§3 Doe – Subproperness

Def A completeness system is a function $\mathbb{D}$ defined on pairs $\langle N, \mathbb{B}\rangle$ s.t. $N = L_\tau^A$ (or $N = L_\tau^{A_1, \ldots, A_n}$) is a countable, full ZFC$^-$ model and $\mathbb{B} \in N$ is a complete BA in N s.t. the following hold:

(a) Each $X \in \mathbb{D}(N, \mathbb{B})$ is a set of G which is $\mathbb{B}$-generic over N;

(b) If $X \in \mathbb{D}(N, \mathbb{B})$ and $a \in \mathbb{B} \setminus \{0\}$, there is $G \in X$ s.t. $a \in G$

(c) Let $X_i \in \mathbb{D}(N, \mathbb{B})$ for $i < \omega$. Then $\bigcap_i X_i \neq \emptyset$ and, in fact, for each $a \in \mathbb{B} \setminus \{0\}$ there is $G \in \bigcap_i X_i$ s.t. $a \in G$

(d) There is a formula φ and an $s \in H_{\omega_1}$ s.t. $\mathbb{D}(N, \mathbb{B}) = \{A_u \mid u \in H_{\omega_1}\}$, where $A_u = \{G \mid G$ is $\mathbb{B}$-generic over N and $H_{\omega_1} \models \varphi[N, \mathbb{B}, G, u, s]\}$.

Note (a), (b) express Shelah's notion of "completeness system". (c) expresses "ω-completeness" and (d) expresses "simplicity" in a sense close to that of Shelah. We shall only make use of completeness systems satisfying (c), (d).

Def Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is Dee-subproper as witnessed by $\theta, \mathbb{D}$ iff $\mathbb{B} \in H_\theta$, $\mathbb{D}$ is a completeness system, and the following holds:

Let $N = L_\tau^A$ be a ZFC^- model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let $\pi : \bar{N} \prec N$, where $\bar{N}$ is countable and full, and $\pi \in N$. Let $\pi(\bar{\theta}, \bar{\mathbb{B}}) = \theta, \mathbb{B}$. Then there is $X \in \mathbb{D}(\bar{N}, \bar{\mathbb{B}})$ s.t. whenever $\bar{G} \in X$, then for all $\bar{s} \in \bar{N}$, $s = \pi(\bar{s})$, there is $b \in \mathbb{B} \setminus \{0\}$ forcing that whenever $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma \in V[G]$ s.t.

- $\sigma : \bar{N} \prec N$
- $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$
- $C_\delta^N(\operatorname{rng} \sigma) = C_\delta^N(\operatorname{rng} \pi)$
- $\sigma''\bar{G} \subset G$.

Def $\mathbb{B}$ is Dee-subproper as witnessed by θ iff it is Dee-subproper as witnessed by $\theta, \mathbb{D}$ for some $\mathbb{D}$.

Def $\mathbb{B}$ is Dee-subproper iff it is Dee-subproper as witnessed by some θ.

Def $\mathbb{B}$ is Dee-subproper as verified by θ iff it is Dee-subproper as witnessed by θ' for all $\theta' \geq \theta$.

It again turns out that if $\mathbb{B}$ is Dee-subproper, then its Dee-subproperness is verified, since if $\theta, \mathbb{D}$ witness the Dee-subproperness of $\mathbb{B}$, then so do $\theta', \mathbb{D}$ for all $\theta' > 2^{\theta}$.

We can, of course, modify our definition of "Dee-subproper as witnessed by $\theta, \mathbb{D}$" to require that $p \in rng(\pi)$ for a fixed parameter p. We then say that $\mathbb{B}$ is "Dee-subproper as witnessed by $\theta, \mathbb{D}, p$. The usual proof shows that every such $\mathbb{B}$ is fully subproper.

Clearly every Dee-subproper forcing is subproper. If, however, we restrict to forcings which are both ω_1-subproper and Dee-subproper, we get a strong iteration theorem, generalizing the corresponding theorem which Shelah proved for proper forcings:

Thm 1 Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS-iteration s.t. $\mathbb{B}_0 = 2$ and:

(a) $\mathbb{B}_{i+1} \neq \mathbb{B}_i$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1} / \dot{G}$ is ω_1-subproper and Dee-subproper)

(c) $\Vdash_{i+1} \mathrm{card}(\delta(\check{\mathbb{B}}_i)) \leq \omega_1$.

Then each $\mathbb{B}_i$ adds no reals.

proof.

Let θ be large enough that $\Vdash_i H_{\check{\theta}} \prec_{\Sigma_n} V$ for all $i < \alpha$, where n is big enough that the formulae "$V_\alpha\, x \in V_\alpha$" and "$A$ is a complete BA and there are $\langle \gamma, \mathbb{D} \rangle$ witnessing the Dee-subproperness of A" are Σ_n.

Then $\Vdash_i H_\theta^{\vee} = V_\theta^{\vee}$ and:

$\Vdash_i \check{\theta}$ verifies the Dee-subproperness of $\check{\mathbb{B}}_{i+1}/\mathring{G}$ for all $i < \alpha$.

Let $N = L_\tau^A$ be a ZFC^- model s.t. N is full, $H_\theta \subset N$, $\theta < \tau$ and $cf(\tau) > \omega$. (There are many such N of cardinality 2^θ.) Let $\pi^\circ : N^\circ \prec N$, where N° is countable and full and $\pi^\circ(\theta^\circ, \mathbb{B}^\circ, \alpha^\circ) = \theta, \mathbb{B}, \alpha$. By our assumption on N we can extend π° to a tower $\pi = \langle \pi^i \mid i \leq \beta \rangle$ of arbitrary countable height β s.t. $\pi^i : N^i \prec N$ and $\pi^i(\theta^i, \mathbb{B}^i, \alpha^i) = \theta, \mathbb{B}, \alpha$.

Hitherto we worked only with pretowers indexed by intervals $[\nu, \beta]$. Now, however, we shall also use sets of the form $\Gamma_\nu = \{0\} \cup [\nu+1, \beta]$ to index pretowers. This necessitates some changes in our definitions. The definition of "Γ_ν-tower" and "Γ_ν-pretower" is obvious. In particular, $\pi \upharpoonright \Gamma_\nu$ is a tower.

Def By a Γ_ν - reformation of π we mean a Γ_ν - pretower σ s.t.

(a) $\sigma \upharpoonright [\nu+1, \beta]$ is a revision of $\pi \upharpoonright [\nu+1, \beta]$

(b) $rng(\sigma^{0,\nu+1}) \subset \bigcup rng(\pi^{\nu,\nu+1})$.

Note This is weaker than saying that σ is a revision of $\pi \upharpoonright \Gamma_\nu$, since that would require that
$\bigcup rng(\sigma^{0,\nu+1}) = \bigcup rng(\pi^{0,\nu+1})$.
Even if σ were a revision of π, it does not follow that $\sigma \upharpoonright \Gamma_\nu$ is a revision of $\pi \upharpoonright \Gamma_\nu$, but $\sigma \upharpoonright \Gamma_\nu$ is a reformation of π.

Note π is a Γ_0 reformation of itself.

We generally write "reformation" to mean "reformation of π".

Def Let σ be a Γ_i reformation and σ' a Γ_j reformation, where $i \leq j < \alpha^0$. σ' respects σ at x iff σ' respects $\sigma \upharpoonright \Gamma_j$ at x.
(We then also say: σ' is an x-reformation of σ.)

Def Let σ, σ', i, j be as above. Let $u \in N^\beta$.
σ' coincides with σ on u iff
σ' coincides with $\sigma \upharpoonright \Gamma_i$ on u.

