Addendum to Corrections and Remarks

After these notes were written, Martin Zeman discovered a way to do the proofs without the use of k-ultrapowers. Zeman's approach seems to me far superior to that which we took. However, it entails doing things in a different order. §7 + §8 of [NFS] cannot be so neatly separated.

In this addendum we present a sketch of Zeman's proof.

Interpolation Lemma Let $n \geq 0$.

Let $\sigma : \bar{M} \xrightarrow{\Sigma_1^{(n)}} M$, where $\bar{M} = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{B} \rangle$, $M = \langle J_{\alpha}^{A}, B \rangle$ are acceptable.

Let $\bar{\beta} \leq \rho^n_{\bar{M}}$ be a cardinal in $\bar{M}$.

Let $\bar{\sigma}_0 : J_{\bar{\beta}}^{\bar{A}} \xrightarrow{\Sigma_0} J_{\tilde{\beta}}^{\tilde{A}}$ cofinally and

$\bar{\sigma}_1 : J_{\tilde{\beta}}^{\tilde{A}} \xrightarrow{\Sigma_0} J_{\alpha}^{A}$

s.t. $\bar{\sigma}_1 \bar{\sigma}_0 = \sigma \upharpoonright J_{\bar{\beta}}^{\bar{A}}$.

There are $\tilde{M} = \langle J_{\tilde{\alpha}}^{\tilde{A}}, \tilde{B} \rangle$, σ_0, σ_1 s.t.

$$\begin{tikzcd} \bar{M} \arrow[rr, "\sigma"] \arrow[dr, "\sigma_0"'] & & M \\ & \tilde{M} \arrow[ur, "\sigma_1"'] & \end{tikzcd}$$

where $\sigma_0 \supset \bar{\sigma}_0$, $\sigma_1 \supset \bar{\sigma}_1$ are $\Sigma_0^{(n)}$-preserving, and $\tilde{M} = \tilde{h}^{n+1}_{\tilde{M}}(\tilde{\beta} \cup rng(\sigma_0))$.

<u>Note</u> If $\bar{\beta} = \rho^n_{\bar{M}}$ and $\bar{M} = \tilde{h}^{n+1}(\bar{\beta} \cup r)$, then $\tilde{M} = \tilde{h}^{n+1}(\tilde{\beta} \cup \sigma_0(r))$. Hence $\rho^n_{\tilde{M}} \leq \tilde{\beta}$. But $\rho^n_{\tilde{M}} \geq \sup \sigma_0 `` \rho^n_{\bar{M}} = \tilde{\beta}$. <u>Hence</u>:

$$(\bar{\beta} = \rho^n_{\bar{M}} \wedge R^n_{\bar{M}} \neq \emptyset) \longrightarrow \tilde{\beta} = \rho^n_{\tilde{M}}$$

Basic Lemma (BL) Let M be a presolid mouse. Let $\sigma : W \longrightarrow M$ witness the goodness of $\langle M, W, \alpha\rangle$. Let $\langle \mathcal{J}^W, \mathcal{J}^M\rangle$ be the coiteration of $\langle M, W, \alpha\rangle$ against M. Let $\theta = lh(\mathcal{J}^M)$. Then:

(a) $\theta \geq 0$ in $\mathcal{J}^W$

(b) W_θ is a simple it. of W in $\mathcal{J}^W$ and a segment of M_θ; hence:

(c) W is a mouse.

Cor 1 Let M be a presolid mouse. Let $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ for all $\omega\rho^n_{\bar{M}} > \nu = \mathrm{crit}(\sigma)$, where $\bar{M}$ is a premouse. Then $\bar{M}$ is a mouse.

unless $n = 0$, $M = \langle J^E_\alpha, \emptyset\rangle$. Then w.l.o.g. assume that σ is cofinal; hence $n = 1$

(prf. σ witnesses goodness of $\langle M, \bar{M}, \nu\rangle$.)

Lemma 2 Let M be a presolid mouse. Let $\bar{M} = \mathrm{core}_{\omega\rho^n_M} M$; $\rho = \omega\rho^n_M = \omega\rho^n_{\bar{M}}$. Then

(a) $\mathbb{P}(\rho) \cap \bar{M} = \mathbb{P}(\rho) \cap M$

(b) $\underline{\Sigma}_1^{(n)}(\bar{M}) \cap \mathbb{P}(\rho) = \underline{\Sigma}_1^{(n)}(M) \cap \mathbb{P}(\rho)$

(Note By (a), $\sigma^{-1}(p_M) = p_{\bar{M}} \in R^*_{\bar{M}}$ and $\bar{M}$ is sound; hence $\bar{M} = \mathrm{core}_\rho(M)$.)

