner = \sigma^{*-1} \ulcorner C \urcorner = \tilde{C}.$) If f

is smooth and $\sigma' = f^* \circ \tilde{\sigma}$, we can

conclude as before that $C_{\omega_2}^N(\text{rng } \sigma') = C$.

Now let $\bar{c} = \sigma^{-1} \circ c$. Define $A_{\bar{c}}$ as before (relativized to N).

Def Let $A \subset A_{\bar{c}}$, $\bar{c} = \langle \bar{\delta}_i \mid i < \omega \rangle$. A is dispersed at $m < \omega$ iff for every $i < \omega$ there is $\bar{G} \in A_{\bar{c}}$ s.t., letting $c_{\bar{G}} = \langle \delta_i \mid i < \omega \rangle$, we have:

$$\delta_m > \bar{\delta}_i \wedge \bigwedge i \geq m \forall h \delta_i \leq \bar{\delta}_h \leq \bar{\delta}_{h+1} \leq \delta_{i+1}$$

By the proof of Lemma 4.1 (b) we know that:

Lemma 5.6 Let $T \in \bar{N}'$, $\iota = \text{st}_m(T)$, $m = |\iota|$.

Then $\{G \in A_{\bar{c}} \mid T \in G\}$ is dispersed at m .

We now prove a technical lemma.

Lemma 5.7 Let $A \subset A_{\bar{c}}$ be dispersed at m .

Let $\sigma(\bar{\iota}) = \iota$. There are $\bar{G} \in A$, $T \in \bar{N}'$ s.t.

(a) $\text{st}_m(T) = \sigma(c_{\bar{G}} \upharpoonright m)$

(b) $\nexists G \ni T$ is \bar{N}' -generic over N , there

is $\sigma' \in N[G]$ s.t.

- $\sigma' : \bar{N} \prec N$

- $\sigma'(\bar{\theta}, \bar{N}', \bar{\iota}) = \theta, N', \iota$

- $C_{\omega_1}^N(\text{rng } \sigma') = C_{\omega_1}^N(\text{rng } \sigma)$

- $\sigma' \circ \bar{G} \subset G$

proof of 5.7.

Define \mathcal{L} on $\langle H_\theta, < \rangle$ as before:

Predicates: \in ; Constants: $\overset{\circ}{f}$, \underline{x} ($x \in H_\theta$)

Axioms: ZFC⁻, $H_{\omega_1} = \underline{H}_{\omega_1}$, $\forall x \in H_\theta$:

$\forall v \in \underline{x} \leftrightarrow \forall_{z \in x} v = \underline{z}$, $\overset{\circ}{f} : \underline{\omega}_2 \rightarrow \underline{\omega}_2$ cofinally;

" $\overset{\circ}{f}$ is smooth" - i.e. $\forall n < \omega$:

$\forall \nu_1, \dots, \nu_n < \underline{\omega}_2 \wedge x \in \underline{D} \langle \langle \nu_i \rangle \rangle \in x \leftrightarrow \langle \overset{\circ}{f}(\nu_i) \rangle \in \underline{F}(x)$.

Set $A^* = \{d \mid \forall \bar{g} \in A \ d = c_{\bar{g}}\}$, $\forall d \in A^*$

set: $\tilde{d} = \tilde{\sigma} "d$ and define $T_d \subset (\omega_2)^{<\omega}$

as before:

Def $t \in T_d \leftrightarrow \mathcal{L} + \bigwedge_{i < |t|} \underline{t}_i = \overset{\circ}{f}(\tilde{\delta}_i)$ is consistent,

where $\tilde{d} = \langle \tilde{\delta}_i \mid i < \omega \rangle$.

Claim There is $d \in A^*$, $T \subset T_d$ s.t.

• $T \in \mathbb{N}'$

• $t = \sigma(d \upharpoonright m)$ where $t = \text{rtm}(T)$

prf.

For each $d \in A^*$ define a game $g(d)$ as follows. At stage i :

I plays an $\gamma_i < \omega_2$ s.t. $\gamma_i > \sup_{h < i} \gamma_h$

II plays, if possible, a ν_i s.t. $\langle \nu_0, \dots, \nu_i \rangle \in T_d$.