Def Let σ, σ', i, j be as above.
σ' is strong over σ iff
(a) σ' respects σ at $\langle \theta, \mathbb{B}, \sigma^0(i), \sigma^0(j) \rangle$
(b) $\sigma' \upharpoonright [j+1, \beta]$ is a $\langle \theta, \mathbb{B}, \sigma^0(i), \sigma^0(j) \rangle, \mathbb{B}_{\sigma^0(i)}$ - revision of $\sigma \upharpoonright [j+1, \beta]$
(c) $C^N_\delta(\mathrm{rng}\, \sigma^0) = C^N_\delta(\mathrm{rng}\, \sigma'^0)$, where $\delta = \delta(\mathbb{B}_{\sigma^0(j)})$
(d) $C^{N^h}_{\delta_h}(\mathrm{rng}\, \sigma^{0h}) = C^{N^h}_{\delta_h}(\mathrm{rng}\, \sigma'^{0h})$, where $\delta_h = \delta(\mathbb{B}^h_{\sigma^{0h}(j)})$, for $h \in \Gamma_i$

Note This is stronger than saying that σ' is a $\langle \theta, \mathbb{B}, \sigma^0(i), \sigma^0(j) \rangle, \mathbb{B}_{\sigma^0(j)}$ - revision of $\sigma \upharpoonright \Gamma_i$.

Def $\langle \sigma, G \rangle$ is a Γ_i - witness iff σ is a Γ_i - reformation, G is $\mathbb{B}_{\sigma^0(i)}$ - generic over N and:
(a) $G^h = (\sigma^h)^{-1}{}'' G$ is $\mathbb{B}^h_{\sigma^{0h}(i)}$ - generic over N^h for $h \in \Gamma_i$
(b) $\sigma \in N[G]$ and $\langle \sigma^{h, j+1} \mid h \in \Gamma^j_i \rangle \in N^{j+1}[G^{j+1}]$ for $i \leq j < \beta$, where $\Gamma^j_i = \Gamma_i \cap (j+1)$.

<u>Note</u> $\Gamma_i^i = \{0\}$; $\Gamma_i^j = \{0\} \cup [i+1, j]$ for $j > i$

<u>Def</u> Let $\overline{G} \subset \mathbb{B}_i^0$. $\langle \sigma, G \rangle$ is a <u>witness for</u> $\overline{G}$ iff $\langle \sigma, G \rangle$ is a witness and $\overline{G} = G^0$.

<u>Def</u> Let $\langle \sigma, G \rangle$ be an Γ_i - witness and $\langle \sigma', G' \rangle$ an Γ_j - witness, where $i \leq j$. $\langle \sigma', G' \rangle$ is <u>strong over</u> $\langle \sigma, G \rangle$ iff

(a) σ' is strong over σ and

(b) $G = G' \cap \mathbb{B}_{\sigma^0(i)}$

$G^h = G'^h \wedge \mathbb{B}_{\sigma^{0h}(i)}$ for $h \in \Gamma_i$,

where $G^h = (\sigma^h)^{-1}{}''G$, $G'^h = (\sigma'^h)^{-1}{}''G'$

(In other words, $\langle \sigma', G' \rangle$ coheres with $\langle \sigma \upharpoonright \Gamma_i, G \rangle$ wrt. $\mathbb{B}_{\sigma^0(i)}$.)

<u>Note</u> Since $\mathbb{B}_0 = 2 = \mathbb{B}_0^h$ for all h, we know that $\langle \pi, \{1\} \rangle$ is a Γ_0 - witness for $\{1\}$.

<u>Def</u> Let $i < \alpha^0$. $\overline{G} \subset \mathbb{B}_i^0$ is <u>good for</u> i iff for all $l \leq i$, $\overline{G}_l = \overline{G} \cap \mathbb{B}_l^0$ is $\mathbb{B}_l^0$ - generic over N^0 and $\overline{G}_l \in N^{l+1}$.

Def Let $i < \alpha^0$. $\bar{G} \subset \mathbb{B}^0_i$ is superb for i iff $\bar{G}$ is good for i and whenever $l \leq j \leq i$, $\langle \sigma, G \rangle$ witnesses $\bar{G}_l$ and $u \subset N^\beta$ is finite, then there is $\langle \sigma', G' \rangle$ witnessing $\bar{G}_j$ s.t. $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$ and σ' coincides with σ on u.

Note This definition appears to be nonsensical, since it makes a statement about generic objects which do not exist in V. A correct formulation would say that the statement holds in $Coll(\alpha, \omega)$ for $\alpha \geq \bar{\bar{N}}$.

Note Taking $l = 0$ in the above definition, it follows that $\bar{G}_j$ has a witness for each $j \leq i$.

Def i is superb iff for all $a \in \mathbb{B}^0_i \setminus \{0\}$ there is $\bar{G} \ni a$ which is superb for i.

Lemma 1.1 If $i < \alpha^0$ is superb, then $\mathbb{B}^0_i$ adds no reals to N^0

proof

Suppose not. There is then $a \in \mathbb{B}^0_i \setminus \{0\}$, $t \in N^0 \mathbb{B}^0_i$ s.t. $a \Vdash (t \subset \check{\omega} \wedge t \notin \check{V})$. Let $\bar{G} \ni a$ be superb for i. Let $\langle \sigma, G \rangle$ witness $\bar{G}$. Let $\sigma^* \supset \sigma$ s.t. $\sigma^* : N^0[\bar{G}] \prec N[G]$; $\sigma^*(\bar{G}) = G$. Then $t^{\bar{G}} \quad N$, since $\bar{G} \in N^{i+1} \subset N$. But since $t^{\bar{G}} \notin N^0$, we have $t^G = \sigma^*(t^{\bar{G}}) \notin N$. Contr!

QED (1.1)

Hence it suffices to show that each $i < \alpha^0$ is superb, since then:

$N^0 \models$ No $\mathbb{B}^0_i$ adds new reals,

hence $N \models$ No $\mathbb{B}_i$ adds new reals,

since $\pi^0 : N^0 \prec N$, $\pi^0(\mathbb{B}^0) = \mathbb{B}$.

We shall actually prove:

<u>Main Claim</u> Let $l \leq i < \alpha^0$. Let $\bar{G}$ be superb for l. Let $a \in \mathbb{B}^0_i$ s.t. $h_l(a) \in \bar{G}$. Then there is $\bar{G}' \ni a$ s.t. $\bar{G} \subset \bar{G}'$ and $\bar{G}'$ is superb for i.

<u>Remark</u> Taking $l = 0$, it follows that i is superb. Hence the main claim proves the Theorem. We shall prove the Main Claim by induction on i. First, however, we prove some preliminary lemmas on superbness.

Lemma 1.2 Let $i < \alpha^0$. The following are equivalent:

(a) $\bar{G}$ is superb for i

(b) Let $l \le j \le i$. Let $l^* \le j^* < \alpha$ and let $a \in \mathbb{B}_{l^*}$, $\dot{\sigma}, \dot{u} \in V^{\mathbb{B}_{l^*}}$ s.t. whenever $G \ni a$ is $\mathbb{B}_{l^*}$-generic and $\sigma = \dot{\sigma}^G$, $u = \dot{u}^G$, then $\langle \sigma, G \rangle$ witnesses $\bar{G}_l$ and $u \subset N^\beta$ is finite and $\sigma^0(j) = j^*$. There $G' \subset \mathbb{B}_{j^*}$, $\sigma' \in V[G']$ s.t. $a \in G'$, $\langle \sigma', G' \rangle$ witnesses $\bar{G}_j$, and is strong over $\langle \sigma, G \rangle$, where $\sigma = \dot{\sigma}^G$ and coinciding with σ on u, where $G = G' \cap \mathbb{B}_{l^*}$.

(c) Let l, j, l^*, j^*, $\dot{\sigma}, \dot{u}$ be as above. There is $b \in \mathbb{B}_{j^*}$ s.t. $h_{l^*}(b) = a$ and whenever $G' \ni b$ is $\mathbb{B}_{j^*}$-generic, there is $\sigma' \in V[G']$ as in (b)

(d) Let $l \le j \le i$, $l^* \le j^* < \alpha$ and let $G \subset \mathbb{B}_{l^*}$, $\sigma \in V[G]$ be s.t. $\langle \sigma, G \rangle$ witnesses $\bar{G}_l$. Let $u \subset N^\beta$ be finite. Let $\sigma^0(j) = j^*$. There is $b' \in \mathbb{B}_{j^*}$ s.t. $h_{l^*}(b') = 1$ and whenever $G' \ni b'$ is $\mathbb{B}_{j^*}$-generic with $G' \supset G$, then there is $\sigma' \in V[G']$ s.t. $\langle \sigma', G' \rangle$ witnesses $\bar{G}_j$, $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$ and σ' coincides with σ on u.

proof of 1.2

(a) → (b) Let $G \ni a$ be $\mathbb{B}_{\ell*}$ - generic. Let $\sigma = \dot\sigma^G$, $u = \dot u^G$. By (a) there exist $\langle \sigma', G' \rangle$ with the desired properties.