proof of Lemma 2

If $\sigma = \mathrm{id}$, there is nothing to prove. Let $\sigma \neq \mathrm{id}$ (hence $n > 0$). Let $\sigma : \bar{M} \to M$ be the core map. Then σ witnesses the goodness of $\langle M, \bar{M}, \rho \rangle$. Let $\langle \mathcal{J}^{\bar{M}}, \mathcal{J}^{M} \rangle$ be the coiteration. Let $\theta = lh(\mathcal{J}^M)$. By ind. on i we see that $\nu_i > \rho$, since ρ is a cardinal in $M_i, \bar{M}_i$ and $\nu_i \geq \rho$. But then $\nu_i \geq \rho^{+M_{i+1}}$ for $i < \theta$. Hence $\mathcal{P}(\rho) \cap M_i \subset M$ for $i \leq \theta$. Hence

$$A^n_{\bar{M}_\theta} = A^n_M \in M.$$

<u>Case 1</u> $\bar{M}_\theta$ is a proper segment of M_θ. Then $A^n_{\bar{M}_\theta} = A^n_{\bar{M}} = A^n_M \in M$. Contr!

<u>Case 2</u> $\bar{M}_\theta = M_\theta$ is a non simple iterate of M in $\mathcal{J}^M$. Let $i+1 \leq \theta$ in $\mathcal{J}^M$ be maximal s.t. $\eta_i < ht(M_\xi)$, where $\xi = T(i+1)$. Then $\mathcal{P}(\rho) \cap \underline{\Sigma}^*(M_\theta) = \mathcal{P}(\rho) \cap \underline{\Sigma}^*(M_{i+1}) = \mathcal{P}(\rho) \cap \underline{\Sigma}^*(M_\xi \| \eta_i) \subset M_\xi$. Hence $A^n_M = A^n_{M_\theta} \in \mathcal{P}(\rho) \cap M_\xi \subset M$. Contr!

<u>Case 3</u> $\bar{M}_\theta = M_\theta$ is simple in $\mathcal{J}^M$.

Then $\rho = \rho^n_{M_{\bar\theta}} = \rho^n_{M_\theta}$. Hence:

(1) $\kappa_i^M \geq \rho$ for all $i \in D^M$ s.t. $i+1 \leq \theta$ in $\mathcal{J}^M$.

Suppose not. Let i be the least counterexample. Then

$\rho = \omega\rho^n_{M_\theta} \geq \omega\rho^n_{M_{i+1}} \geq \sup \Pi^M_{\sigma_i} {}''\rho \geq \nu_i > \rho$.

Contr! QED (1)

(a), (b) are then immediate, since (1) holds on the $\bar{M}$ side by (BL).

QED (Lemma 2)

A modification of this yields:

<u>Lemma 3</u> Let M be a presolid mouse. Let $\sigma : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ for $\omega\rho^n_{\bar{M}} > \nu = \mathrm{crit}(\sigma)$, where ν is cardinal preserving in M (i.e. If $\tau < \nu$ is a card. in $J_\nu^{E^M}$, then in M), and $\bar{M}$ is a premouse. Suppose $\bar{M}$ is a premouse and that $\omega\rho^{n+1} \leq \nu < \omega\rho^n$ in $\bar{M}$, where $\bar{M} = \tilde{h}^{n+1}_{\bar{M}}(\nu \cup \bar{p})$ for a finite $\bar{p} \subset M$. Then one of the following holds.

(a) $\bar{M} \in M$

(b) $\mathbb{P}(\rho) \cap \bar{M} = \mathbb{P}(\rho) \cap M$ and $\underline{\Sigma}_1^{(n)}(\bar{M}) \cap \mathbb{P}(\rho) = \underline{\Sigma}_1^{(n)}(M) \cap \mathbb{P}(\rho)$, where $\rho = \omega\rho^{n+1}_{\bar{M}} = \omega\rho^{n+1}_{M}$

(c) $E_\nu^M \neq \emptyset$, $\bar{M} \in \mathrm{Ult}^*(M, E_\nu^M)$ and $\bigvee m \; \omega\rho^m_M = \tau(E_\nu^M)$.

[<u>Note</u> We use the notation:
$\kappa(E_\nu) = \mathrm{crit}(E_\nu)$, $\tau(E_\nu) = \kappa^{+J_\nu^E}$,
$\lambda(E_\nu) = \mathrm{lh}(E_\nu) = E_\nu(\kappa)$.]

[<u>Note</u> In case (c) E_ν is a superstrong extender in M.]

prf. of Lemma 3.

Let $\mathcal{Y}^{\bar{M}}$, $\mathcal{Y}^{M}$, θ be as in the proof of Lemma 2. Note that $\bar{M}$ is codable by a $\underline{\Sigma}_1^{(n)}(\bar{M})$ subset $a \subset \nu$. Assume $\bar{M} \notin M$. We show that (b) or (c) holds. Then $a \notin M$, since otherwise $a \in J^{E^M}_{\sigma(\nu)}$ and $\bar{M}$ would be reconstructible from a in $J^{E^M}_{\sigma(\nu)}$.