If $i < n$, II must choose $\nu_i = \sigma(\tilde{\delta}_i)$.

If $i > n$, II must choose $\nu_i > \gamma_i$.

If at some i , II has no move, then I wins.

Otherwise II wins.

Subclaim II has a winning strategy for some $d \in A^*$.

prf.

Suppose not. For every $d \in A^*$, I then has a winning strategy S_d . Let C be the set of $\lambda < \omega_2$ s.t. for all $d \in A^*$, if $\nu_0, \dots, \nu_n < \lambda$ in any sequence of moves by II, then $S_d(\nu^*) < \lambda$.

C is then club in ω_2 . But then, setting $F(\xi) =$ the least $\lambda \in C$ s.t. $\xi < \lambda$, there is $m < \omega$ s.t. $\forall i \geq m \ \delta_i > \sup_{h < i} F(\delta_h)$

($\langle \delta_i \mid i < \omega \rangle = C$ being our good sequence with $\bar{c} = \sigma^{-1} \cap C$.) Since A is dispersed at m , there is $d \in A^*$ s.t.

$$\delta_m > \bar{\delta}_m \wedge \forall i \geq m \ \forall j \ (\delta_i \leq \bar{\delta}_j < \bar{\delta}_{j+1} \leq \delta_{i+1}),$$

But then $\langle \sigma(\delta_i) \mid i < \omega \rangle$ is a sequence of moves which defeats S_d . Contr!

QED (Subclaim)

Now let S be II's winning strategy for $d \in A^*$. Let $T =$ the set of finite sequences obtained by applying S to a finite sequence of moves by I. T has the desired properties.

QED (Claim)

Now let $G \ni T$ be \mathbb{N}' -generic over N . As before, letting $e_G = \langle \delta_i' \mid i < \omega \rangle$, $\tilde{d} = \langle \tilde{\sigma}_i \mid i < \omega \rangle$, there is a unique smooth f s.t. $f(\tilde{\sigma}_i) = \delta_i'$ for $i < \omega$. Moreover, $f \in N[G]$ (hence $f^* \in N[G]$). But then $\sigma' = f^* \circ \tilde{\sigma}$ has the desired properties.

QED (Lemma 5.7)

We are now ready to prove:

Thm 6 N' is ω_1 -subproper.

The proof will again stretch over several lemmas. In the following let N be viable, $d < \omega_1$, and let $\pi = \langle \pi^i \mid i < d \rangle$ be a tower for N with $N^i = N^{\pi^i}$ and $\pi(\theta_i, N^i) = \theta_i, N'$ for $i < d$. We must show that, for any finite $u \subset N^d$ and any $\bar{T} \in N'^0$, π has a $\langle \theta_i, N' \rangle, N', G$ -revision σ which coincides with π on u and is s.t. $\bar{T} \in G^0$.

We first show the existence of a somewhat simpler sort of revision:

Def By an excellent revision of π we mean a pair $\langle c, \sigma \rangle$ s.t.

- $c = \langle c^i \mid i \leq d \rangle$
- $\sigma = \langle \sigma^i \mid i \leq d \rangle$ is a revision of π
- $\langle c^d, \sigma^d \rangle$ is excellent for N
- $\langle c^i, \sigma^{i+1} \rangle$ is excellent for N^{i+1} ($i < d$)
- $\sigma \in N[c^d]$
- $\langle \sigma^{h+1} \mid h \leq i \rangle \in N^{i+1}[c^i]$ ($i < d$).

(Hence $c^d \subset \omega_2$ is good and tame for N and $c^i \subset \omega_2^{N^{i+1}}$ is good and tame for N^{i+1} .)

It follows that $\langle c^i \mid i < d \rangle \in N$ and $\langle c^h \mid h < i \rangle \in N^{i+1}$ for $i < d$.

Lemma 6.1 Let $c' \in \omega_2$ be good and tame for N .
 Let $u \in N^d$ be finite. There is $\langle c, \sigma \rangle \in N[c']$
 which is an excellent revision of π
 coinciding with π on u .