(b) → (c)

For any such a there is $\langle \sigma', G' \rangle$ with the stated properties, by (b). Hence there is $b_a \in G'$ which forces the existence of σ' with these properties. Since $b_a, a \in G'$, we can assume w.l.o.g $b_a \subset a$. Now let A be a maximal antichain in:

$\{ h_{\ell*}(b_{a'}) \mid a' \subset a, a' \neq 0 \}$

For $c \in A$ pick $b(c) = b_{a'}$ s.t. $c = h_\ell(b_{a'})$.

Set: $b = \bigcup \{ b(c) \mid c \in A \}$.

Then $h_{\ell*}(b) = \bigcup A = a$ and b has the desired property. QED (b)→(c)

(c) → (d) Let $\langle \sigma, G \rangle$ be given. There is then $a \in \mathbb{B}_{\ell*}$, $\dot\sigma \in V^{\mathbb{B}_{\ell*}}$, $\dot u \in V^{\mathbb{B}_{\ell*}}$ s.t. $\sigma = \dot\sigma^G$, $u = \dot u^G$, and a forces $\langle \sigma, \ \rangle$ to witness $\overline{G}_\ell$ and u to be finite and $\sigma(\dot i) = \dot j^*$. Let b be as in (c). Let $b' = b \cup \neg a$. Then b' has the desired property.

QED (c) → (d)

(d) → (a) Let $\langle G, \sigma \rangle$ witness $\bar{G}_\ell$ with $\sigma^\circ(\ell) = \ell^\nabla$ and $\sigma^\circ(j) = j^*$. Let b' be as in (d). Extend G to a $\mathbb{B}_{j^*}$ - generic $G' \ni b'$. The conclusion is immediate. QED (Lemma 1.2)

All of these definitions and results depend on an arbitrarily chosen $\beta \geq \alpha^\circ$. Since we have been dealing with a single $i < \alpha^\circ$, we could replace β by any $\nu \leq \beta$ s.t. $\nu \geq i$, thus obtaining the notion of "ν-superbness". It will turn out that ν-superbness is equivalent to β-superbness whenever $\nu \geq i$. Since we are including the case $\nu = i$, however, we must slightly revise our definitions. Recall that $\Gamma_i^\nu = \Gamma_i \cap (\nu+1)$.

The definitions of "Γ_i^ν - tower" and "Γ_i^ν - pretower" are obvious. In particular, $\pi \upharpoonright \Gamma_i^\nu$ is a tower.

The definition of "reformation" must be modified as follows:

Def Let $i < \alpha^0$, $i \leq \nu \leq \beta$. By a Γ_i^ν-reformation of $\pi \upharpoonright (\nu+1)$ we mean a Γ_i^ν-pretower σ s.t.

(a) $\sigma \upharpoonright [i+1, \nu]$ is a revision of $\pi \upharpoonright [i+1, \nu]$ if $\nu > i$.

(b) $\mathrm{rng}\, \sigma^{0, i+1} \subset \bigcup \mathrm{rng}\, (\pi^{i, i+1})$ if $\nu > i$

(c) $\mathrm{rng}\, \sigma^0 \subset \bigcup \mathrm{rng}\, \pi^i$ if $\nu = i$.

The definition of "σ' respects σ at x" is virtually unchanged. "Coincidence" must be redefined as follows:

Def Let σ, σ' be reformations of $\pi \upharpoonright (\nu+1)$, where σ is a Γ_i^ν-pretower, σ' is a Γ_j^ν-pretower, and $i \leq j$.
Let $u \subset N^{\hat{\nu}}$, where $\hat{\nu} = \max \Gamma_j^\nu$.
σ' coincides with σ on u iff σ' coincides with $\sigma \upharpoonright \Gamma_j^\nu$ on u.

(Note $\hat{\nu} = \nu$ if $\nu > i$; $\hat{\nu} = 0$ if $\nu = i$.)

The definition of "strong must be altered

The definition of "strong over" is as before with $\Gamma_i^\nu, \Gamma_j^\nu$ in place of Γ_i, Γ_j (except that (b) is understood as holding vacuously if $\nu = j$).

The definition of "witness" is also virtually unchanged, as is the def. of "$\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$".

The definition of "superb" requires a small subtle change:

Def Let $i < \alpha^0$, $i \leq \nu \leq \beta$. $\bar{G} \subset \mathbb{B}_i^0$ is ν-__superb for i__ iff $\bar{G}$ is good and whenever $l \leq j \leq i$, $u \subset N^{\hat{\nu}}$ is finite, $\langle \sigma, G \rangle$ witnesses $\bar{G}_l$ and $u \subset N^{\hat{\nu}}$ ... then there is $\langle \sigma', G' \rangle$ witnessing $\bar{G}_j$ s.t. $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$ and σ' coincides with σ on u, where $\hat{\nu} = \max \Gamma_j^\nu$.

(Hence $\hat{\nu} = \nu$ unless $\nu = j = i$ and $\hat{\nu} = 0$.)

In (b), (c) of Lemma 1.2 we must similarly replace "$u \subset N^\beta$" by "$u \subset N^{\hat{\nu}}$". The proof of Lemma 1.2 then goes through as before.

<u>Lemma 1.3</u> Let $\nu \leq \beta$. The set:

$\{\langle \bar{G}, i\rangle \mid i \leq \nu, i < \alpha^0$ and $\bar{G}$ is ν-suspect for $i\}$

is uniformly N-definable in the parameters $\theta, \mathbb{B}, \pi \upharpoonright (\nu+1)$.

proof (sketch).
We first note that, since $\mathbb{B} \in H_\theta \subset N$, the various terms $\dot{\sigma}$, $\dot{\imath}$ etc. involved in the definition can be taken as lying in N. However, the definition will involve statements of the form:

$$a \Vdash_{\mathbb{B}_\ell} \dot{\sigma}^k : \check{N}^k \prec \check{N},$$

which are not expressible in N. We note that in all such cases it will also be forced that $a \Vdash \operatorname{rng} \dot{\sigma}^k \subset \check{N}^*$, where $N^* = \bigcup \pi^\nu$. But $N^* \in N$ and $N^* \prec N$, so we can replace $\check{N}$ by $\check{N}^*$ in the above statement.

QED (Lemma 1.3)

But $\pi^{\nu+1} : N^{\nu+1} \prec N$ and $\sigma^{\nu+1}(\tilde{\pi}) = \pi \upharpoonright (\nu+1)$, where

$$\tilde{\pi} = \langle \pi^{h,\nu+1} \mid h \leq \nu\rangle.$$

Hence:

<u>Corollary 1.4</u> Let $\nu < \beta$. The set

$\{\langle \bar{G}, i\rangle \mid i \leq \nu, i < \alpha^0$ and $\bar{G}$ is ν-superb for $i\}$

is uniformly $N^{\nu+1}$- definable in the parameters $\theta^{\nu+1}$, $\mathbb{B}^{\nu+1}$, $\tilde{\pi}$, where $\tilde{\pi} = \langle \pi^{h,\nu+1} \mid h \leq \nu\rangle$.

Using this we prove:

<u>Lemma 1.5</u> Let $\nu < \beta$. Let $i < \alpha^0$ s.t. $i \leq \nu$. Let $\bar{G} \subset \mathbb{B}^0_i$. Then $\bar{G}$ is superb for i iff iff $\bar{G}$ is ν- superb for i.

prf.

($\rightarrow$) By Cor 1.4 it suffices to show that $\bar{G}$ is ν- superb in $N^{\nu+1}$. We use criterion (b) in Lemma 1.2. Let $l \leq j \leq i$, $l^* \leq j^*$

Let $a \in \mathbb{B}^{\nu+1}_{l^*}$, $\dot{\sigma}, \dot{u} \in (N^\nu)^{\mathbb{B}^{\nu+1}_{l^*}}$ s.t.