We shall prove (c)

Now suppose that Case 1 or Case 2 holds. Then $\nu_0 = \nu$ and $E^M_\nu \neq \emptyset$, since otherwise $\nu_i > \nu$ whenever $i \in D$ and $i+1 \leq \theta$ in $\mathcal{Y}^M$. Arguing as before we would have: $\mathbb{P}(\nu) \cap M_i \subset M$ for $i \leq \theta$ in $\mathcal{Y}^M$.

Hence exactly as before we would get: $a \in M$. Contr! Since $\nu_i > \nu$ for $i > 0$, we can certainly argue as before to get:

$a \in M_1$, where $\pi^M_{01} : M \xrightarrow[E_\nu]{*} M_1$.

Now suppose that $\tau = \tau(E_\nu) \neq \omega\rho^m$ in M for any $m < \omega$. Then

$\omega\rho^{m+1} \leq \kappa < \tau < \omega\rho^m$ in M for some m and, letting $\rho = \rho^m_{M_1}$, we have: ρ is a cardinal in M_1 + $\rho > \tau$. Hence $a \in J^{E^{M_1}}_\rho$. But if $\bar\rho = \rho^m_M$, then

$$\pi_{01} : J^{E^M}_{\bar\rho} \xrightarrow[E_\nu]{} J^{E^{M_1}}_\rho \text{ is a } \Sigma_0$$

ultrapower. Hence $a \in M$, since $E_\nu, J^{E^M}_{\bar\rho} \in J^{E^M}_{\sigma(\nu)}$ + hence $J^{E^{M_1}}_\rho \in J^{E^M}_{\sigma(\nu)}$. Contr!

QED (Cases 1, 2).

Now let Case 3 hold. We prove (b)

We again have:

$\omega\rho^{n+1}_{M_\theta} = \omega\rho^{n+1}_{\bar M_\theta} = \omega\rho^{n+1}_{\bar M} \leq \nu$. Clearly

$\mathbb{P}(\omega\rho^{n+1}_{\bar M}) \cap \bar M = \mathbb{P}(\omega\rho^{n+1}_{\bar M_\theta}) \cap \bar M_\theta$,

since $\kappa_i^{\bar M} \geq \nu$ whenever $i \in D$ and $i+1 \leq \theta$ in $\mathcal{T}^{\bar M}$. For the same reason we need only show:

<u>Claim</u> $\kappa_i^M \geq \omega\rho_M^{n+1}$ whenever $i \in D$ and $i+1 \leq \theta$ in $\mathcal{J}^M$.

Suppose not. Then:

$$\omega\rho_{M_\theta}^{n+1} \geq \omega\rho_{M_{i+1}}^{n+1} \geq \nu_i \geq \nu \geq \omega\rho_{\bar{M}}^{n+1}.$$

Hence $\overbrace{\ }^{i=0 \text{ and}}$ $\nu = \nu_i = \omega\rho_{\bar{M}}^{m} = \omega\rho_{M_\theta}^{n+1}$. But then $\omega\rho_M^{m+1} = \tau_i$. Now let $a \subset \tau_i$ be $\Sigma^*(M)$ in s, s.t. $a \notin M$. Let a' be $\Sigma^*(M_\theta)$ in $\pi_{0\theta}^M(s)$ by the same def. Then $a' \cap \tau_i = a$, since $\mathrm{crit}(\pi_{0\theta}^M) \geq \tau_i$. Hence $a \in J_\nu^{E^{M_\theta}} = J_\nu^{E^M}$, since $\tau_i < \omega\rho_{M_\theta}^{n+1}$. Contr!

QED (Lemma 3)

The proof of condensation (using Σ^*-iterations) gives us:

<u>Lemma 4</u> Let M be a presolid mouse. Let $\sigma : \bar{M} \xrightarrow[\Sigma_0^{(n)}]{} M$, where $\bar{M}$ is a premouse which is sound and solid above $\nu = \mathrm{crit}(\sigma)$, where $\omega\rho^{n+1} \leq \nu < \omega\rho^n$ in $\bar{M}$, and ν is cardinally absolute in M. Then one of the following hold

(a) $\bar{M} = \mathrm{core}_\nu(M)$ and σ is the core map.

(b) $\bar{M} = M\|\eta$ for an $\eta < \mathrm{ht}(M)$.

(c) $\bar{M} = \mathrm{Ult}^*(M\|\eta, E^M_\mu)$, where:

  (i) $\nu < \eta < \mathrm{ht}(M)$ and $\omega\rho^\omega_{M\|\eta} < \nu$

  (ii) $\mu \leq \omega\eta$

  (iii) $\nu = \kappa^{+M\|\eta}$, where $\kappa = \mathrm{crit}(E^M_\mu)$

  (iv) E^M_μ is generated by $\{\kappa\}$.

(d) $\bar{M} = M_1\|\eta$, where $\eta < \mathrm{ht}(M_1)$ and $\pi : M \xrightarrow[E^M_\nu]{*} M_1$.

