Correction to II

The theory of k-Ultrapowers was not properly worked out in II. We redo it here.

§1

 (Based on NFS §2)

<u>k - Ultrapowers</u> Let $k \le \omega$.

<u>Def</u> Let N be an acceptable model.

$\pi : N \longrightarrow_F^k M$ iff

(a) M is transitive

(b) $\pi : N \longrightarrow_{\Sigma_0^{(n)}} M$ for $n \le k$ s.t. $\omega\rho_N^n > \kappa$,

where:

(c) $\kappa = \mathrm{crit}(\pi)$

(d) $F = \langle \lambda \cap \pi(X) \mid X \in \mathcal{P}(\kappa) \cap M \rangle$, where $\kappa < \lambda \le \pi(\kappa)$, λ is p.r. closed, and;

(e) N = the closure of $\mathrm{rng}(\pi) \cup \lambda$ under Σ_0 fcns and good $\Sigma_1^{(n)}$ fcns for $\omega\rho_N^{n+1} > \kappa$ s.t. $n < k$.

<u>Def</u> $\Gamma_k = \Gamma_k(\kappa, N) =_{df}$ the set of $f : \kappa \to N$ s.t. $f \in N$ or f is a good $\underline{\Sigma}_1^{(n)}(N)$ map where $\omega\rho_N^{n+1} > \kappa$ and $n < k$.

If $\pi : N \longrightarrow_F^k M$ it follows exactly as before (§2 p. 4 of NFS) that $\pi(f)$ is uniquely defined for $f \in \Gamma_k$.

Hence:

Lemma 1.3' $M = \{\pi(f)(\alpha) \mid f \in \Gamma_k,\ \alpha < \lambda\}$.

Set: $H^M_m = H_{\omega\rho^m_M}$ for $m < k$ s.t. $\omega\rho^{m+1}_N > \kappa$

$H^M_m = \bigcup \pi''(H^N_{\omega\rho^m_N})$ for $m = k$, $\rho^m_N > \kappa$

or $m < k$ s.t. $\omega\rho^{m+1}_N \leq \kappa < \omega\rho^m_N$.

(i.e. "m is maximal") N *)

Then Fos Thm holds in the form:

Lemma 1.4' $M \models \varphi(\pi(f_1)(\alpha_1), \ldots, \pi(f_m)(\alpha_m)) \leftrightarrow$

$\leftrightarrow \vec{\alpha} \in F(\{\vec{\xi} \mid N \models \varphi(f_1(\xi_1), \ldots, f_m(\xi_m))\})$,

if φ is $\Sigma^{(m)}_0$, $\omega\rho^m_N > \kappa$, $m \leq k$,

where in M the Σ^* language is interpreted by $\langle H^M_i \mid i \leq m\rangle$.

(Note To prove this we use:

Let $\pi(f)(\alpha) \in H^M_m$, where m is maximal in the above sense.*) Then $\pi(f)(\alpha) = \pi(f')(\alpha)$ for an $f' \in N$.)

*) m is maximal iff
$m = \max\{m \mid m \leq k \wedge \omega\rho^m_N > \kappa\}$. It follows that
$\rho^m_N = \min\{\rho^m_N \mid m \leq k \wedge \omega\rho^m_N > \kappa\}$

Note Lemma 1.4 of NFS §2 was wrongly formulated.

Construction of k-Ultrapower:

Let $N = \langle J_\alpha^A, B \rangle$ be acceptable.
Let $\kappa < \alpha$. Let F be an extender at κ, λ on N.

$\mathbb{D} = \mathbb{D}^k(N, F)$ defined by;

$\mathbb{D} = \langle D, \tilde{=}, \tilde{\in}, \tilde{A}, \tilde{B} \rangle$ where:

$D = \{\langle \alpha, f \rangle \mid f \in \Gamma_k, \ \alpha < \lambda\}$, $\Gamma_k = \Gamma_k(\kappa, 1$

$\langle \alpha, f \rangle \tilde{=} \langle \beta, g \rangle \longleftrightarrow \langle \alpha, \beta \rangle \in F(\{\langle \xi, \zeta \rangle \mid f(\xi) = g($
$\tilde{\in}$ $\qquad \in$

etc.