$G \ni a$ is $\mathbb{B}^{\nu+1}_{l^*}$- generic over $N^{\nu+1}$, then, letting $\sigma = \dot{\sigma}^G$, $u = \dot{u}^G$; $u \subset N^{\hat{\nu}}$ where $\hat{\nu} = \max \Gamma^\nu_j$ is finite and $\langle \sigma, G\rangle$ witnesses $\bar{G}_l$, and $\sigma^0(l) = l^*$, $\sigma^0(j) = j^*$.

<u>Claim</u> There is $G' \subset \mathbb{B}^{\nu+1}_{j^*}$ s.t. G' $\mathbb{B}^{\nu+1}_{j^*}$ - generic over $N^{\nu+1}$ and there is $\sigma' \in N^{\nu+1}[G']$ s.t. $\langle \sigma', G'\rangle$ witnesses $\bar{G}_j$ and, letting $G = G' \cap \mathbb{B}^{\nu+1}_{i^*}$, $\sigma = \dot{\sigma}^G$, $u = \dot{u}^G$, we have: $\langle \sigma', G'\rangle$ is strong over $\langle \sigma, G\rangle$ and σ' coincides with σ on u.

Define $\tilde{\ell} < \tilde{j} < \alpha$ by: $\tilde{\ell} = \pi^{\nu+1}(\ell^*)$, $\tilde{j} = \pi^{\nu+1}(j')$.

By β-subproperness there is $\hat{\tilde{a}} \in \mathbb{B}_{\tilde{\ell}}$ and $\ddot{\sigma} \in N^{\mathbb{B}_{\tilde{\ell}}}$ s.t. whenever $\tilde{G} \ni \hat{\tilde{a}}$ is $\mathbb{B}_{\tilde{\ell}}$-generic and $\tilde{\sigma} = \ddot{\sigma}^{\tilde{G}}$, then:

- $\tilde{\sigma}$ is a $\langle \theta, \mathbb{B}, \tilde{\ell}, \tilde{j} \rangle, \mathbb{B}_{\tilde{\ell}}, \tilde{G}$-revision of $\pi \upharpoonright [\nu+1, \beta]$
- $a \in \tilde{G}^{\nu+1}$
- $\tilde{\sigma}$ coincides with $\pi \upharpoonright [\nu+1, \beta]$ on $\tilde{u}$, where $\tilde{u} = \pi^{\nu+1,\beta} {}'' \sigma^{\hat{\nu}} {}'' u$.

Extend the domain of $\tilde{\sigma}$ to Γ_{ℓ} by setting: $\tilde{\sigma}^h = \tilde{\sigma}^{\nu+1} \circ \sigma^h$ for $h \in \Gamma_{\ell}^{\nu+1}$. Then $\langle \tilde{\sigma}, \tilde{G} \rangle$ witnesses $\overline{G}_{\ell}$.

By superbness there is $\langle \tilde{\sigma}', \tilde{G}' \rangle$ s.t. $\langle \tilde{\sigma}', \tilde{G}' \rangle$ witnesses $\overline{G}_j$ and is strong over $\langle \tilde{\sigma}, \tilde{G} \rangle$ and $\tilde{\sigma}'$ coincides with $\tilde{\sigma}$ on $\tilde{u}$. But then $\langle \sigma', G' \rangle$ has the desired properties, where $\sigma' = (\tilde{\sigma}')^{\nu+1}$, $G' = (\tilde{G}')^{\nu+1}$.

QED $(\rightarrow)$

($\leftarrow$) Let $\langle G, \sigma \rangle$ witness $\bar{G}_\ell$, where $\sigma^0(\ell) = \ell^*$, $\sigma^0(j) = j^*$. Let $u \subset N^\beta$ be finite.

<u>Claim</u> There are G', σ' s.t. G' is $\mathbb{B}_{j^*}$ - generic, $\sigma' \in N[G']$, $\langle \sigma', G' \rangle$ witness $\bar{G}_j$ and is strong over $\langle \sigma, G \rangle$. Moreover σ' coincides with σ on u.

prf.

Set: $\tilde{G} = G^{\nu+1}$, $\tilde{\sigma} = \langle \sigma^{h, \nu+1} \mid h \in \Gamma_\ell^{\nu+1} \rangle$

Then $\langle \tilde{G}, \tilde{\sigma} \rangle$ witnesses $\bar{G}_\ell$ in the sense of $N^{\nu+1}$. Set $\tilde{u} = (\sigma^{\hat{\nu}})^{-1} {}'' u$, where $\hat{\nu} = \max \Gamma_j^\nu$. We apply Lemma 1.2 (d) to $\langle \tilde{G}, \tilde{\sigma} \rangle$. Let $\tilde{\sigma}^0(\ell) = \tilde{\ell}$, $\tilde{\sigma}^0(j) = \tilde{j}$. There is $\hat{b} \in \mathbb{B}_{\tilde{j}}^{\nu+1}$ s.t. whenever $\tilde{G}' \ni \tilde{b}'$ is $\mathbb{B}_{\tilde{j}}^{\nu+1}$ - generic over $N^{\nu+1}$ and $\tilde{G}' \supset \tilde{G}$, there is $\tilde{\sigma}' \in N^{\nu+1}[\tilde{G}']$ s.t. $\langle \tilde{\sigma}', \tilde{G}' \rangle$ witness $\bar{G}_j$, is strong over $\langle \tilde{\sigma}, \tilde{G} \rangle$ and $\tilde{\sigma}'$ coincides with $\tilde{\sigma}$ on $\tilde{u}$.

Moreover $h_{\tilde{\ell}}(\tilde{b}') = 1$.

<u>Subclaim</u> There is $\langle \sigma', G' \rangle$ s.t.

(a) σ' is a $\langle \theta, \mathbb{B}, i^*, j^* \rangle$, $\mathbb{B}_{j^*}$, G - revision of $\sigma \restriction [j+1, \beta]$

(b) σ' coheres with $\sigma \restriction [\nu+1, \beta]$ wrt. $\mathbb{B}_{\ell^*}$

(c) σ' coincides with $\sigma \restriction [\nu+1, \beta]$ on u.

Moreover:

(d) $C^N_\delta(\mathrm{rng}(\sigma')^{\nu+1} \circ \tilde{\sigma}^0) = C^N_\delta(\mathrm{rng}\,\sigma^0)$

where $\delta = \delta(\mathbb{B}_{j^*})$

(e) $C^{N^h}_{\delta_h}(\mathrm{rng}(\sigma')^{\nu+1,h} \circ \tilde{\sigma}'^0) = C^{N^h}_{\delta_h}(\mathrm{rng}\,\sigma^{0,h})$

for $\nu < h \leq \beta$, where $\delta_h = \delta(\sigma^{0h}(\mathbb{B}_j))$.

proof

We first note that for any $t \in N^\beta$, there is $\langle \sigma', G' \rangle$ satisfying (a) – (c) with the additional property that σ' coincides with $\sigma \upharpoonright [\nu+1, \beta]$ at t. This follows by §2 Lemma 2. (Let $a \in G$, $\dot{\sigma} \in N^{\mathbb{B}_{\ell^*}}$ s.t. a forces $\sigma = \dot{\sigma}^G$ to be s.t. $\langle G, \sigma \rangle$ is a witness for $\bar{G}_\ell$ and $\sigma^0(\ell) = \ell^*$, $\sigma^0(j) = j^*$; There is $a' \in \mathbb{B}_{j^*}$ s.t. $h_{\ell^*}(a') = a$ and a' forces that if $G' \ni a$ is $\mathbb{B}_{j^*}$ - generic, there is $\sigma' \in N^{\mathbb{B}_{j^*}}$ satisfying (a) – (c) and coinciding with $\sigma \upharpoonright [\nu+1, \beta]$ at t. Let $G' \supset G$ be Set $b = a' \cup (1 \setminus a)$. Let $G' \supset G$ be $\mathbb{B}_{j^*}$ - generic s.t. $b \in G'$. Then $a' \in G'$ and the conclusion follows.)