<u>Lemma 5</u> Let M be a mouse. Then M is solid.

pf. Suppose not.

Let M be a counterexample with $\mathrm{ht}(M)$ minimal. Then M is presolid. Assume w.l.o.g. that M is sound. (If not, replace M by $N = \mathrm{core}(M)$. Then N is sound and $p_N^{-1} = \sigma^{-1}(p_M)$ by Lemma 2. N is a counterexample, since otherwise $\sigma(W_\nu^N)$ is a generalized witness for $\sigma(\nu)$ for $\nu \in p_N$.)

where σ is the core map

Let $\nu \in p_M$ be maximal s.t. $W_\nu^M \notin M$. By Lemma 3 we know: E_ν is a superstrong extender in M and $\omega\rho^m = \tau = \tau($ in M for some m. Thus $m \leq n+1$, where $\omega\rho^{n+1} \leq \nu < \omega\rho^n$ in M. For $\alpha < \nu$ define $W_{\alpha,\nu} = W^M_{\alpha,\nu} =_{\mathrm{Df}}$ the transitivation of $\tilde{h}^{n+1}(\alpha \cup p_M \setminus (\nu+1))$. (i.e. $W_{\alpha,\nu}$ is defined from α, $p_M \setminus (\nu+1)$ as W_ν from ν, $p_M \setminus (\nu+1)$.) Then:

(1) $W_{\alpha,\nu} \in M$ for all $\alpha < \nu$.

pf. $W_{\alpha,\nu} \in M_1 = \mathrm{Ult}^*(M, E_\nu)$ by Lemma 3

But $\overline{\overline{W}}_{\alpha,\nu} \leq \alpha < \nu$ in M_1, where ν is a cardinal in M_1. Hence $W_{\alpha,\nu} \in J_\nu^{E^{M_1}} = J_\nu^{E^M}$. QED (1)

We call $\langle W, \pi \rangle$ a <u>generalized</u> α, ν witness to $\nu \in M$ iff $W \in M$ and $W = \tilde{h}^{n+1}(\alpha \cup p)$ for some p s.t.

$$M \models \varphi(\vec{\xi}, p_M \setminus (\nu+1)) \rightarrow W \models \varphi(\vec{\xi}, p)$$

for all $\Sigma_1^{(n)}$ - formulae and all $\vec{\xi} < \alpha$. It is clear by the usual argument that, if M has a generalized α, ν witness, then $W_{\alpha,\nu} \in M$. It is also clear, that if $W_{\alpha,\nu} \in M$ for all $\alpha < \nu$ then $\nu \in P_M$.

<u>Case 0</u> $\sup \sigma'' \omega\rho_W^n < \omega\rho_M^n$.

Set:

$\bar{\sigma} : \bar{W} \overset{\sim}{\longleftrightarrow} h_{M^{n,p_M \restriction n}}(\nu \cup q)$,

where $q = (p_M \cap \omega\rho_M^n) \setminus (\nu+1)$.

Let $W^{\bar{p}} = M^{n, p_M \restriction n}$, where $\bar{p} \in R_{\bar{W}}^n$.

Then $\underline{W = W_\nu^M}$ and σ is the unique $\sigma \supset \bar{\sigma}$ s.t. $\sigma(\bar{p}) = p$.

$\Sigma_1^{(n)}$-preserving

Now let $\rho = \sup \sigma'' \rho_M^n$

$\tilde{M} = \langle J_\rho^{E^M}, A \cap J_\rho^{E^M} \rangle$, where

$M^{n,p_n,r_n} = \langle J_{\rho_M^n}^E, A \rangle$. Then

$\tilde{M} \in J_{\rho_M^n}^E$ is amenable and

$h_{M^{n,p_n}}(\nu \cup p) = h_{\tilde{M}}(\nu \cup p) \in M^{n,p_n}$

But then W is codable by an $a \subset \nu$ s.t. $a \in M^{n,p_n}$; hence $a \in J_{\sigma(\nu)}^{E^M}$, where $\sigma(\nu)$ is regular in M. Hence W is reconstructible from a in $J_{\sigma(\nu)}^{E^M}$. Hence $W \in M$.

QED (Case 0)

<u>Case 1</u> Case 0 fails and $\omega\rho^{n+2} \geq \tau = \tau(E_\nu)$ in M.

Let $\sigma : \bar{M} \longleftrightarrow \tilde{h}^{n+2}(p_M \cup \kappa)$

Then σ is $\Sigma_2^{(n)}$-preserving, since $p_M \upharpoonright n+1 \in R_M^{n+1}$. But each $\xi \in p_M \setminus (\nu+1)$ satisfies the $\Sigma_2^{(n)}$-statement:

$\bigvee W$ W is a generalized witness to ξ.