(Hence $N[c'] = N[c]$ by Lemma 5.1.)

proof. By induction on d ,

Case 1 $d = 0$. Trivial by Cor. 5.5

Case 2 $d = \beta + 1$

Let $\langle c', \sigma' \rangle$ be excellent for N, N^d with
 $\sigma' \upharpoonright u = \pi^d \upharpoonright u$. (This exists in $N[c']$ by Cor. 5.5.)

Set $\tilde{u} = (\pi^d)^{-1} \upharpoonright u$, $\tilde{\pi} = \langle \pi^{i^d} \mid i \leq \beta \rangle$.

There exists $c' \in H_{\omega_1}$ which is good and
 tame for N^d (e.g. an \mathbb{N}^d -generic
 sequence). Hence, by the induction
 hypothesis there is $\langle \tilde{c}, \tilde{\sigma} \rangle \in H_{\omega_1}$ which
 is an excellent revision of $\tilde{\pi}$ coinciding
 with $\tilde{\pi}$ on \tilde{u} .

$$c^i = \begin{cases} c'' & \text{if } i = d \\ \tilde{c}^i & \text{if } i < d \end{cases} ; \quad \sigma^i = \begin{cases} \sigma' & \text{if } i = d \\ \sigma' \circ \tilde{\sigma}^i & \text{if } i < d. \end{cases}$$

$\langle c, \sigma \rangle$ has the desired properties.

QED (Case 2)

Case 3 α is a limit ordinal.

Let $\langle \alpha_i \mid i < \omega \rangle$ be monotone and cofinal in α s.t. $\alpha_0 = 0$ and $\alpha_i = \beta_i + 1$ for $i > 0$.

We successively pick $\langle c(i), \sigma(i) \rangle$ ($i < \omega$) s.t. $\langle c(i), \sigma(i) \rangle$ is an excellent revision of the tower $\pi(i) = \langle \pi^v, \alpha_{i+1} \mid \alpha_i \leq v < \alpha_{i+1} \rangle$ for $N^{\alpha_{i+1}}$. Setting

$\tilde{\pi}(i) = \langle \pi^v, \alpha_{i+1} \mid v \leq \beta_{i+1} \rangle$ and defining

$\langle \tilde{c}(i), \tilde{\sigma}(i) \rangle$ by:

$$\langle \tilde{c}(0), \tilde{\sigma}(0) \rangle = \langle c(0), \sigma(0) \rangle$$

$$\tilde{c}(i+1)^v = \begin{cases} \tilde{c}(i+1)^v & \text{if } \alpha_i \leq v < \alpha_{i+1} \\ \tilde{c}(i)^v & \text{if } v < \alpha_i \end{cases}$$

$$\tilde{\sigma}(i+1)^v = \begin{cases} \sigma(i+1)^v & \text{if } \alpha_i \leq v < \alpha_{i+1} \\ \sigma(i+1)^{\alpha_i} \circ \tilde{\sigma}(i)^v & \text{if } v < \alpha_i \end{cases}$$

we see that $\langle \tilde{c}(i), \tilde{\sigma}(i) \rangle$ is an excellent revision of $\tilde{\pi}(i)$ for $i < \omega$.

While doing this construction we ensure that $\sigma(i)$ coincides with $\pi(i)$ on u_i , where $u_i \subset N^{\beta_{i+1}}$ is a finite set chosen so as to satisfy the conditions:

- $U_h \subset U_i$ for $h \leq i$

- $(\pi^{\beta_{i+1}, d})^{-1} U \subset U_i$

- Letting $\langle x_i \mid i < \omega \rangle$ be a fixed enumeration of N^d we have:

- $(\pi^{\beta_{i+1}, d})^{-1}(x_h) \in U_i$

for $h < i$, whenever defined.