Note that $\bar{c} = C^{N^{\nu+1}}_{\delta_{\nu+1}}(\mathrm{rng}\,\sigma^{0,\nu+1}) \in$ $\in N^{\nu+1}[G^{\nu+1}]$ and there is $\bar{t} \in N^{\nu+1}\,\mathbb{B}^{\nu+1}_{\hat{\ell}}$ s.t. $\bar{t}^{G^{\nu+1}} = \bar{c}$. ~~Let~~ σ' coincide with $\sigma \upharpoonright [\nu+1, \beta]$ on $t = \sigma^{\nu+1}(\bar{t})$.

hen, letting $c = C^N_\sigma(\mathrm{rng}\,\sigma^0)$, we have:

(1) $c = \sigma^{*0}(\bar{c}) = \sigma^{*0}(\bar{t}^{\bar{G}^{\nu+1}}) = t^G$

where σ^* is the canonical completion of σ. However:

(2) $\bar{c} = C^{N^{\nu+1}}_{\sigma_{\nu+1}}(\mathrm{rng}\,\tilde{\sigma}'^0)$

Letting σ'^* be the canonical completion of σ' we have:

(3) $(\sigma'^*)^{\nu+1}(\bar{c}) = C^N_\sigma(\mathrm{rng}((\sigma'^*)^{\nu+1}(\tilde{\sigma}'^0))) =$

$C^N_\sigma(\mathrm{rng}(\sigma'^{\nu+1} \circ \tilde{\sigma}'^0))$

But:

(4) $(\sigma'^*)^{\nu+1}(\bar{c}) = (\sigma'^*)^{\nu+1}(\bar{t}^{G'^{\nu+1} \cap \mathbb{B}^{\nu+1}_\ell}) =$

$= \sigma'^{\nu+1}(\bar{t})^{G' \cap \mathbb{B}_{\ell\nu}} = t^G = c$.

This proves (d). The proof of (e) is entirely similar. QED (Subclaim)

Extend the domain of σ' to Γ_j by setting $\sigma'^h = \sigma'^{\nu+1}\tilde{\sigma}'^h$ if $h \in \Gamma_j^\nu$. $\langle \sigma', G' \rangle$ is easily seen to have the desired properties. QED (Lemma 1.5)

Combining Lemmas 1.4 and 1.5 we easily get:

Lemma 1.6

(a) $\{\langle \bar{G}, i\rangle \mid i < \alpha^0 \wedge \bar{G}$ is superb for $i\}$ is uniformly H_{ω_1} definable in the parameter $\langle \pi^{h\beta} \mid h < \beta\rangle$.

(b) $\{\langle \bar{G}, i\rangle \mid i < \nu \wedge \bar{G}$ is superb for $i\}$ is uniformly $H_{\omega_1}^{N^{\nu+1}}$-definable in the parameter $\langle \pi^{h\nu} \mid h \leq \nu\rangle$ for $\nu \leq \beta$

We are now ready to prove the Main Claim. We proceed by induction on i. The case $i=0$ is trivial. There remain two cases.

Case 1 $i = \eta + 1$

By the induction hypothesis it suffices to prove the claim for $l = \eta$. Let $\bar{G}$ be superb for η. We know:

$$\Vdash^{H_\theta}_{\pi^0(\eta)} V_{\xi, \mathbb{D}} \; (\langle \xi, \mathbb{D} \rangle \text{ witnesses the Dee-subproperness of } \check{\mathbb{B}}_{\pi^0(i)} / \dot{G})$$

Hence the same holds for $H_\theta^{N^0}$, and there are $\bar{\xi}, \bar{\mathbb{D}} \in H_\theta^{N^0[\bar{G}]}$ s.t.

$$H_\theta^{N^0[\bar{G}]} \models \langle \bar{\xi}, \bar{\mathbb{D}} \rangle \text{ witnesses the sub-properness of } \mathbb{B}_i^0 / \bar{G}.$$

$\bar{G}$ adds no new reals to N^0, since η is superb. Thus $H_{\omega_1}^{N^0} = H_{\omega_1}^{N^0[\bar{G}]}$ and $\bar{\mathbb{D}} \in N^0$ is defined by:

$$\bar{\mathbb{D}}(M, \mathbb{A}) = \{ A_u \mid u \in H_{\omega_1}^{N^0} \} \text{ where}$$

$$A_u = \{ G \mid H_{\omega_1}^{N^0} \models \varphi[M, \mathbb{A}, G, u, a] \}$$

where $a \in H_{\omega_1}^{N^0}$.

But then $\mathbb{D} = \pi^0(\bar{\mathbb{D}})$ has the same definition from φ, a, with H_{ω_1} in place of $H_{\omega_1}^{N^0}$.

It is clear that, in fact, $\sigma(\mathbb{D}) = \mathbb{D}$ for any $\sigma : N^0 \prec N$.

Set: $\tilde{N}^0 = L_{\tau^0}^{A^0, \bar{G}}$, where $N^0 = L_{\tau^0}^{A^0}$.

Set: $\tilde{\mathbb{B}}^0 = \mathbb{B}_i^0 / \bar{G}$. Then $\tilde{N}^0, \tilde{\mathbb{B}}^0 \in N^i$, since $\bar{G} \in N^i$. We know:

$\mathbb{D}(\tilde{N}^0, \tilde{\mathbb{B}}^0) = \{A_u \mid u \in H_{\omega_1}\}$ where

$A_u = \{G \mid H_{\omega_1} \models \varphi[\tilde{N}^0, \tilde{\mathbb{B}}^0, G, u, a]\}$.

Set $X = \bigcap\{A_u \mid u \in N^i\}$. Since N^i is countable we know that $X \neq \emptyset$ and, in fact, for every $a \in \tilde{\mathbb{B}}^0 \setminus \{0\}$ there is $G \in X$ with $a \in G$. Fix $a \in \mathbb{B}_i^0 \setminus \{0\}$ with $h_\eta(a) \in \bar{G}$ and set

$\tilde{G}^0 =$ the N-least $G \in X$ s.t. $a/\bar{G} \in G$.

Since $\pi^i : N^{i+1} \prec N$, it follows that $\tilde{G}^0 \in N^{i+1}$. Set:

$\bar{G}' = \bar{G} * \tilde{G}^0 =_{\mathrm{Df}} \{b \in \mathbb{B}_i^0 \mid b/\bar{G} \in \tilde{G}^0\}$.

Then $\bar{G}'$ is $\mathbb{B}_i^0$-generic over N^0.
Moreover, $a \in \bar{G}'$.

<u>Claim</u> $\bar{G}'$ is superb for i.

By Lemma 1.5 it suffices to show that $\bar{G}'$ is i-superb for i.

We must show that if $l \le j' \le i$, $\langle \sigma, G \rangle$ is an i-witness for $\bar{G}_l'$ and $u \in N^{\hat{j}}$ is finite, then there is an i-witness $\langle \sigma', G' \rangle$ for $\bar{G}_j'$ which is strong over $\langle \sigma, G \rangle$ and s.t. σ' coincides with σ on u. By the induction hypothesis, however, we need only prove this for $l = \eta$, $j' = i$. (Hence $\hat{j} = \max \Gamma_i' = 0$.)

Let $\langle \sigma, G \rangle$ be a Γ_η^i - witness for $\bar{G}$. (Hence $\sigma = \{\langle \sigma^0, 0 \rangle, \langle \sigma^i, i \rangle\}$.) Let $u \subset N^0$ be finite. Let $\tilde{\sigma}$ = the canonical completion of σ. Then:

$$\tilde{\sigma}^0 : N^0[\bar{G}] \prec N[G], \; \tilde{\sigma}^0(\bar{G}) = G.$$

Set: $\tilde{N} = L_\tau^{A,G}$. Hence $\tilde{\sigma}^0 : \tilde{N}^0 \prec \tilde{N}$.