Hence the same holds in $\bar{M}$ of $\sigma^{-1}(\xi)$, Hence, letting $\sigma(\bar{\nu}) = \nu$, $\bar{M}$ is solid above $\bar{\nu}$ and $\sigma(p_{\bar{M}} \setminus (\bar{\nu}+1)) = p_M \setminus (\nu+1)$. Similarly it follows that for all $\alpha < \bar{\nu}$, $\bar{\nu}$ has a generalized $\alpha, \bar{\nu}$ witness in $\bar{M}$. Hence $\bar{\nu} \in P_{\bar{M}}$. But $\bar{M} \in M$, since $\bar{M}$ is codable by a $\underline{\Sigma}_1^{(n+1)}(M)$ set $a \subset \kappa < \omega\rho_M^{n+2}$. M is a presolid mouse by Lemma 1. Hence by the minimality of $ht(M)$, $\bar{M}$ is sound. But then $\sigma(W_{\bar{\nu}}^{\bar{M}})$ is a generalized witness for $\nu \in P_M$. Contr! QED (Case 1)

<u>Case 2</u> The above cases fail. Then $\omega\rho^{n+2} \leq \kappa < \omega\rho^{n+1} = \tau$.

Let $\pi : M \xrightarrow[E_\nu]{*} M'$. By Lemma 3 we have: $W_\nu^M \in M'$. Now let $\bar{\pi} : J_{\rho_M^n}^E \xrightarrow[E_\nu]{} J_{\rho^*}^{E^*}$. There is $\bar{\sigma} : J_{\rho^*}^{E^*} \xrightarrow[\Sigma_0]{} J_{\rho'}^{E^{M'}}$ defined by:

where $\rho' = \rho_{M'}^n$.

$\bar\sigma(\bar\pi(f|\alpha)) = \pi'(f|\alpha)$ for $f \in M$, $f : \kappa \longrightarrow J^E_{\rho^n_M}$. But then we can interpolate an M^* s.t.

$$\begin{tikzcd} M \arrow[rr, "\pi'"] \arrow[dr, "\pi^*"'] & & M' \\ & M^* \arrow[ur, "\sigma^*"'] & \end{tikzcd}$$

where $\pi^* \supset \bar\pi$, $\sigma^* \supset \bar\sigma$ are $\Sigma_0^{(n)}$-preserving and $\rho^* = \rho^n_{M^*}$, $M^* = h^{(n+1)}_{M^*}(\nu)$

(Note M^* is the "n-ultrapower" of M by E_ν.) Thus $W^M_\nu \notin M^*$, since otherwise $W^M_\nu \in Ult(J^E_{\tau^+}, E_\nu) \in M$.

Since π^* is $\Sigma_1^{(n)}$-preserving, $\pi^*(W^M_\xi)$ is a generalized witness to $\pi^*(\xi) \in P_{M^*}$ for $\nu < \xi \in P_M$.

Hence $\pi^*(P_M \setminus (\nu+1)) = P_{M^*} \setminus (\nu+1)$.

But π^* takes ν cofinally to $\pi^*(\nu)$, since $cf(\nu) = \tau$ in M and π^* takes τ cofinally to ν. For each $\alpha < \nu$, $\pi^*(W^M_{\alpha,\nu})$ is a generalized $\pi^*(\alpha), \pi^*(\nu)$ witness to $\pi^*(\nu)$. Hence $\pi^*(\nu) \in P_{M^*}$ and:

Note that $\sigma^* \upharpoonright (\nu+1) = \mathrm{id}$. Hence

(2) $\pi^*(\rho_M \setminus \nu) = \rho_{M^*} \setminus \nu$

Note that $\sigma^* \upharpoonright (\nu+1) = \mathrm{id}$. Hence

(3) $\sup \sigma^{*} {}'' \omega\rho^* < \omega\rho^n_{M'}$,

since otherwise σ^* is $\Sigma_1^{(n)}$-preserving $\nu \cup (\rho_M \setminus \nu) \subset \mathrm{rng}(\sigma^*)$ and M' is sound above ν. Hence $\sigma^* = \mathrm{id}$, $M^* = M'$ and $W_\nu^M \in M^*$. Contr!

We now show:

<u>Claim</u> M^* is solid.

<u>Case A</u> $n = 0$.

Set $\tilde{\alpha} = \sup \sigma'' \alpha^*$, where $M = \langle J_\alpha^E, F \rangle$, $M^* = \langle J_{\alpha^*}^{E^*}, F^* \rangle$, $M' = \langle J_{\alpha'}^{E'}, F' \rangle$.

Then $M' | \tilde{\alpha} = \langle J_{\tilde{\alpha}}^{E'}, F' \cap J_{\tilde{\alpha}}^{E'} \rangle$ is amenable, and $M' | \tilde{\alpha} \in M'$.

Hence M^* is codable by an $a^* \subset \nu^* = \pi^*(\nu)$, since $M^* = h_{M^*}(\nu^* \cup (\rho_{M^*} \setminus \nu))$. But clearly $a^* \in M'$, since

$\mathrm{rng}(\sigma) = h_{M'|\tilde{\alpha}}(\nu^* \cup (p_{M'} \setminus \pi'(\nu)))$.