Łoś Thm for Σ_0-fmlae follows as before (Using §2 Lemma 2.2 as before.)

Lemma 2.3 as before.

Assume $\tilde{\in}$ well fnd.

$[\ ] : \mathbb{D} \xrightarrow{\sim} M$, M transitive defined as before. As before ~~etc~~ define $\pi : N \xrightarrow[\Sigma_0]{} M$ by $\pi(x) = [\langle 0, \mathrm{const}_x \rangle]$.

As before: $\pi(f)$ is defined for $f \in \Gamma_k$ and $[f, \alpha] = \pi(f)(\alpha)$. As before we get:

Lemma 1.4′ Let $\bar{H} = H^n_N$, $H = \cup\pi''\bar{H}$, where n is maximal – i.e. $n = \max\{n \leq k \mid \omega\rho^n_N > \kappa\}$. Then $\pi \upharpoonright \bar{H} : \bar{H} \longrightarrow_{F} H$.

. . . . .

As before, we define a pseudo interpretation of the Σ^* – language in M by defining domains H^M_n $(n \leq k, \omega\rho^n_N > \kappa)$ for the variables v^n.

For $n < k$, $\omega\rho^n > \kappa$ we first set:

$\Gamma^n_k = \{f \in \Gamma_k \mid rng(f) \subset H^n_N\}$

For maximal $n \leq k$, $\omega\rho^n > \kappa$ set:

$\Gamma^n_k = \{f \in \Gamma_k \mid rng(f) \in H^n_N\}$

(Hence $\Gamma^n_k \subset H^n_N$ for n maximal.) As before we then set:

$$H_n = H^M_n = \{[\alpha, f] \mid \langle \alpha, f\rangle \in D \wedge f \in \Gamma_n\}.$$

H_n is then transitive as before and we get Łoś Thm as before:

Lemma 3.1′ Let φ be a $\Sigma^{(n)}_0$ fmla for an n s.t. $n \leq k$ and $\omega\rho^n_N > \kappa$ or a $\Sigma^{(n)}_1$ – fmla for an n s.t. $n < k$ and $\omega\rho^{n+1}_N > \kappa$. Then

$M \models \varphi([\alpha_1, f_1], \ldots, [\alpha_n, f_n]) \longleftrightarrow$

$\longleftrightarrow \vec{\alpha} \in F(\{\vec{\xi} \mid N \models \varphi(f_1(\alpha_1), \ldots, f_n(\alpha_n))\})$

Lemma 3.2′ $\pi : N \longrightarrow_{\Sigma_0^{(n)}} M$ for $n \leq k$, $\omega\rho_N^n > u$

Cor 3.3′ $\pi : N \longrightarrow_{\Sigma_2^{(n)}} M$ for $n < k$, $\omega\rho_N^{n+1} > u$

The prfs are as before. (These lemmas are now proven only in the sense of the pseudointerpretation.)

As before:

Cor 3.4′ M is an acceptable model.

Cor 3.5′ Set $\omega\rho_n = On \cap H_n$. Let $M = \langle J_{\alpha'}^{A'}, B' \rangle$. Then $H_n = J_{\rho_n}^{A'}$.

Lemma 3.6′ $\rho_n = \rho_M^n$ for $\omega\rho_N^{n+1} > u$, $n < k$;

$\rho_n \leq \rho_M^n$ for $\omega\rho_N^n > u$, $n \leq k$.

Hence:

$\pi : N \longrightarrow_{\Sigma_0^{(n)}} M$ for $\omega\rho_N^n > u$, $n \leq k$ (with the normal interpretation of the Σ^*-language.)