We know that $\sigma^0(\bar{\mathbb{D}}) = \mathbb{D}$ and that there is $\xi < \theta^0$ s.t.

$$H_{\theta^0}^{\tilde{N}^0} \models (\langle \xi, \bar{\mathbb{D}} \rangle \text{ witnesses the Dee-subproperness of } \bar{\mathbb{B}}^0)$$

Hence in $H_\theta^{\tilde{N}} = H_\theta^{V[G]} = H_\theta[G]$ we have:

$\langle \sigma^0(\xi), \mathbb{D} \rangle$ witnesses the Dee-subproperness of $\tilde{\mathbb{B}} = \mathbb{B}_{\sigma^0(i)}/G$.

But then this statement holds in $V[G]$, since $H_\theta^{\tilde N} \prec_{\Sigma_n} H_\theta^{V[G]}$ for a large n. Since $\theta > (2^{\overline{\overline{\sigma^0(3)}}})^+$, we conclude that $\langle \theta, \mathbb{D} \rangle$ witness the Dee-subproperness of $\widehat{\mathbb{B}}$ in $V[G]$. But then there is $\sigma \in H_{\omega_1}$ s.t. for all $g \in A_\sigma$ and all finite $u' \in \tilde N^0$ there is $b \in \widehat{\mathbb{B}} = \mathbb{B}^{0}_{\sigma(i)}/G$ s.t. whenever $\tilde G \ni b$ is $\widehat{\mathbb{B}}$-generic, then there is $\tilde\sigma' \in V[G][\tilde G]$ with:

- $\tilde\sigma' : \tilde N^0 \prec \tilde N$
- $\tilde\sigma' \restriction u' = \tilde\sigma^0 \restriction u'$
- $C_{\tilde\delta}^{\tilde N}(\operatorname{rng} \tilde\sigma') = C_{\tilde\delta}^{\tilde N}(\operatorname{rng} \tilde\sigma^0)$ where $\tilde\delta = \delta(\widehat{\mathbb{B}})$ in $V[G]$
- $\tilde\sigma' {}'' g \subset \tilde{G}$.

This means, in particular, that $\operatorname{rng} \tilde\sigma' \subset \bigcup \operatorname{rng}(\tilde\sigma^0) = N^*$ where $N^* \in N$, $N^* \prec N$. Hence the condition $\tilde\sigma' : \tilde N^0 \prec \tilde N$ can be equivalently formulated as:
$\tilde\sigma' : \tilde N^0 \prec N^*$.

Since $\tilde{\sigma}^i : \tilde{N}^i \prec \tilde{N}$ with $\tilde{\sigma}^i(\tilde{\sigma}^{0,i}) = \tilde{\sigma}^0$ and $\tilde{\sigma}^i(\bigcup \mathrm{rng}\, \tilde{\sigma}^{0i}) = N^*$, it follows that there is such a $v \in H^{N^i}_{\omega_1}$. But then $X \subset A_v$. Hence, letting $u' = u \cup \{i, \eta, \theta^0, \mathbb{B}^0\}$, there is $b \in \tilde{\mathbb{B}}$ s.t. whenever $\tilde{G} \ni b$ is $\tilde{\mathbb{B}}$-generic, then the above holds with $g = \tilde{G}$. Then $\tilde{\sigma}' \in V[G][\tilde{G}] = V[G']$, where $G' = G * \tilde{G} = \{b \in \mathbb{B}_{\sigma^0(i)} \mid b/G \in \tilde{G}\}$. Set: $\sigma'^0 = \tilde{\sigma}' \upharpoonright N^0$; $\sigma' = \{\langle \sigma'^0, 0\rangle\}$.

It suffices to show:

<u>Claim</u> $\langle \sigma', G'\rangle$ witnesses $\overline{G}$, and is strong over $\langle \sigma, G\rangle$, and σ' coincides with σ on u.

The verification is straightforward. We first note that by the usual proof:

$C^N_\delta(\mathrm{rng}\, \sigma'^0) = C^N_\delta(\mathrm{rng}\, \sigma^0)$, where $\delta = \delta(\mathbb{B}_{\sigma^0(i)})$, (Cf. proof of §1 Lemma 1)

Hence $\mathrm{rng}(\sigma'^0) \subset \bigcup \mathrm{rng}(\sigma^0) \in \bigcup \mathrm{rng}(\sigma^i) = \bigcup \mathrm{rng}(\pi^i)$

Thus σ' is an i-reformulation of $\pi \upharpoonright i$. $\langle \sigma', G' \rangle$ witnesses $\bar{G}$, since $\bar{G} = G'^{0} = (\sigma'^{0})^{-1}{}''G$. σ' is obviously an $\langle \theta^0, \mathbb{B}^0, \eta, i \rangle$-reformulation of σ, since $\sigma'^{0}(\theta^0, \mathbb{B}^0, \eta, i) = \theta, \mathbb{B}, \sigma^0(\eta), \sigma^0(i)$. But σ' coheres with σ w.r.t. $\mathbb{B}_{\sigma^0(\eta)}$, since $G' \supset G$ and $\bar{G}' \supset \bar{G}$. Hence $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$. But $\sigma'^{0} \upharpoonright u = \sigma^0 \upharpoonright u$. Hence σ' coincides with σ on u. QED (Case 1)

Case 2 $i = \lambda$, Lim (λ).

Def $A = A_\lambda =$ the set of $\bar{G} \subset \mathbb{B}^\circ_\lambda$ s.t. $\bar{G}$ is $\mathbb{B}^\circ_\lambda$-generic over N° and $\bar{G}_\ell$ is superb for ℓ for all $\ell < \lambda$, where $\bar{G}_\ell = \mathbb{B}^\circ_\ell \cap \bar{G}$.

We prove:

Claim 1 Let $\ell < \lambda$ and let $\bar{G}$ be superb for ℓ. Let $a \in \mathbb{B}^\circ_\lambda \setminus \{0\}$ s.t. $h_\ell(a) \in \bar{G}$. There is $\bar{G}' \in A$ s.t. $a \in \bar{G}'$.

Claim 2 Every element of A is superb for λ.

We first prove Claim 1. Let $\langle \bar{\xi}_i \mid i < \lambda \rangle$ be monotone and cofinal in λ s.t. $\bar{\xi}_0 = \ell$. Since we are doing an RCS-iteration, the set:

$$\Delta = \{ b \in \mathbb{B}^\circ_\lambda \mid b = \bigcap_{\nu < \lambda} h_\nu(b) \}$$

is dense in $\mathbb{B}^\circ_\lambda$. Call an ultrafilter G on $\mathbb{B}^\circ_\lambda$ good if $G^+ \subset G$, where $G^+ = \{ b \in \Delta \mid \bigwedge \nu < \lambda \; h_\nu(b) \in G \}$. Exactly as in §1 Thm 3 (Case 1) we choose a "master sequence" $\langle b_i \mid i < \omega \rangle$ s.t.

(a) $b_i \in \mathbb{B}_{\bar{\xi}_i} \setminus \{0\}$, $b_i = h_{\bar{\xi}_i}(b_j)$ for $i \leq j$, $b_i \subset h_{\bar{\xi}_i}(a)$ (where w.l.o.g. $a \in \Delta$)

(b) If G is a good ultrafilter and $b_i \in G$ for $i < \omega$, then G is $\mathbb{B}^\circ_\lambda$-generic.

We then successively pick $\bar{G}_i \subset \mathbb{B}^{\circ}_{\bar{\xi}_i}$ s.t. $\bar{G}_0 = \bar{G}$, $\bar{G}_{i+1} \supset \bar{G}_i$ is superb for $\bar{\xi}_{i+1}$ s.t. $b_{i+1} \in \bar{G}$. Set $\tilde{G} = \bigcup_i \bar{G}_i$. Then
$\tilde{G}^{++} = \{ b \in \mathbb{B}^{\circ}_{\lambda} \mid \forall b' \in \tilde{G}^{+} \ b' \subset b \} \in A$.

QED (Claim 1).