Hence $\alpha^* \in J^{E'}_{\pi'\sigma(\nu)}$, where $\pi'\sigma(\nu)$ is regular in M'. Hence $M^* \in M'$ and $\sigma^* \in M^*$, since

$$\sigma^*(h_{M^*}(i, \langle \vec{\xi}, p_{M^*} \setminus \nu^* \rangle)) = h_{M'|\tilde{\alpha}}(i, \langle \vec{\xi}, p_{M'} \setminus \pi'(\nu) \rangle)$$

for $\vec{\xi} < \nu$. By the minimality of $ht(M)$, M clearly satisfies:

(*) For all $\tilde{\alpha}, \sigma, N^*, N$, if $\tilde{N} = \langle J^E_{\tilde{\alpha}}, F \cap J^E_{\tilde{\alpha}} \rangle$ is amenable, N is a premouse, and $\sigma : N \xrightarrow[\Sigma_1]{} \tilde{N}$ where $\mathrm{crit}(\sigma) \geq \rho^1_N$, then N is solid.

(Note In this case $N \in M$ is a mouse by Lemma 1, since $\sigma : N \xrightarrow[\Sigma_0]{} N$.)

Clearly (*) is a Π_1 statement

But then (*) holds in M'. Hence M^* is solid. QED (Case A)

<u>Case B</u> $n > 0$.

Let $\tilde{\rho} = \sup \sigma^{*}{}''\rho^{*}$. Interpolate $\tilde{M}$ s.t.

$$\begin{tikzcd} M^{*} \arrow[rr, "\sigma^{*}"] \arrow[dr, "\tilde{\sigma}_0"'] & & M' \\ & \tilde{M} \arrow[ur, "\tilde{\sigma}_1"'] & \end{tikzcd}$$

where $\tilde{\sigma}_0 \supset \sigma^{*} \restriction \omega\rho^{*}$, $\tilde{\sigma}_1 \supset \mathrm{id} \restriction \omega\tilde{\rho}$ are $\Sigma_0^{(n)}$-preserving and $\tilde{\rho} = \rho^{n}_{\tilde{M}}$.

Clearly $\tilde{M}$ is sound and solid above $\omega\tilde{\rho}$. Moreover $\tilde{M}$ is a premouse since $\tilde{\sigma}_1$ is $\Sigma_0^{(n)}$-preserving. But $\tilde{M} \neq \mathrm{core}_{\omega\tilde{\rho}}\, M'$, since $\tilde{\rho} = \rho^{n}_{\tilde{M}} < \rho^{n}_{M'}$.

Hence by Lemma 4 one of the following must hold:

(a) $\tilde{M} = M' \| \eta$ for an $\eta < ht(M')$

(b) $\tilde{M} = Ult^{*}(M' \| \eta, E^{M'}_{\mu})$ for an $\eta < ht(M')$ and a $\mu \leq \omega\eta$ s.t. $\omega\rho^{\omega}_{M'\|\eta} \leq crit(E^{M'}_{\mu})$

(c) $\tilde{M} = M'' \| \eta$ for an $\eta < ht(M'')$, where $M'' = Ult^{*}(M', E^{M'}_{\omega\tilde{\rho}})$.

By Lemma 3, $\tilde{M} \in \mathbb{M}$ if (a) or (b) holds. Note that in Case (b), the embedding $k : M' \| \eta \xrightarrow[E_\mu^{M'}]{} \tilde{M}$ is easily seen to be an element of M', since

$$k(\tilde{h}^{n+1}(\xi, p_{M'\|\eta})) = \tilde{h}^{n+1}(\xi, p_{\tilde{M}})$$

for $\xi < \operatorname{crit}(E_\mu^{M'})$.

By the minimality of $\operatorname{ht}(M)$, M must satisfy:

(∗∗) For all $\omega\eta$, $\mu \le \omega\eta$, k, $\hat{M}$, M^*, π, if $\hat{M} = M\|\eta$ or $k : M\|\eta \xrightarrow[E_\mu]{} \hat{M}$ and $\pi : M^* \longrightarrow \hat{M}$ is Σ_0^n-preserving for all $\omega\rho^n_{M^*} > \operatorname{crit}(\pi)$, then M^* is solid.

But (∗∗) is a Π_1 statement and hence holds in M'. In Case (c) it holds in M'' as well. Hence M^* is solid.