- Letting $\langle \langle z_i, \nu_i \rangle \mid i < \omega \rangle$ be a fixed enumeration of $\{ \langle z, \nu \rangle \mid \nu < d \wedge z \in N^d \}$,

we have: At $h < i$, $\nu_h < d_i$, then

$$\pi^{d_i, \beta_{i+1}}(\tilde{\sigma}^{(i-1)\nu_h}(z_h)) \in U_i.$$

It is clear that we can set:

$$\tilde{c}^\nu = \tilde{c}^{(i)\nu} \text{ for } \nu < d_{i+1}$$

$$\tilde{\sigma}^{\nu\tau} = \tilde{\sigma}^{(i)\nu\tau} \text{ for } \nu \leq \tau < d_{i+1}.$$

$\langle \tilde{\sigma}^{\nu\tau} \mid \nu \leq \tau < d \rangle$ is then a commutative, continuous system. By our construction we have ensured that:

- At $x \in N^d$, there are $\nu < d$, $\bar{x} \in N^\nu$

s.t. $x = \pi^{\nu d}(\bar{x})$ and

$$\pi^{\nu\tau}(\bar{x}) = \tilde{\sigma}^{\nu\tau}(\bar{x}) \text{ for } \nu \leq \tau < d,$$

- At $x \in N^r$, $r < \alpha$, there is $\tau \geq r$ and $t \leq \tau < \alpha$ and for all ξ , if $\tau \leq \xi < \alpha$, then $\tilde{\sigma}^{\tau \xi}(x) = \pi^{\tau \xi}(\tilde{\sigma}^{\tau \tau}(x))$.

Arguing as before (cf. the proof of §2 Theorem 5), it follows that

$\langle N^r \mid r < \alpha \rangle$, $\langle \tilde{\sigma}^{\tau \tau} \mid \tau \leq \tau < \alpha \rangle$ has a unique direct limit of the form:

$$N^\alpha, \langle \tilde{\sigma}^{\tau \tau} \mid \tau < \alpha \rangle.$$

Now let $\langle c^*, \sigma^* \rangle \in N[c']$ be excellent.

Set:

$$c^r = \begin{cases} c^* & \text{if } r = \alpha \\ \tilde{c}^r & \text{if } r < \alpha \end{cases}$$

$$\sigma^r = \begin{cases} \sigma^* & \text{if } r = \alpha \\ \sigma^* \cdot \tilde{\sigma}^r & \text{if } r < \alpha. \end{cases}$$

Then $\langle c, \sigma \rangle$ has the desired properties,

QED (Lemma 6.1)

We now define:

Def $\langle G, \sigma \rangle$ is a superb revision of π iff the following hold:

- σ is a revision of π
- G is \mathbb{N}' -generic over N
- $G^i = \sigma^{-1} \text{'' } G$ is \mathbb{N}'^i -generic over N^i ($i \leq \alpha$)
- Set: $C^\alpha = C_G$; $C^i = C_{G^{i+1}}$ for $i < \alpha$. Then $\langle C, \sigma \rangle$ is an excellent revision of π .

Remark If $\langle G, \sigma \rangle$ is a superb revision of π , then σ is a $\langle \theta, \mathbb{N}' \rangle, \mathbb{N}', G$ -revision of π . Hence the assertion of Thm 6 will follow from the conjunction of Lemma 6.1 and:

Lemma 6.2 Let $\langle C, \sigma \rangle$ be an excellent revision of π . Set $\bar{C}^\alpha = (\sigma^\alpha)^{-1} \text{'' } C^\alpha$, $\bar{C}^i = (\sigma^{i, i+1})^{-1} \text{'' } C^{i+1}$ ($i < \alpha$). Define $A_{\bar{C}^\alpha}$ relative to N and $A_{\bar{C}^i}$ relative to N^{i+1} for $i < \alpha$. Let $u \subset N^\alpha$ be finite. Let $\bar{T} \in \mathbb{N}'^0$. Then there exist $T \in \mathbb{N}'$, $\bar{G} \in A_{\bar{C}^0}$ s.t.

- $\text{stm}(T) = \sigma^0(\text{stm}(\bar{T}))$
- If $G \exists T$ is \mathbb{N}' -generic over N , then there is $\sigma' \in \mathcal{N}[G]$ s.t. $\langle G, \sigma' \rangle$ is a superb revision of π s.t. $G^0 = \bar{G}$, and σ' coincides with σ on u .