(However, Łoś Thm for $\Sigma_0^{(n)}$ holds only in the pseudointerpretation, since $\rho_n < \rho_M^n$ is possible for n maximal.)

<u>Cor 3.6'</u> Let $n < k$, $\rho_N^{n+1} > \kappa$. Then $\pi``P_N^n \subset P_M^n$

<u>Cor 3.7'</u> $\pi : N \xrightarrow[F]{k} M$

The prfs are exactly as before.

Using the same proof as before we also get;

<u>Lemma 4.4'</u> Let $n \le k$ be maximal s.t. $\omega\rho_N^n > \kappa$.

Then:

If $R_N^n \neq \emptyset$, then

(a) $\pi : N \xrightarrow[\Sigma_0^{(n)}]{} M$ cofinally

(b) $\pi``R_N^n \subset R_M^n$.

(Thus $\rho_M = \rho_n^M$ in this case.)

[Lemmas 4.1 - 4.3 use essentially that $\omega\rho^{n+1} \le \kappa$ where n is maximal; hence $\pi : N \xrightarrow[F]{*} M$]

In place of our old Lemma 5.1 we have the stronger form:

<u>Note</u> Steel can't use Lemma 4.4', since his ultraproducts require a soundness condition.

<u>Lemma 5.1'</u> Let $n \leq k$ be maximal. Let F be Σ_1 - amenable wrt. N and assume that $\omega\rho_N^{n+1} < \omega\rho_N^n$. Then

$\pi : N \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally.

prf.

For $\omega\rho_N^{n+1} \leq \kappa$ it is proven, so assume $\kappa < \omega\rho_N^{n+1}$. (Hence $n = k$).

The case $n = 0$ is trivial, so assume $n > 0$. Suppose not.

Let $\bar\rho = \rho_N^{n+1}$, $\rho = \pi(\rho)$.

Let $\bar H = H_N^n$, $H = H_M^M = \bigcup_{u \in \bar H} \pi(u)$.

Let $\bar A$ be $\Sigma_1^{(n)}(N)$ in $\bar p$ s.t.

$\bar A \subset \omega\bar\rho$, $\bar A \notin N$. Then $\bar A$ is

$\Sigma_1(\langle \bar H, \bar B \rangle)$ where $\bar B = \bar B' \cap \omega\rho_N^n$ and B' is $\Sigma_1^{(n-1)}(N)$ in $\bar p$.

$\bar Q = \langle \bar H, \bar B \rangle$ is amenable. Let

$Q = \langle H, B \rangle$, where $\pi \restriction \bar H : \bar Q \longrightarrow_{\Sigma_0} Q$.

(Hence Q is amenable).

Since $\pi : N \longrightarrow_{\Sigma_2^{(n-1)}} M$, it follows easily that $B = B' \cap \omega\rho^m_M$, where B' has the same $\Sigma_1^{(n-1)}(M)$ definition in p. Let $A \subset \omega\rho$ be $\Sigma_1(Q)$ by the same def. as $\bar{A}$ (in $\bar{Q}$). Since $\rho^m_M < \rho^n_M$, we have: $B \in M$; hence $Q \in M$ and $A \in M$. Hence $A \in H$, since $\omega\rho \in H$. Let $A = \pi(f)(\alpha)$, $\alpha < lh(F)$, $f \in \bar{H}$, $f : \kappa \longrightarrow \bar{H}$. For $\nu < \omega\bar{\rho}$ we have:

$$\nu \in \bar{A} \longleftrightarrow \pi(\nu) \in A = \pi(f)(\alpha)$$
$$\longleftrightarrow \{\xi < \kappa \mid \nu \in f(\xi)\} \in F_\alpha$$

where F_α is $\Sigma_1(N)$. But $F_\alpha \subset (H_{\kappa^+})^N$, where $\kappa^+ \leq \omega\bar{\rho} = \omega\rho^{n+1}_N < \omega\rho^n_N \leq \omega\rho^1_N$. Hence $F_\alpha \in N$. Hence $\bar{A} \in N$. Contr! QED (Lemma 5.1)