QED (Claim)

Now let $W^* = W^{M^*}_{\nu^*}$, $\nu^* = \pi^*(\nu)$.
W^* is coded by an $a^* \subset \nu^*$ which is $\Sigma_1^{(n)}(W^*)$ in $r^* = \sigma^{*-1}(p_{M^*} \setminus (\nu^*+1)) \cup \{\nu^*\}$, where $\sigma^* : W^* \longrightarrow M^*$ is the collapsing map. W is coded by an $a \subset \nu$ which is $\Sigma_1^{(n)}(W)$ in $r = \sigma^{-1}(p_M \setminus (\nu+1)) \cup \{\nu\}$ by the same def. Since
$\pi^* : M \xrightarrow{\Sigma_1^{(n)}} M^*$ and $\pi(r) = r^*$,
we conclude:

$$\xi \in a \longleftrightarrow \pi^*(\xi) \in a^*$$
$$\longleftrightarrow \langle \xi, \alpha \rangle \in F(\{\langle \xi, \eta \rangle \mid \xi \in f(\eta)\}),$$

where $a^* = \pi^*(f)(\alpha)$, $f \in M$.
Hence $a \in M$. Hence $a \in J^{EM}_{\sigma(\nu)}$ and $W \in M$ is reconstructible from a in $J^{EM}_{\sigma(\nu)}$. QED (Lemma 5)

Lemma 6 Let M be a mouse. Let $M, \bar{M}, \sigma, \nu$ be as in Lemma 4. If (a), (b), (c) do not hold, then we have:

(d) $\bar{M} = M_1 \| \eta$ for an $\eta < ht(M_1)$, where $M_1 = Ult(E_\nu, M)$.

[Hence $\bar{M} = Q \| \eta$, where $\hat{Q} = (J^E_{\tau+})^M$, $\tau = \tau(E_\nu)$ and $\hat{Q} = Ult(\hat{Q}, E_\nu) \in M$]

prf. Suppose not.

We know: $\bar{M} = M' \| \eta$ for an $\eta < \nu^{+M'}$, where $M' = Ult^*(M, E_\nu)$.

~~Let~~ $\omega\rho^{n+1} \leq \nu < \omega\rho^n$ in $\bar{M}$.

know: $\tau = \omega\rho^{m+1}_M$ for ~~some~~ m, since otherwise $(J^E_{\nu+})^{M'} = (J^E_{\nu+})^Q = (J^E_\nu)^{M_1}$. Clearly $m \leq n$, since $\rho^n_M \geq \sup \sigma''\omega\rho^n_{\bar{M}} > \sigma(\nu)$.

Let $\hat{\sigma} : \hat{M} \to M$ be the core map.

Let $\hat{M} = core_\tau(M)$. Let $\bar{\pi} : \hat{M}^m \xrightarrow[E_\nu]{} N$. Define $\bar{\sigma} : N \to M'^m$ by:

$\bar{\sigma}(\bar{\pi}(f)(\alpha)) = \pi' \hat{\sigma}(f)(\alpha)$

for $f \in N$, $f : \kappa \to N$, $\alpha < \lambda = lh(E_\nu$

Then $\bar\sigma \upharpoonright (\nu+1) = id$. Hence

$\bar\sigma \upharpoonright \nu^{+N} = id$. Hence:

(1) $\nu^{+N} < \nu^{+M'}$,

since otherwise Q is an initial segment of N and $\bar M = M' \| \eta =$
$= N \| \eta = Q \| \eta = M_1 \| \eta$. Contr!

We can interpolate an M^* s.t.

$$\begin{tikzcd} \hat M \arrow[rr, "\pi'\hat\sigma"] \arrow[dr, "\pi^*"'] & & M' \\ & M^* \arrow[ur, "\sigma^*"'] & \end{tikzcd}$$

where $\pi^* \supset \bar\pi$, $\sigma^* \supset id \upharpoonright \nu^{+N}$

are $\Sigma_0^{(m)}$-preserving,

$M^* \| m, \bar\pi(p_{\hat M}) \upharpoonright m = N$ and

$M^* = \tilde h^{(m+1)}(\nu \cup \bar\pi(p_{\hat M}))$. (Note

M^* is the "m-ultraproduct" of $\hat M$.) Set $\gamma = \tau^{+\hat M}$. Then

$\gamma = \tau^{+M}$, since $\mathbb{P}(\tau) \cap M = \mathbb{P}(\tau) \cap \hat M$.

Set: $\gamma^* = \pi^*(\gamma)$, $\gamma' = \sigma^*(\gamma^*)$.

Then $\gamma^* = \nu + M^*$, $\gamma' = \nu + M'$, and $\sigma^* \upharpoonright \gamma^* = \mathrm{id}$.

<u>Case 1</u> $\gamma^* = \gamma'$.

Then $\eta < \gamma^*$. Hence $\bar{M} = M' \| \eta =$
$= M^* \| \eta = Q \| \eta = M_1 \| \eta$. Contr.!

<u>Case 2</u> $\gamma^* < \gamma'$.

Then $\sigma^* : M^* \xrightarrow[\Sigma_0^{(m)}]{} M'$, where $\gamma^* = \mathrm{crit}(\sigma^*)$ and $\rho^{m+1}_{M^*} \leq \gamma^* < \rho^m_{M^*}$.

Moreover, letting $p^* = \pi^*(p_M \setminus \nu)$, we have $p^* = P_{M^*} \setminus \nu$, since $M^* = h^m_{M^*}(\nu \cup p^*)$ and the $\nu \in p^*$ have generalized witnesses. Hence M^* is sound above ν. Hence M^* is sound above γ and we apply Lemma 4 to $\sigma : M^* \to M'$. Cases (c), (d) cannot hold, since γ^* is a double successor in M^*. But $M^* \neq \mathrm{core}_{\gamma^*}(M')$, since otherwise:

$\rho_{M^*}^{m+1} = \rho_{M'}^{m+1} = \gamma' > \gamma^* \geq \omega\rho_{M^*}^{m+1}$.

Contr! Thus Case (b) holds and $M^* = M' \| \eta$ for some $\eta < \gamma'$, since M^* is sound above γ^*. Thus we have $\bar{\eta}, \eta < \gamma'$ s.t.

$$M^* = M' \| \eta \quad , \quad \bar{M} = M' \| \bar{\eta} .$$

<u>Claim</u> $\bar{\eta} < \eta$

Suppose not.

<u>Case A</u> $\eta < \bar{\eta}$.

Let $a \subset \tau = \omega\rho_{\hat{M}}^{m+1}$ be $\Sigma_1^{(m)}(\hat{M})$ in $q = p_{\hat{M}} \cup \{\tau\}$ s.t. a codes $\hat{M}$ and let a^* be $\Sigma_1^{(m)}(M^*)$ in $q^* = \pi^*(q^*)$ by the same def.

Then $a^* \subset \nu$ codes M^*. But $a^* \in \bar{M}$. Hence $\tilde{a} = \nu \cap \sigma(a^*) \in M$. Clearly $\xi \in a \longleftrightarrow \pi^*(\xi) \in a^*$.

But $\pi^* \upharpoonright \hat{Q} : \hat{Q} \xrightarrow[E_\nu]{} Q$; Hence $\pi^* \upharpoonright \hat{Q} \in M$ and $a \in M$. Hence $A_M^n = A_{\hat{M}}^n \in M$. Contr! QED (Case A)

Case B $\eta = \bar{\eta}$.

Then $\bar{M} = M^*$. It follows exactly as in Case A that $a^* \notin \bar{M}$. Using this we derive a contradiction.

Clearly $\nu = \omega\rho_{\bar{M}}^{\omega}$, since ν is a cardinal in M. Hence $\bar{M}$ is sound. Moreover $p_{\bar{M}} = q^* = \pi^*(\hat{q})$, where $\hat{q} = p_{\hat{M}} \setminus \tau$. Let $p_{\hat{M}} \setminus \tau = \{\eta_i \mid i < n\}$, where $\eta_0 > \dots > \eta_{n-1}$. ~~Let~~

$\eta_i^* = \pi^*(\eta_i)$, $\eta_i' = \sigma^*(\eta_i^*)$.

Let $\eta_i'' = \hat{\sigma}(\eta_i)$. Then $p_M \setminus \tau =$

$= \{\eta_i'' \mid i < n\}$.

Claim $\sigma(q^*) = p_M \setminus \tau$.

prf. Suppose not. Let i be least s.t. $\eta_i'' \neq \eta_i'$. Suppose $\eta_i' < \eta_i''$.

Then $\sigma(q^*) \subset \operatorname{rng}(k)$, where $k : W^M_{\eta_i''} \to M$ is the collapsing map.

It follows that $a^* \in M$;

since $\eta_i'' > \eta_i' > \nu$ and hence $\nu \cup q^*) \subset \mathrm{rng}(k)$. Hence a^* is $\Sigma_1^{(m)}(W^M_{\eta_i''})$. Contr!

Now let $\eta_i'' < \eta_i'$. $\sigma(W^{\bar{M}}_{\tilde{\eta}_i})$ is a generalized witness for η_i'. Hence $\eta_i' \in P_M$. Contr! QED (Claim)

But then $p_M \cup \tau \subset \mathrm{rng}(\sigma)$. Hence $\mathrm{rng}(\hat{\sigma}) \subset \mathrm{rng}(\sigma)$. Set:

$\tilde{\sigma} = \sigma^{-1}\hat{\sigma}$. Then $\tilde{\sigma} : \hat{M} \longrightarrow_{\Sigma_1^{(m)}} \bar{M}$, $\tilde{\sigma} \upharpoonright \tau = \mathrm{id}$, $\tilde{\sigma}(p_{\hat{M}} \setminus \tau) = p_{\bar{M}}$, where $\tau = \omega\rho_{\hat{M}}^{m+1}$. Hence $\hat{M} = \mathrm{core}_\tau(\bar{M})$.

But $\bar{M}$ is sound. Hence $\hat{M} = \bar{M}$.

But $\tau = \omega\rho_{\hat{M}}^{m+1} < \nu = \omega\rho_{\bar{M}}^{\omega}$. Contr!

QED