We now turn to the proof of Claim 2. We shall closely imitate the proof in Case 2 of §1 Thm 3. Let $\bar{G} \in A$ be given. By Lemma 1.5 it suffices to show that $\bar{G}$ is λ-superb for λ. We employ criterion (b) of Lemma 1.2. Let $l < j \leq \lambda$, $l^* \leq j^* < \alpha$. Let $a \in \mathbb{B}_{l^*}$, $\dot{\sigma} \in V^{\mathbb{B}_{l^*}}$ s.t. whenever $G \ni a$ is $\mathbb{B}_{l^*}$-generic and $\sigma = \dot{\sigma}^G$, then $\langle \sigma, G \rangle$ witnesses $\bar{G}_l$ and $\sigma(l) = l^*$, $\sigma(j) = j^*$. Let $\dot{u} \in V^{\mathbb{B}_{l^*}}$ s.t. $a \Vdash (\dot{u} \subset \check{N}^{\hat{\nu}}$ is finite). ($\hat{\nu} = \max \Gamma_j^{\lambda}$)

<u>Claim</u> There is $\langle \sigma', G' \rangle$ s.t.

- G' is $\mathbb{B}_{j^*}$-generic and $a \in G'$
- $\langle \sigma', G' \rangle$ witnesses $\bar{G}_j$
- $\langle \sigma', G' \rangle$ is strong over $\langle \sigma, G \rangle$, where $G = G' \cap \mathbb{B}_{l^*}$, $\sigma = \dot{\sigma}^G$
- σ' coincides with σ on $u = \dot{u}^G$.

Since $\bar{G}_j$ is superb for j when $j < \lambda$, it suffices to prove the claim for $j = \lambda$. Hence $\hat{\nu} = 0$. Similarly, we can choose $\ell < \lambda$ as large as we like, so we ensure that

(*) If $cf(\lambda) < \tilde{\delta} = \sup_{\nu < \lambda} \delta(\mathbb{B}^0_\nu)$ in N^0, then $cf(\lambda) < \delta(\mathbb{B}^0_\nu)$ for a $\nu < \ell$ in N^0

We are assuming $a \Vdash_{\mathbb{B}_{\ell^*}} (\dot{\sigma}^0(\check{\ell}) = \check{\ell}^* \wedge \dot{\sigma}^0(\check{\lambda}) = \check{\lambda}^*)$.

We may also assume w.l.o.g. that there are u, η s.t. $\lambda \in u$ and:

$a \Vdash_{\mathbb{B}_{\ell^*}} (\dot{u} = \check{u} \wedge \sup \dot{\sigma}^{0}{}''\check{\lambda} = \check{\eta})$,

We then get:

(1) Let $\langle \sigma'; G' \rangle$ be as in the above Claim for a $j < \lambda$. Then $\sup \sigma'{}''\lambda = \eta$.

proof

<u>Case A</u> $cf(\lambda) < \delta(\mathbb{B}^0_\nu)$ in N^0, for a $\nu < \ell$.

Then $cf(\lambda) \leq \omega_1$ in $N^0[\bar{G}_\ell]$. Let $\bar{f} =$ = the $N^0[\bar{G}_\ell]$ - least $\bar{f} : \omega_1 \to \lambda$ s.t. $\lambda = \sup \bar{f}{}''\omega_1$. Let $\langle \sigma, G \rangle, \langle \sigma'; G \rangle$ be as in the above claim and let $f =$ = the $N[G]$ - least $f : \omega_1 \to \lambda$ s.t. $\lambda = \sup f{}''\omega_1$. Let σ^* be the unique extension of σ^0 s.t. $\sigma^* : N[\bar{G}_\ell] \prec N[G]$ and $\sigma(\bar{G}_\ell) = G$. Let $\sigma^{*\prime}$ be the unique extension of σ'^{0} s.t. $\sigma^{*\prime} : N[\bar{G}_l] \prec N[G]$ and $\sigma^{*\prime}(\bar{G}_l) = G$. (This exists since $\sigma'^{\prime\prime}\bar{G}_l \subset G$.) Clearly $\sigma^*(\bar{f}) = \sigma^{*\prime}(\bar{f}) = f$. Hence $\sup \sigma^{0}{}''\lambda = \sup \sigma'^{0}{}''\lambda = \sup f''\omega_1^{N^0}$.

QED (Case A)

<u>Case B</u> Case A fails. Then λ is regular in N^0 and $\lambda > \delta(\mathbb{B}^0_\nu)$ for $\nu < \lambda$. It follows easily that $\sup \sigma^{0}{}''\lambda = \sup \lambda^* \cap C^N_{\delta(\mathbb{B}_{j^*})}(\operatorname{rng} \sigma^0) =$

$$= \sup \lambda^* \cap C^N_{\delta(\mathbb{B}_{j^*})}(\operatorname{rng} \sigma'^{0}) = \sup \sigma'^{0}{}''\lambda.$$

QED (1)

Fix a sequence $\langle \eta_i \mid i < \omega \rangle$ monotone and cofinal in η s.t. $\eta_0 = l^*$. We also ~~fix~~ a sequence $\langle \bar{\xi}_i \mid i < \omega \rangle$ monotone and cofinal in l s.t. $\bar{\xi}_0 = l$.

We construct $a_i, \dot{\tau}_i$ $(i<\omega)$ s.t.

I (a) $a_i \in \mathbb{B}_{\eta_i}$, $h_{\eta_l}(a_i) = a_l$ for $l \le i$, $a_0 = a$

(b) $\dot{\tau}_i \in V^{\mathbb{B}_{\eta_i}}$ with $a \Vdash_{\mathbb{B}_{\rho^+}} \dot{\tau}_0 = \dot{\sigma}^0$

II Let $G \ni a_i$ be $\mathbb{B}_{\eta_i}$-generic. Set:

$G_\nu = G \cap \mathbb{B}_\nu$ $(\nu \le \eta_i)$, $\tau_l = \dot{\tau}_l^{G_{\eta_l}}$ $(l \le i)$. Then:

(a) $\tau_i : N^0 \prec N$

(b) $\tau_i(\theta^0, \mathbb{B}^0, \lambda) = \theta, \mathbb{B}, \lambda^+$

(c) $\sup \tau_i{}''\lambda = \eta$

(d) $\tau_i(\bar{\xi}_l) = \tau_h(\bar{\xi}_l)$ for $h \le i$, $l \le m_h$, where

m_h = the least m s.t. $\tau_h(\bar{\xi}_m) > \eta_{h+1}$

Simultaneously we define $\dot{\Gamma}_i \in V^{\mathbb{B}_{\eta_i}}$ s.t.

III Let $G \ni a_i$ be as in II. Set: $\Gamma_l = \dot{\Gamma}_l^{G_l}$ $(l \le i)$.

Then $\Gamma_i = \langle\langle \xi_j, \dot{\sigma}_j, b_j, \dot{u}_j \rangle \mid j \le m_i \rangle$ where

(a) $\Gamma_h \subset \Gamma_i$ for $h < i$.

(b) $\xi_j = \tau_i(\bar{\xi}_j)$ for $j < m_i$

(c) $b_j \in \mathbb{B}_{\xi_j} \setminus \{0\}$, $b_h = h_{\xi_h}(b_j)$ for $h \le j \le m_i$

(d) $\dot{\sigma}_j, \dot{u}_j \in V^{\mathbb{B}_{\xi_j}}$ for $j \le m_i$

Moreover, we have:

IV Let G, Γ_i be as in III. Let $G' \supset G$ be $\mathbb{B}_{m_i}$-generic s.t. $b_{m_i} \in G'$. Set:

$G'_\nu = G' \cap \mathbb{B}_\nu$ ($\nu \le \xi_{m_i}$), $\sigma_h = \dot{\sigma}_h^{G'_{\xi_h}}$, $u_h = \dot{u}_h$ for $h \le m_i$. Then for all $j \le m_i$:

(a) σ_j is a Γ_j^λ-reformation of $\pi \upharpoonright (\lambda+1)$, and $\sigma_0 = \sigma = \dot{\sigma}^{G_{\ell*}}$

(b) $\langle \sigma_j, G'_{\xi_j} \rangle$ witnesses $\bar{G}_j$.

(c) $\langle \sigma_j, G'_{\xi_j} \rangle$ is strong over $\langle \sigma_h, G'_h \rangle$ for $h \le j$

(d) $u_j \subset N^0$ is finite

(e) σ_j coincides with σ_h on $\sigma_h^0 {}'' u_h$ for $h \le j$

(f) $u \cup \bigcup_{h<j} u_h \subset u_j$

(g) $x_h, w_h \in u_j$ for $h < j$, where $\langle x_i \mid i < \omega \rangle$ is a fixed enumeration of N^0 and w_j = the N^0-least w s.t. $\bar{\bar{w}} = \delta = \delta(\mathbb{B}_j^0)$ in N^0 and $\sigma^0(x_j) \in \sigma_j^0(w_j)$,

(h) $\bar{\xi}_0, \ldots, \bar{\xi}_{m_h} \in u_j$ if $h \le i$, $j \le m_h$

(i) If $j = m_i$, then $\bar{\xi}_0, \ldots, \bar{\xi}_{m_i^+} \in u_j$, where m_i^+ = the least n s.t. $\sigma_{j-1}(\bar{\xi}_n) > \eta_h$ where

$\eta_{h-1} < \sigma_i(\bar{\xi}_{m_i}) \le \eta_h$;

Finally:

$\underline{V}$ Let G be as in $\underline{II}$ where $i = k+1$. Then:

(a) $h_{\eta_i}(b_{n_k}) \in G$

(b) $\tau_i = (\dot\sigma^G_{n_k - 1})^\circ$ $\underline{*)}$

Note $\dot\sigma_{n_{k-1}} \in V^{\mathbb{B}_{\eta_i}}$ since $\xi_{n_k - 1} \le \eta_i < \xi_{n_k}$.

Moreover $b_{n_k - 1} = h_{\xi_{n_k - 1}}(b_{n_k}) \in G$. Hence

$\underline{IV}$ (a) – (h) hold at $j \le n_k - 1$ with:

G in place of G', since G extends to a

$G' \supset G$ as in $\underline{IV}$.

$\underline{*)}$ Note: We recall once more our convention that, if $\mathbb{A} \subseteq \mathbb{B}$, we arrange that $V^{\mathbb{A}} \subseteq V^{\mathbb{B}}$ – i.e, if $t \in V^{\mathbb{A}}$, then $t \in V^{\mathbb{B}}$ and $t^B = t^A$ if B is $\mathbb{B}$-generic and $A = B \cap \mathbb{A}$. (If the reader is uncomfortable with this convention, he can, of course, define: $\tau_i = (\dot\sigma^{G \cap \mathbb{B}_{\xi_{n_{k-1}}}}_{n_{k-1}})^\circ$.)

Assuming I–V, we now finish the proof of the Main Claim. Set: $\tilde{a} = \bigcap_{i<\omega} a_i$.

Then $\tilde{a} \in \mathbb{B}_\eta \subset \mathbb{B}_{\lambda^*}$, $h_{\ell^*}(\tilde{a}) = a$. Let $G \ni \tilde{a}$ be $\mathbb{B}_{\lambda^*}$-generic. Set $G_\nu = G \cap \mathbb{B}_\nu$ for $\nu \leq \eta$. Set $\sigma = \dot{\sigma}^{G_{\ell^*}}$. We know that $u = \dot{u}^{G_{\ell^*}}$.

<u>Claim</u> There is $\sigma' \in V[G]$ s.t.

- $\langle \sigma', G \rangle$ witnesses $\bar{G} = \bar{G}_\lambda$
- $\langle \sigma', G \rangle$ is strong over $\langle \sigma, G_{\ell^*} \rangle$
- σ' coincides with σ on u,

where σ' is a $\Gamma^{\neg\lambda}_\lambda$-reformation of $\pi \restriction (\lambda+1)$.

(Note that then $\sigma' = \{\langle \sigma'^0, 0 \rangle\}$.)

Set: $\tau_i = \dot{\tau}_i^{G_{\eta_i}}$, $\Gamma_i = \dot{\Gamma}_i^{G_{\eta_i}}$ for $i < \omega$.

Then II (a)–(d) hold for all i, and so do III (a)–d. Set:

$$\Gamma = \bigcup_{i<\omega} \Gamma_i = \langle \langle \xi_j, \sigma_j, b_j, u_j \rangle \mid j < \omega \rangle.$$

By V we have $b_{m_k - 1} = h_{\xi_{m_k - 1}}(b_{m_k}) \in G$ for all $k < \omega$. Hence $b_j \in G_{\xi_j}$ for $j < \omega$ and, letting:

$\sigma_j^i = \sigma_j^{i\,G_{\xi_i}}$, IV (a)–(i) hold for $i < \omega$.

Since $x_\delta^i \in U_j^i$, we can define a new map $\tilde{\sigma} : N^0 \prec N$ by:

$\tilde{\sigma}(x) = \sigma_i^0(x)$ if $\sigma_i^0(x) = \sigma_h^0(x)$ for all $h \geq i$.

Then:

(1) $\mathrm{rng}(\tilde{\sigma}) \subset C^N_{\tilde{\delta}}(\mathrm{rng}\,\sigma^0)$,

where $\tilde{\delta} = \sup_{\nu < \eta} \delta(\mathbb{B}_\nu) \leq \delta(\mathbb{B}_{\lambda^*})$,

since $\tilde{\sigma}(x) = \sigma_i^0(x) \in C^N_{\delta(\mathbb{B}_{\xi_\delta})}(\mathrm{rng}\,\sigma^0)$.

(2) $C^N_{\tilde{\delta}}(\mathrm{rng}\,\tilde{\sigma}) = C^N_{\tilde{\delta}}(\mathrm{rng}\,\sigma^0)$

follows by the usual proof, using that $w_i \in U_i$.

Set $\sigma' = \{\langle \tilde{\sigma}, 0 \rangle\}$. Since σ is a reformulation of $\pi \upharpoonright (\lambda+1)$, we have: $\mathrm{rng}\,\tilde{\sigma} \subset \bigcup \mathrm{rng}(\sigma^0) \subset \bigcup \mathrm{rng}\,\pi^\lambda$ by (1), and hence:

(3) σ' is a reformulation of $\pi \upharpoonright (\lambda+1)$,

(4) $\tilde{\sigma}(\ell) = \ell^*$, $\tilde{\sigma}(\lambda) = \lambda^*$, $\tilde{\sigma} \upharpoonright u = u$,

since $\ell = \bar{\xi}_0 \in u_0$ and $\lambda^* \in u \subset u_0$;

(6) $\tilde{\sigma}'' \bar{G}_{\bar{\xi}_i} \subset G_{\xi_i}$ for $i < \omega$.

proof.

Let $b \in \bar{G}_{\bar{\xi}_i}$. Let $\tilde{\sigma}(b) = \sigma_j^0(b)$ for a $j \geq i$ s.t. $b \in u_j$. Then

$\tilde{\sigma}(b) = \sigma_j^0(b) \in G_{\xi_j} \cap \mathbb{B}_{\xi_i} = G_{\xi_i}$, since

$\tilde{\sigma}(\mathbb{B}_{\bar{\xi}_i}) = \mathbb{B}_{\xi_i}$. QED (6)

(7) $\tilde{\sigma}^{-1}{}'' G \subset \bar{G}$

proof.

Let $\Delta = \{ b \in \mathbb{B}_{\lambda^*} \mid b = \bigcap_{\nu < \lambda^*} h_\nu(b) \}$. Δ is dense in G; hence it suffices to show that $\tilde{\sigma}^{-1}{}'' \Delta \subset \bar{G}$. Let $b \in \Delta$, $b = \tilde{\sigma}(\bar{b})$

Then $\bar{b} = \bigcap_{\nu < \lambda} h_\nu(\bar{b})$ and $\tilde{\sigma}(h_{\bar{\xi}_i}(\bar{b})) =$

$h_{\xi_i}(b) \in G_{\xi_i}$. By genericity, however, (6) gives us $\tilde{\sigma}^{-1}{}'' G_{\xi_i} \subset \bar{G}_{\bar{\xi}_i}$; hence $h_{\bar{\xi}_i}(\bar{b}) \in \bar{G}_{\bar{\xi}_i}$ for $i < \omega$. Hence

$\bar{b} \in \bar{G}$. QED (7)