As a corollary of the proof we get:

Cor 5.1.1′ Let $1 \leq n < k$ be maximal. Let F be Σ_1-amenable wrt. N. Let $\omega\rho_N^{n+1} < \omega\rho_N^n$. Then

$$\omega\rho_M^{n+1} \leq \pi(\omega\rho_N^{n+1}) < \omega\rho_M^n .$$

prf.
If $\omega\rho^k \leq \kappa$, it follows by Cor 4.2 and the proof of Lemma 5.1 of NFS §2. Otherwise it follows by the above proof.
QED

Combining the results we have:

Def $k \leq \omega$ is good for N iff either $k = 0$, $k = \omega$, or $1 \leq k < \omega$ and $\omega\rho^{k+1} < \omega\rho^k$ in N.

Cor 6.2 of §2 NFS can be generalized to:

Lemma 6.2′ Let F be close to N. Let k be good for N. Then

$$\pi : N \longrightarrow_{\Sigma_0^{(k)}} M \quad (\text{cofinally, if } k < \omega)$$

Moreover, k is good for M.

<u>prf. of 6.2'</u>

The case $k = \omega$ or $\omega\rho^{k+1} \leq \kappa$ is given by Cor 2 and Lema 6.1 of §2 NFS.
The case $k < \omega$ and $\omega\rho^{k+1} > \kappa$ follows by the above. QED (6.2)

We even get a generalisation of Lemma 6 of §2 NFS:

<u>Def</u> Let $k \leq \omega$. $P_N^{\langle k \rangle}$ = the set of $p \in N$ s.t. for all $n \leq k$ s.t. $n < \omega$ there is A which is $\Sigma_1^{(k)}(N)$ in p with:

(a) $A \cap H_N^{n+1} \notin N$ if $n < k$

(b) If $n = k > 0$, then $\langle H_N^n, A \cap H_A^n \rangle$ is not amenable.

(Thus $P^{\langle \omega \rangle} = P^*$ and $P^{\langle 0 \rangle} = N$)

Note that $P^{\langle k \rangle}$ is non empty when k is good for N.

<u>Lemma 6.4'</u> Let F be close to N and k good for N. Then $\pi''P_N^{\langle k \rangle} \subset P_M^{\langle k \rangle}$.

prf. of Lemma 6.4′

For $k=\omega$ (hence also for $\omega\rho^{k+1} \leq \kappa$) this follows from NFS §2 Cor 6.4. For $k=0$ trivial. For $\omega\rho^{k+1} > \kappa$, $k>0$, pick $p \in P_N^{\langle k \rangle}$ and let $\bar{A}$ be $\Sigma_1^{(k)}(N)$ in p s.t. $\langle H_N^k, \bar{A} \cap H_N^k \rangle$ is not amenab. W.l.o.g. suppose that $A \cap \omega\bar{\rho} \notin N$ for a $\bar{\rho} < \rho_N^n$. The proof of Lemma 5 can be repeated using $\bar{\rho}$ in place of ρ_N^{k+1} to show that there is A which is $\Sigma_1^{(k)}(M)$ in $\pi(p)$ with $A \cap \omega\rho \notin M$, where $\rho = \pi(\bar{\rho})$. Thus $p \in P_M^{\langle k \rangle}$. (In this proof we must, however, use the real ρ_N^{k+1} to show: $F_\alpha \in N$.)

QED (6.4′)

We also note that Lemma 5.2 holds in the form:

<u>Lemma 5.2'</u> Let F be Σ_1-amenable wrt. N, let k be good for **N** and let $B \subset \kappa$ be $\underline{\Sigma}_1^{(n)}(M)$, where n is maximal s.t. $n \leq k$, $\omega\rho^n > \kappa$. Then B is $\underline{\Sigma}_1^{(n)}(N)$

<u>Cor 5.3'</u> Let F, n be as above. Then $\underline{\Sigma}_1^{(n)}(M) \cap \mathcal{P}(J_\kappa^A) = \underline{\Sigma}_1^{(n)}(N) \cap \mathcal{P}(J_\kappa^{\bar{A}})$, where $M = \langle J_\alpha^A, D \rangle$, $N = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{D} \rangle$.

The proofs are exactly as before.

As before we get:

<u>Cor 6.5'</u> Let F be close to N and let k be good for N. Then $\mathcal{P}(\kappa) \cap \underline{\Sigma}_1^{(n)}(N) = \mathcal{P}(\kappa) \cap \underline{\Sigma}_1^{(n)}(M)$ for all $n \leq k$.

Lemma 7 of §2 NFS also generalizes with no change of proof:

<u>Lemma 7'</u> Let F be Σ_1 amenable wrt. N. Let k be good for N. The statement of Lemma 7 holds for maximal $n \leq k$ s.t. $\omega\rho_N^n > \kappa$.

We can also generalize Lemma 8:

<u>Lemma 8′</u> Let $\langle N_i \mid i < \theta \rangle$, $\langle \pi_{ij} \mid i \leq j < \theta \rangle$ be s.t. N_0 is acceptable and k is good for N_0, and:

(a) N_i is transitive

(b) $\pi_{ij} : N_i \longrightarrow N_j$; $\pi_{ij}\pi_{hi} = \pi_{hj}$, $\pi_{ii} = id$; $N_\lambda, \langle \pi_{i\lambda} \mid i < \lambda \rangle$ = the direct limit of $\langle N_i \mid i < \lambda \rangle$, $\langle \pi_{ij} \mid i \leq j < \lambda \rangle$ for limit $\lambda < \theta$

(c) If $i+1 < \theta$ and N_i is acceptable and k is good for N_i , then $\pi_{i,i+1} : N_i \xrightarrow[F_i]{k} N_{i+1}$, where F is close to N_i.

Then for all $j < \theta$:

(i) N_j is acceptable and k is good for N_j

(ii) $\pi_{ij} : N_i \longrightarrow_{\Sigma_0^{(k)}} N_j$ (cofinally iff $k < \omega$

(iii) π_{ij} " $P^{\langle k \rangle}_{N_i} \subset P^{\langle k \rangle}_{N_j}$ for $i \leq j$

(iv) Let $\kappa_i = crit(F_i)$. If $\kappa_i \leq \kappa_h$ for $i \leq h < j$, then $\mathbb{P}(\kappa_i) \cap \underline{\Sigma}_1^{(n)}(N_i) = \mathbb{P}(\kappa_i) \cap \underline{\Sigma}_1^{(n)}($

for $n \leq k$.

(v) If $n < k$ and $\kappa_h < \rho^{n+1}_{N_h}$ for $i \leq h < j$, then $\pi_{ij} : N_i \longrightarrow_{\Sigma_2^{(n)}} N_j$ and π_{ij} " $P^n_{N_i} \subset P^n_{N_j}$

(vi) If $n \leq k$ is max s.t. $\omega\rho^n_{N_h} > \kappa_h$ for $i \leq h < j$, then $\pi_{ij} : N_i \longrightarrow_{\Sigma_0^{(n)}} N_j$ cofinally

Also:

(vii) Let $0 < k < \omega$. Then $\pi_{ij}(\rho^{k+1}_{N_i}) \geq \rho^{k+1}_{N_j}$ for $i \leq j$.

(It is this which guarantees the goodness of k for N_j)

The proof of Lemma 8′ is again by a straightforward induction on j.

§2 Extendability

Def Let M be acceptable, $k \le \omega$. Let F be an extender on M at κ, ν.
M is k-extendible by F iff there are π, N s.t. $\pi : M \xrightarrow[F]{k} N$.

Def $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$ defined as before.

Lemma 1′ Let $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$. Let $\pi : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ for all $n \le k$ s.t. $\omega\rho^n_{\bar{M}} > \bar{\kappa}$. Let M be l-extendible by F, where $l \ge k$. Then $\bar{M}$ is k-extendible by $\bar{F}$. Let $\sigma : M \xrightarrow[F]{l} N$ and $\bar{\sigma} : \bar{M} \xrightarrow[\bar{F}]{k} \bar{N}$. Define a pseudo interpretation of the Σ^*-language over $\bar{N}$ by setting: $\bar{H}_n = H^{\bar{N}}_n =_{Df} = H^n_{\bar{N}}$ for $n < k$ s.t. $\omega\rho^{n+1}_{\bar{M}} > \kappa$ and $\bar{H}_n = H^{\bar{N}}_n = \bigcup \bar{\sigma}'' H^n_{\bar{N}}$ for $n \le k$ maximal s.t. $\omega\rho^n > \kappa$. In the sense of this interpretation

There is a unique π' s.t. $\pi' : \bar{N} \xrightarrow[\Sigma_0^{(n)}]{} N$ for all $n \leq k$ s.t. $\omega\rho^n_{\bar{M}} > \bar{\kappa}$, $\pi'\bar{\sigma} = \sigma\pi$, and $\pi' \upharpoonright \nu = g$. π' is defined by:

$$\pi'(\bar{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$$

for $\alpha < \bar{\nu}$ and $f \in \Gamma_k(\bar{\kappa}, \bar{M})$.

The proof is exactly as before.

(Note The formulation of Lemma 1 in §3 NFS is wrong, since we forgot to mention the pseudo interpretation and did not make an assumption (e.g. $\bar{F}$ is close to $\bar{M}$ and k good for $\bar{M}$) which would guarantee that $H^{\bar{N}}_n = H^n_{\bar{N}}$.)

As a corollary we obviously have:

<u>Lemma 1.1'</u> Let $\bar{M}, M, \bar{F}, F, \pi, g, \pi'$, k, l be as above, where $\bar{F}$ is close to $\bar{M}$ and k is good for $\bar{M}$. Then $\pi' : \bar{N} \xrightarrow[\Sigma_0^{(n)}]{} N$ for $n \leq k$ s.t. $\omega\rho^n_{\bar{N}} > \bar{\kappa}$,

In particular, we have $\pi' : \bar{N} \xrightarrow[\Sigma_0^{(k)}]{} N$ if $\omega\rho^k_{\bar{M}} > \bar{\kappa}$. Combining this with Lemma 2 of §3 NFS we get:

<u>Lemma 2'</u> Assume:

(a) $\langle \pi, g \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow^{*} \langle M, F \rangle$

(b) $\pi : \bar{M} \xrightarrow[\Sigma_0^{(k)}]{} M$, where k is good for $\bar{M}$.

(c) $\bar{F}, F$ are weakly amenable.

(d) F is Σ_1-amenable w.r.t. M.

Let $l \geq k, \sigma, N, \bar{\sigma}, \bar{N}, \pi'$ be as above. Then

$$\pi' : \bar{N} \xrightarrow[\Sigma_0^{(k)}]{} N,$$

By Lemma 1' we also have:

<u>Lemma 1.2'</u> Let $\bar{M}, \bar{F}, M, F, \pi, g, \pi'$, $\bar{N}, N$ be as in Lemma 1', where $R^n_{\bar{M}} \neq \emptyset$ for $n \leq k$ max. s.t. $\omega\rho^n_{\bar{M}} > \bar{\kappa}$. Then $\pi' : \bar{N} \xrightarrow[\Sigma_0^{(n)}]{} N$.

All of the copying theorems for k-iterations given in $\underline{\text{II}}$ seem to go through on the <u>additional</u> assumption that k is good for $\overline{M}$.