We prove Lemma 6.2 by induction on d .

Case 1 $d=0$.

Then $c = \langle c^0 \rangle$, $\sigma = \langle \sigma^0 \rangle$ where $\langle c^0, \sigma^0 \rangle$ is excellent for N . We know that

$A = \{G \in A_{\bar{c}^0} \mid \bar{T} \in G\}$ is dispersed at $m = |\text{ratm}(\bar{T})|$. The conclusion follows by Lemma 5.7.

Case 2 $d = \beta + 1$.

By the induction hypothesis there are $\bar{G} \in A_{\bar{c}^0}$,

$\tilde{T} \in \mathbb{N}'^d$ s.t. $\bar{T} \in \bar{G}$ and

- $\text{ratm}(\tilde{T}) = \sigma^{0,d}(\text{ratm}(\bar{T}))$

- $\forall \tilde{G} \ni \tilde{T}$ is \mathbb{N}'^d -generic over N^d ,

there is $\tilde{\sigma}' \in N^d[\tilde{G}]$ s.t. $\langle \tilde{c}, \tilde{\sigma}' \rangle$ is a superb-revision of $\tilde{\pi} = \langle \pi^{i,d} \mid i \leq \beta \rangle$

and $\tilde{\sigma}'$ coincides with $\tilde{\sigma} = \langle \sigma^{i,d} \mid i \leq \beta \rangle$ on $\tilde{u} = (\sigma^{\beta,d})^{-1} \llbracket u$.

But $\tilde{A} = \{G \in A_{\tilde{c}} \mid \tilde{T} \in G\}$ is dispersed at $m = |\text{ratm}(\tilde{T})|$. Since $\langle c^d, \sigma^d \rangle$ is excellent for N , there are $\tilde{G} \in \tilde{A}$, $T \in \mathbb{N}'$ s.t.

- $\text{ratm}(T) = \sigma^d(\text{ratm}(\tilde{T})) = \sigma^0(\text{ratm}(\bar{T}))$

- $\forall G \ni T$ is \mathbb{N}' -generic over N , then there

is $\sigma' \in N[G]$ s.t. $\sigma': N^d \prec N$; $\sigma'(0^d, \mathbb{N}'^d) = \theta_1 \mathbb{N}'$;

$\sigma' \upharpoonright u = \sigma^d \upharpoonright u$, $C_{\omega_1}^N(\text{rng } \sigma') = C_{\omega_1}^N(\text{rng } \sigma^d) = C_{\omega_1}^N(\text{rng } \tilde{\pi})$, and $\sigma' \llbracket \tilde{G} \subset G$.

Now let $G \ni T$ be (N') -generic over N .

Let $\sigma' \in N[G]$, $\bar{G} \in \bar{A}$ be as above. Since $\bar{G} \ni \bar{T}$, there is $\bar{\sigma}' \in N^d[\bar{G}]$, $\bar{G} \in A_{\bar{c}_0}$ as above.

$$\text{Set: } \sigma''^v = \begin{cases} \sigma' & \text{if } v = d \\ \sigma' \bar{\sigma}'^v & \text{if } v < d \end{cases}$$

Then $\langle G, \sigma'' \rangle$ is a superb revision of π and σ'' coincides with σ on u . QED (Case 2)

Case 3 d is a limit ordinal.

We know that there is $d < \omega_1$ which is good and tame for N and i.t.

$$d \in N[c^d], \quad d \in \text{rng}(\sigma^0).$$

Hence $N[c^d] = N[d]$ and we may suppose w.l.o.g. that $c^d = d$. Set:

$$d = \langle \delta_i \mid i < \omega \rangle$$

$$d^v = (\sigma^v)^{-1} \text{"} d = \langle \delta_i^v \mid i < \omega \rangle \quad (v \leq d).$$

The notion of a superb d -revision $\langle \bar{G}, \bar{\sigma} \rangle$ is defined like that of a superb revision with N^d in place of N and $\bar{\pi} = \langle \pi^{i,d} \mid i \leq d \rangle$ in place of π .

Equivalently: