§2. Combinatorial properties of L

Thm 1 (Everyone) $V = L \to \diamond_\kappa$.
Moreover, if $V = L$ and $B \subset \kappa$ is Mahlo in κ, then there is $\langle S_\alpha \mid \alpha \in B \rangle$ s.t. for all $S \subset \kappa$, $\{\alpha \mid S \cap \alpha = S_\alpha\}$ is Mahlo in κ.

proof.
By induction on $\alpha \in B$ define $S_\alpha, C_\alpha \subset \alpha$ s.t.

$\langle S_\alpha, C_\alpha \rangle$ = the least pair $\langle S, C \rangle$ (in $<_L$) s.t. $C \subset \alpha$ is closed, unbounded in α and $\bigwedge \gamma \in C \;\; S \cap \gamma \neq S_\gamma$. if such $\langle S, C \rangle$ exists.

Claim If $S \subset \kappa$, then $\{\alpha \mid X \cap \alpha = S_\alpha\}$ is Mahlo.

Suppose not. Then there is a pair $\langle S, C\rangle$ s.t. C is closed, unbounded in κ, $S \subset \kappa$, and $\bigwedge \alpha \in C \quad S \cap \alpha \neq S_\alpha$. Let $\langle S, C\rangle$ be the least such pair (in $<_L$).

Then $\langle S, C\rangle$ is L_{κ^+}-definable from B.

Define a sequence $N_\nu \prec L_{\kappa^+}$ by:

$N_0 =$ the smallest $N \prec L_{\kappa^+}$ s.t. $N \cap \kappa$ is transitive & $B \in N$

$N_{\nu+1} =$ the smallest $N \prec L_{\kappa^+}$ s.t. $N \cap \kappa$ is transitive and $N_\nu \cup \{N_\nu\} \subset N$

$$N_\lambda = \bigcup_{\nu \in \lambda} N_\nu \quad \text{if } Lim(\lambda).$$

Set: $\alpha_\nu = \kappa \cap N_\nu$. Then $\langle \alpha_\nu \,{}_{<\kappa}\rangle$ is a normal function.

Hence there is $\alpha = \alpha_\alpha$ s.t. $\alpha \in B$.

Let $\pi : N_\alpha \xleftrightarrow{\sim} L_\beta$. Then

$\pi(\kappa) = \alpha$; $\pi(\langle S, C\rangle) = \langle S \cap \alpha, C \cap \alpha\rangle$,

and:

$\langle S\cap\alpha, C\cap\alpha\rangle$ = the least pair $\langle S', C'\rangle$ s.t. $S', C' \subset \alpha$, C' is closed, unbonded in α, and $\bigwedge \gamma \in C'$ $S' \cap \gamma \neq S_\gamma$.

Hence: $\langle S\cap\alpha, C\cap\alpha\rangle = \langle S_\alpha, C_\alpha\rangle$.

Hence: $\alpha \in C \wedge S\cap\alpha = S_\alpha$.

Contradiction! QED

<u>Note</u> The hypothesis $V = L$ in Thm 1 may be replaced by:

$\bigvee A \subset \kappa$ $V = L[A]$.

Theorem 2 (Jensen)

$$V = L \wedge \gamma < \kappa . \to \diamondsuit^{+}_{\kappa \gamma} .$$

proof.

Define $S_x \subset \mathcal{P}(\mathrm{lub}(x))$ $(\bar{\bar{x}} < \gamma, x \subset \kappa)$ as follows:

$S_x = \emptyset$ if $\bar{\bar{x}} < \omega$; otherwise:

$S_x = \mathcal{P}(\mathrm{lub}(x)) \cap M_x$, where

M_x = the smallest $M \prec L_\kappa$ s.t. $x \cup \{x\} \cup \{\gamma\} \subset M$.

Thus $\bar{\bar{S}}_x \leq \bar{\bar{x}}$.

Claim If $X \subset \kappa$, there is an unbounded $B \subset \kappa$ s.t. whenever $\alpha = \mathrm{lub}(x)$ is a limit pt. of $B \cap x$, then $X \cap \alpha$, $B \cap \alpha \in M_x$.

proof. Suppose not. Let X be the least $X \subset \kappa$ (in $<_L$) for which the claim fails. Then X is L_{κ^+}-definable from the parameter γ.

Define $N_\nu \prec L_{\kappa^+}$ $(\nu < \kappa)$ by:

$N_0 =$ the smallest $N \prec L_{\kappa^+}$ s.t. $N \cap \kappa$ is transitive and $\gamma \in N$

$N_{\nu+1} =$ the smallest $N \prec L_{\kappa^+}$ s.t. $N \cap \kappa$ is transitive and $N_\nu \cup \{N_\nu\} \subset N$.

$$N_\lambda = \bigcup_{\nu < \lambda} N_\nu \quad \text{for limit } \lambda.$$

Set: $\alpha_\nu = \kappa \cap N_\nu$. Then $\langle \alpha_\nu \mid \nu < \kappa \rangle$ is a normal fcn.

Set: $\sigma_\nu : N_\nu \xrightarrow{\sim} L_{\beta_\nu}$. Then $\alpha_\nu < \beta_\nu < \alpha_{\nu+1}$; $\alpha_\nu = \sigma_\nu(\kappa)$; $X \cap \alpha_\nu = \sigma_\nu(X)$.

Set: $B = \{\beta_\nu \mid \nu < \kappa\}$.

Now let $x \subset \kappa$, $\omega \le \overline{\overline{x}} < \gamma$ and let $\alpha = \text{lub}(x)$ be a limit pt. of $B \cap x$.

<u>Claim</u> $\alpha \cap X, \alpha \cap B \in M_x$.

proof. Set: $M = M_x$.

Let $\eta = \sup\{\nu \mid \beta_\nu \in x\}$.

Then $\alpha = \alpha_\eta$. It suffices to prove: $\beta_\eta \in M$, since ~~$X \cap \alpha_\eta$ is definable from L_{β_η}~~, γ

~~the way~~

$X \cap \alpha_\eta$ is L_{β_η} - definable from γ and $B \cap \alpha_\eta$ is definable from L_{β_η}, γ the way that B was defined from L_{κ^+}, γ.

We first prove:

(*) $\mathcal{P}(\alpha_\eta) \cap M \not\subset L_{\beta_\eta}$

proof. $\alpha_\eta > \gamma$, but $cf(\alpha_\eta) = cf(X) < \gamma$. Hence there is $a \in M$ s.t. $a \subset \alpha_\eta$ is cofinal in α_η but of order type $< \alpha_\eta$. Then $a \notin L_{\beta_\eta}$, since α_η is regular in L_{β_η}. QED (*)

For $\nu \in \eta \cap M$ set: $\beta^{(\nu)} = \langle \beta_\iota \mid \iota \leq \nu \rangle$. Then $\beta_\nu \in M \rightarrow \beta^{(\nu)} \in M$ $(\nu < \eta)$ since $\langle \beta_\iota \mid \iota < \nu \rangle$ is definable from L_{β_ν}, γ the way $\langle \beta_\iota \mid \iota < \kappa \rangle$

was defined from $L_{\varkappa^+}, \gamma$. ~~At~~ It follows that:

$$\nu \in \eta \cap M \rightarrow \beta^{(\nu)} \in M,$$

since there is a $\tau \geq \nu, < \eta$ s.t. $\beta_\tau \in M$ and $\beta_\nu = \beta^{(\tau)}(\nu) \in M$.

Set: $\sigma_{i\nu} = \sigma_i \circ \sigma_\nu^{-1}$ $(i < \nu < \varkappa)$.

For $\nu \in \eta \cap M$, set:

$$\sigma^{(\nu)} = \langle \sigma_{i\tau} \mid i < \tau \leq \nu \rangle.$$

Then $\sigma^{(\nu)} \in M$, since $\cancel{\langle \sigma_{i\tau} \mid i < \tau \leq \nu \rangle}$ ~~and~~ $\langle \sigma_{i\nu} \mid i < \nu \rangle$ is definable from L_{β_ν}, γ as $\langle \sigma_i \mid i < \varkappa \rangle$ was defined from $L_{\varkappa^+}, \gamma$ and $\sigma_{i\tau} = \sigma_{i\nu} \sigma_{\tau\nu}^{-1}$ for $i < \tau < \nu$.

We now show that β_η is definable in a canonical fashion from $\langle \beta_\iota \mid \iota < \eta \rangle$. We note that

$$\langle L_{\beta_\iota}, \in \rangle, \quad \sigma_{\iota\nu} \quad (\iota < \nu < \eta)$$

is a directed system of elementary imbeddings and is definable in a canonical fashion from $\langle \beta_\iota < \eta \rangle$. Let $\langle U, E \rangle$ be the direct limit of this system + let

$g_\iota : \langle L_{\beta_\iota}, \in \rangle \longrightarrow \langle U, E \rangle$ be the natural projections. It is clear that $\langle U, E \rangle$ is isomorphic to $\langle L_{\beta_\eta}, \in \rangle$ and that, if f is the isomorphism, then $\sigma_{\iota\eta} = f g_\iota$.

Now let $\pi : M \xrightarrow{\sim} L_\delta$.

Set: $\tilde{\beta}^{(\nu)} = \pi(\beta^{(\nu)})$; $\tilde{\sigma}^{(\nu)} = \pi(\sigma^{(\nu)})$ for $\nu \in M \cap \eta$. Then $\tilde{\beta}^{(\nu)}$ is an initial segment of $\tilde{\beta}^{(\tau)}$ for $\nu < \tau$; similarly for $\tilde{\sigma}^{(\nu)}$. If we set:

$$\tilde{\beta} = \bigcup_{\nu \in M \cap \eta} \tilde{\beta}^{(\nu)} \quad ; \quad \tilde{\sigma} = \bigcup_{\nu \in M \cap \eta} \tilde{\sigma}^{(\nu)} ,$$

then $\tilde{\beta}$ is a function defined on $\pi(\eta)$, $\tilde{\sigma}$ is a function defined on $\{\langle \nu, \tau \rangle \mid \nu < \tau < \pi(\eta)\}$ and:

$$\tilde{\beta}_\iota = \pi(\beta_{\pi^{-1}(\iota)})$$
$$\tilde{\sigma}_{\iota\nu} = \pi(\sigma_{\pi^{-1}(\iota)\,\pi^{-1}(\nu)}) .$$

$\langle L_{\tilde{\beta}_\iota}, \in \rangle$, $\tilde{\sigma}_{\iota\nu}$ $(\iota < \nu < \pi(\eta))$ is a directed system of elementary imbeddings. Let

$\langle \tilde{U}, \tilde{E} \rangle$ be the direct limit of this system. Let $\tilde{g}_\iota : \langle L_{\tilde{\beta}_\iota}, \in \rangle \to \langle \tilde{U}, \tilde{E} \rangle$ be the natural projections. Then there is an elementary imbedding h defined by:

$$\begin{tikzcd}
\langle L_{\beta_{\pi^{-1}(\iota)}}, \in \rangle \arrow[r, "g_{\pi^{-1}(\iota)}"] & \langle U, E \rangle \\
\langle L_{\tilde{\beta}_\iota}, \in \rangle \arrow[u, "\pi^{-1}"] \arrow[r, "\tilde{g}_\iota"] & \langle \tilde{U}, \tilde{E} \rangle \arrow[u, "h"']
\end{tikzcd}$$

Thus $\langle \tilde{U}, \tilde{E} \rangle$ is well founded. Let $\tilde{f} : \langle \tilde{U}, \tilde{E} \rangle \overset{\sim}{\longleftrightarrow} \langle U, \in \rangle$, where U is transitive. Then $U = L_{\bar{\beta}}$ for some $\bar{\beta}$, since

$\langle \tilde{U}, \tilde{E} \rangle$ is a model of $V = L$.

Set: $\bar{\sigma}_\iota = \bar{f}\bar{g}_\iota$; $\pi^* = f h f^{-1}$.

Then $\bar{\sigma}_\iota \bar{\sigma}_\tau^{-1} = \tilde{\sigma}_{\iota\tau}$ $(\iota < \tau < \pi(\eta))$

and:

$$\begin{tikzcd}
L_{\beta_{\pi^{-1}(\iota)}} \arrow[r, "\sigma_{\pi^{-1}(\iota)\eta}"] & L_{\beta_\eta} \\
L_{\tilde{\beta}_\iota} \arrow[u, "\pi^{-1}"] \arrow[r, "\bar{\sigma}_\iota"] & L_{\bar{\beta}} \arrow[u, "\pi^*"']
\end{tikzcd}$$

Set: $\tilde{\alpha}_\iota$ = the largest $\alpha \in L_{\tilde{\beta}_\iota}$ s.t. α is a cardinal in $L_{\tilde{\beta}_\iota}$

$\bar{\alpha}$ = the largest $\alpha \in L_{\bar{\beta}}$ s.t. α is a cardinal in $L_{\bar{\beta}}$.

It is easily seen that:

$\tilde{\alpha}_\iota = \pi(\alpha_{\pi^{-1}(\iota)})$; $\bar{\sigma}_\iota(\tilde{\alpha}_\iota) = \bar{\alpha}_\iota$;

$\bar{\sigma}_\iota \upharpoonright L_{\tilde{\alpha}_\iota} = \mathrm{id} \upharpoonright L_{\tilde{\alpha}_\iota}$ for $\iota < \pi(\eta)$.

Hence $\bar{\alpha} = \sup\limits_{\iota<\eta} \tilde{\alpha}_\iota$. Since

$\alpha_\eta = \sup\limits_{\iota \in \eta \cap M} \alpha_\iota$, we conclude:

$\bar{\alpha} = \pi(\alpha_\eta)$. Note that:

$\sigma_{\iota\eta}(\alpha_\iota) = \alpha_\eta$; $\sigma_{\iota\eta} \upharpoonright L_{\alpha_\iota} = \mathrm{id} \upharpoonright L_{\alpha_\iota}$

for $i < \eta$.

Using these facts, we get:

(a) $x \in L_{\bar{\alpha}} \longrightarrow \pi^*(x) = \pi^{-1}(x)$.

proof. Let $x \in L_{\tilde{\alpha}_\nu}$ $(\nu < \pi(\eta))$

~~Then $\pi^*(x)$~~ Let $\nu = \pi(\nu')$.

Then $\pi^*(x) = \sigma_{\nu'\eta}\, \pi^{-1}\, \bar{\sigma}_\nu^{-1}(x) = \pi^{-1}(x)$

QED (a)

(b) $\pi^*(\bar{\alpha}) = \alpha_\eta$

proof. $\pi^*(\bar{\alpha})$ = the largest $\alpha \in L_{\beta_\eta}$ s.t. α is a cardinal in L_{β_η} = $= \alpha_\eta$, since π^* is an elementary imbedding into L_{β_η}.

(c) If $a \subset \bar{\alpha}$, $a \in L_{\bar{\beta}} \cap L_\delta$, then
$\pi^*(a) = \pi^{-1}(a)$.

proof.

(i) $\pi^*(a), \pi^{-1}(a) \subset \alpha_\eta$, since $a \subset \bar{\alpha}$.

~~(ii) $\pi^*(a)$~~ Hence:

(ii) $\pi^*(a) = \bigcup_{\nu \in M \cap \eta} \pi^*(a) \cap \nu =$

$= \bigcup_{\nu < \pi(\eta)} \pi^*(a \cap \nu)$

(iii) $\pi^{-1}(a) = \bigcup_{\nu < \pi(\eta)} \pi^{-1}(a \cap \nu)$

(similarly)

But $\pi^*(a \cap \nu) = \pi^{-1}(a \cap \nu)$ for $\nu < \pi(\eta)$ by (a). QED (c)

(d) $\delta > \bar{\beta}$.

proof. Suppose not. Then

$\mathcal{P}(\alpha_\eta) \cap M \subset L_{\beta_\eta}$, since $\pi^* \pi$ maps M into L_{β_η} and $\pi^* \pi(a) = a$ for $a \subset \alpha_\eta$, $a \in M$, by (c). QED (d)

Hence $\bar{\beta} \in L_\delta$. It follows that $\tilde{\beta} = \langle \tilde{\beta}_i \mid i < \pi(\eta) \rangle \in L_\delta$, since $\tilde{\beta}$ is definable from $L_{\bar{\beta}}$, $\pi(\gamma)$ as $\langle \beta_i \mid i < \kappa \rangle$ was defined from L_{κ^+}, γ. Similarly,

$$\tilde{\sigma} = \langle \tilde{\sigma}_{i\nu} \mid i < \nu < \pi(\eta) \rangle \in L_\delta .$$

It is easily seen that:

$$\pi^{-1}(\tilde{\beta}) = \langle \beta_\iota \mid \iota < \eta \rangle$$

$$\pi^{-1}(\tilde{\sigma}) = \langle \sigma_{\iota\nu} \mid \iota < \nu < \eta \rangle .$$

Since $L_{\bar{\beta}} \simeq$ the direct limit of $L_{\tilde{\beta}_\iota}, \tilde{\sigma}_{\iota\nu}$ $(\iota < \nu < \pi(\eta))$, we have:

~~$L_{\pi(\tilde{\beta})}$ the direct limit of~~

~~$L_{\beta_\iota}, \sigma_{\iota\nu}$ $(\iota < \nu < \eta$~~ $)$.

~~Hence~~

$$L_{\pi(\bar{\beta})} \simeq \text{the direct limit of}$$

$$L_{\beta_\iota}, \sigma_{\iota\nu} \quad (\iota < \nu < \eta).$$

Hence $\pi(\bar{\beta}) = \beta_\eta \in M$. QED

<u>Note</u> The hypothesis $V = L$ can be replaced by: $\bigvee A \subset \kappa$ $V = L[A]$ in Theorem 2.

We now turn to the question: For what κ does $\Diamond^+_{\kappa\kappa}$ hold, assuming $V = L$? We define a class of large cardinals which we call <u>ineffable</u> and prove the following theorems (in ZFC):

$$\kappa \text{ ineffable} \longrightarrow \Diamond_\kappa$$

$$\kappa \text{ ineffable} \longrightarrow \neg \Diamond^+_\kappa$$

$$V = L \wedge \kappa \text{ not ineffable} \longrightarrow \Diamond^+_{\kappa\kappa}.$$

(Hence $V = L \rightarrow \Diamond^+_\kappa \longleftrightarrow \Diamond^+_{\kappa\kappa}$).

The ineffable cardinals appear to be of interest in their own right. The definition of ineffability is due independently to Jensen and Kunen.

<u>Def</u> Let κ be a regular cardinal. κ is <u>ineffable</u> iff whenever $\langle A_\alpha \mid \alpha < \kappa \rangle$ is a sequence s.t. $A_\alpha \subset \alpha$ for $\alpha < \kappa$, then there is an $X \subset \kappa$ s.t. X is Mahlo in κ and $\alpha, \beta \in X \wedge \alpha < \beta . \rightarrow A_\alpha = \alpha \cap A_\beta$.

<u>Theorem 3</u> (Jensen) κ is ~~in~~ ineffable iff whenever $M_\nu = \langle |M_\nu| ; \ldots \rangle$ $(\nu < \kappa)$ is a sequence of ~~relational systems with~~ models of a ~~languag~~ language $\mathcal{L}$ with fewer than κ symbols • s.t. $|M_\lambda| = \bigcup_{\nu < \lambda} |M_\nu|$ for limit $\lambda < \kappa$, then there is a Mahlo set $X \subset \kappa$ s.t.

$$\alpha, \beta \in X \wedge \alpha < \beta . \rightarrow M_\alpha \prec M_\beta .$$

(The proof is straightforward)

<u>Theorem 4</u> (Kunen) The following are equivalent:

(i) κ is ineffable

(ii) If $f : [\kappa]^2 \to \kappa$ s.t. $f(\alpha,\beta) < \alpha$ for $\alpha < \beta < \kappa$, then there is a Mahlo set $X \subset \kappa$ s.t. X is homogeneous for f.

(iii) If $f : [\kappa^2] \to 2$, then there is a Mahlo set $X \subset \kappa$ s.t. X is homogeneous for f.

proof.

(i) $\to$ (ii) Define $\bar{f}_\alpha : \alpha \to \alpha$ by:

$$\bar{f}_\alpha(\nu) = f(\nu, \alpha).$$

By ineffability, there is a Mahlo set $X' \subset \kappa$ s.t.

$$\alpha, \beta \in X' \wedge \alpha < \beta \to \bar{f}_\alpha = \bar{f}_\beta \upharpoonright \alpha .$$

Define $\bar{f} : X' \to \kappa$ by:

$\bar{f}(\nu) = \bar{f}_\alpha(\nu)$ for $\alpha > \nu$, $\alpha \in X$.

Since $\bar{f}(\alpha) < \alpha$ for $\alpha \in X'$ and X' is Mahlo, there exists a $\nu_0 < \kappa$ s.t. $X = \{\alpha \in X' \mid \bar{f}(\alpha) = \nu\}$ is Mahlo. But X is homogeneous for f.

QED (i)$\to$(ii)

(ii) $\to$ (iii) (trivial)

(iii) $\to$ (i) Let $A_\alpha \subset \alpha$ $(\alpha < \kappa)$

Order ~~P(κ) by~~ the bounded subsets of κ by:

$a \prec b \leftrightarrow \bigvee\alpha\,(a \cap \alpha = b \cap \alpha \wedge \alpha \in b \setminus a)$.

Define $f : [X]^2 \to 2$ by:

$$f(\alpha, \beta) = \begin{cases} 0 & \text{if } A_\alpha \prec A_\beta \\ 1 & \text{if not.} \end{cases}$$

Let $X \subset \kappa$ be Mahlo in κ and homogeneous for f. Since $<$ admits no infinite descending κ-sequence, we have: $f''[X]^2 = \{0\}$. Define $g : \kappa \to 2$ by:

$$g(\nu) = \begin{cases} 1 & \text{if } \bigvee \alpha \in X \; (\nu \in A_\alpha \wedge \\ & \wedge\; A_\alpha \cap \nu = \{\iota < \nu \mid g(\iota) = 1\} \\ 0 & \text{if not} \end{cases}$$

By induction on ν, there exists a least $\alpha_\nu \in X$ s.t.

$$\bigwedge \beta \geq \alpha_\nu \; (\beta \in X \to A_\beta \cap \nu = \{\iota < \nu \mid g(\iota) = 1\}).$$

Let $C = \{\alpha \mid \alpha = \sup_{\nu < \alpha} \alpha_\nu\}$.

Then C is closed, unbounded in κ. Hence $C \cap X$ is Mahlo in κ. It is easily seen that:

$$\alpha \in C \cap X \rightarrow \alpha = \alpha_\alpha$$

$$\rightarrow A_\alpha = \{\nu < \alpha \mid g(\nu) = 1\}$$

Hence $\beta, \alpha \in C \wedge \alpha < \beta . \rightarrow A_\alpha = \alpha \cap A_\beta$.

QED

Theorem 5 (Kunen) If κ is ineffable, then κ is Π^1_2 indescribable.

Theorem 6 (Jensen, Kunen) The least ineffable cardinal is smaller than the least $\kappa \rightarrow (\omega)^{<\omega}$ and larger than the least cardinal which is Π^n_m indescribable for $n, m < \omega$.

The proofs of these theorems are straightforward.

Theorem 7 (Jensen, Kunen) If κ is ineffable, then κ is ineffable in L.

proof.

Let $\langle A_\alpha \mid \alpha < \kappa \rangle \in L$ s.t. $A_\alpha \subset \alpha$.

Let $A \subset \kappa$ be s.t. $\{\alpha \mid A_\alpha = \alpha \cap A\}$ is Mahlo. We claim: $A \in L$.

Since κ is Π^1_1 indescribable, it suffices to show: $\bigwedge \alpha < \kappa \; A \cap \alpha \in L$. For this it suffices that $A \cap \alpha \in L$ for arbitrarily large $\alpha < \kappa$.

But $A \cap \alpha = A_\alpha \in L$ on a Mahlo set.

QED

~~Theorem 8 (Jensen) If κ is ineffable, then~~

Theorem 8 (Jensen) Let M be a model of ZFC + let κ be ineffable in M. Then there is a Cohen extension N s.t. GCH holds in N and κ is ineffable in N.

We omit the proof of this theorem because it is rather long and because it is not relevant to our present concerns.

~~Note One might want to consider the cardinals obtained by requiring~~

Note Suppose we weaken the definition of ineffability by requiring only that the set $X \subset \kappa$ be unbounded in κ

rather than Mahlo. Call such cardinals <u>almost</u> <u>ineffable</u>. Then the least almost ineffable cardinal is not ineffable. However, the ~~an~~ analogues of ~~Th~~ Thm 3, Thm 6 - Thm 8 hold for almost ineffable cardinals. In place of Thm 5 we have: If κ is almost ineffable, then κ is Π^1_1-indescribable.

<u>Theorem 9</u> (Jensen, Kunen) If κ is ineffable, then $\neg KH_\kappa$.

proof.

Let $B \subset \mathcal{P}(\kappa)$ be s.t. $\overline{\overline{B|\alpha}} \leq \overline{\overline{\alpha}}$ for $\alpha < \kappa$. Let b_ν^α $(\nu < \alpha)$ enumerate $B|\alpha$. Set:

$R_\alpha = \{\langle \iota, \nu\rangle \mid \iota \in b_\nu^\alpha\}$. Then $R_\alpha \subset \alpha^2$.

By ineffability, there is an $R \subset \kappa^2$ s.t. $\{\alpha < \kappa \mid R \cap \alpha^2 = R_\alpha\}$ is Mahlo in κ. Set: $b_\nu = R''\{\nu\}$ $(\nu < \kappa)$. <u>We claim</u>:

$$B \subset \{b_\nu \mid \nu < \kappa\}.$$

Suppose not. Let $b \in B$, $b \neq b_\nu$ $(\nu < \kappa)$. Then there is a closed, unbounded $A \subset \kappa$ s.t.

$\alpha \in A \longrightarrow \bigwedge_{\gamma < \alpha} \ b \cap \alpha \neq b_\gamma \cap \alpha$.

Since $\{\alpha < \kappa \mid R \cap \alpha^2 = R_\alpha\}$ is Mahlo, there is an $\alpha \in A$ s.t. $R \cap \alpha^2 = R_\alpha$. Then $b_\gamma \cap \alpha = b_\gamma^\alpha$ for $\gamma < \alpha$. Hence $b \cap \alpha \notin B|\alpha$. Hence $b \notin B$. Contradiction! QED

[<u>Note</u> If $\diamondsuit''_\kappa$ is the weaker version of $\diamondsuit^+_\kappa$ mentioned in §1 p. 2, then an entirely analogous proof shows: If κ is ineffable, then $\neg \diamondsuit''_\kappa$.]

Theorem 10 (Jensen) If $V = L$ and κ is not ineffable, then $\diamondsuit^+_{\kappa\kappa}$ holds.

proof.

Some sequence $\langle A_{\alpha < \kappa} \rangle$ is a counterexample to ineffability (i.e. $A_\alpha \subset \alpha$ and there is no Mahlo set $X \subset \kappa$ s.t.

$$\alpha, \beta \in X \wedge \alpha < \beta . \rightarrow A_\alpha = \alpha \cap A_\beta \;).$$

Let $\langle A_{\alpha < \kappa} \rangle$ be the least such counterexample (in $<_L$). Then $\langle A_{\alpha < \kappa} \rangle$ is L_{κ^+}-definable.

We define $S_x \subset \mathcal{P}(\mathrm{lub}(x))$ $(x \subset \kappa,\ \overline{\overline{x}} < \kappa)$ as follows:

$S_x = \emptyset$ if $\overline{\overline{x}} < \omega$.

$S_x = \mathcal{P}(\mathrm{lub}(x)) \cap M_x$, otherwise,

where M_x is the smallest $M \prec L_\kappa$ s.t. $x \cup \{x\} \subset M$ and $A_{\mathrm{lub}(x)} \in M$.

We must show that, whenever $X \subset \kappa$, there is an unbounded $B \subset \kappa$ s.t. if $\alpha = \mathrm{lub}(x)$ is a limit pt. of $B \cap x$, then $X \cap \alpha$, $B \cap \alpha \in S_x$.

Suppose not. Let X be the least $X \subset \kappa$ (in $<_L$) for which this fails. Then X is L_{κ^+} definable

We imitate the proof of Thm 2, defining a sequence $N_\nu \prec L_{\kappa^+}$ $(\nu < \kappa)$ as follows:

N_0 = the smallest $N \prec L_{\kappa^+}$ s.t. $N \cap \kappa$ is transitive

$N_{\nu+1}$ = the smallest $N \prec L_{\kappa^+}$ s.t. $N \cap \kappa$ is transitive and $N_\nu \cup \{N_\nu\} \subset N$

$N_\lambda = \bigcup_{\nu<\lambda} N_\nu$ for $Lim(\lambda)$.

We define $\alpha_\nu, \beta_\nu, \sigma_\nu$ as before by: $\alpha_\nu = \kappa \cap N_\nu$; $\sigma_\nu : N_\nu \overset{\sim}{\longleftrightarrow} L_{\beta_\nu}$.

Set: $B = \{\beta_\nu \mid \nu < \kappa\}$.

Let $\alpha = \text{lub}(x)$ be a limit point of $B \cap x$. ~~We claim~~:

<u>Claim</u> $B \cap \alpha, X \cap \alpha \in M_x$.

proof. Set: $M = M_x$.

Let $\eta = \sup\{\nu \mid \beta_\nu \in x\}$.

Then $\alpha = \alpha_\eta$. As before, it suffices to show: $\beta_\eta \in M$.

The proof is exactly as before except that a new proof is required for:

(*) $\mathbb{P}(\alpha) \cap M \not\subset L_{\beta_\eta}$.

proof of (*).

Suppose not. Then $A_{\alpha_\eta} \in L_{\beta_\eta}$.

Set: $X = \{\gamma < \alpha_\eta \mid A_\gamma = \gamma \cap A_{\alpha_\eta}\}$.

We consider two cases:

<u>Case 1</u> X is Mahlo in α_η in the sense of the model L_{β_η}.

Set $X^* = \sigma_\eta^{-1}(X)$. Then X^* is Mahlo in κ and

$$\alpha, \beta \in X^*, \ \alpha < \beta \ . \rightarrow A_\alpha = \alpha \cap A_\beta .$$

Contradiction!

<u>Case 2</u> X is not Mahlo.

Then there is $C \in L_{\beta_\eta}$ s.t. C is closed, unbounded in α_η and $A_\gamma \neq \gamma \cap A_{\alpha_\eta}$ for $\gamma \in C$. Set:

$$C^* = \sigma_\eta^{-1}(C) \ ; \ A^* = \sigma_\eta^{-1}(A_{\alpha_\eta}) .$$

Then C^* is closed, unbounded in κ, $A^* \subset \kappa$ and $A_\gamma \neq \gamma \cap A^*$ for $\gamma \in C^*$. However, $\alpha_\eta \in C^*$, since $C^* \cap \alpha_\eta = \sigma_\eta(C^*) = C$ is unbounded in α_η. Hence $A_{\alpha_\eta} \neq \alpha_\eta \cap A^*$.

But $\alpha_\eta \cap A^* = \sigma_\eta(A^*) = A_{\alpha_\eta}$.

Contradiction!

QED

[<u>Note</u> The hypothesis "$V = L$ and κ is not ineffable" can be replaced by:

~~$\bigvee B \bigvee \langle A_\alpha \mid \alpha < \kappa \rangle \ (V =$~~

$$\bigvee B \subset \kappa \ \bigvee \langle A_{\alpha<\kappa} \rangle \ (V = L[B] \wedge$$
$$\wedge \ \langle A_{\alpha<\kappa} \rangle \text{ is a counterexample to ineffability } \wedge$$
$$\wedge \ \bigwedge \alpha < \kappa \ \ A_\alpha \in L[B \cap \alpha] \ ) \ ]$$

We now turn to the proof that if κ is ineffable, then $\diamondsuit_\kappa$ holds. We shall show, in fact, that $\diamondsuit_\kappa$ holds for an even larger class of cardinals which we call <u>subtle</u>.

<u>Def</u> Let κ be regular. κ is <u>subtle</u> iff for every closed, unbounded $C \subset \kappa$ and every sequence $\langle A_\alpha \mid \alpha \in C \rangle$, if $A_\alpha \subset \alpha$ for $\alpha \in C$, then

$$\bigvee \alpha\, \beta \in C \ (\alpha < \beta \wedge A_\alpha = \alpha \cap A_\beta).$$

The least subtle cardinal is smaller than the least almost ineffable cardinal but still larger than the least cardinal which is Π^n_m indescribable for $n, m < \omega$. Another example of a cardinal which is subtle but not ineffable is the least $\kappa \to (\omega)^{<\omega}$.

<u>Theorem 11</u> (Kunen) If κ is subtle, then $\diamondsuit_\kappa$ holds.

proof.

By induction on limit $\alpha < \kappa$ define $\langle S_\alpha, C_\alpha \rangle$ s.t. $S_\alpha \subset \alpha$, C_α is closed, unbounded in α and $\bigwedge \gamma \in C_\alpha \; S_\gamma \neq \gamma \cap S_\alpha$ if possible.

<u>Claim</u> If $S \subset \kappa$, then $\{\alpha < \kappa \mid S \cap \alpha = S_\alpha\}$ is Mahlo.

Suppose not. Then there is a pair $\langle S, C \rangle$ s.t. $S \subset \kappa$, C is closed, unbounded in κ and

$$\bigwedge \alpha \in C \quad S_\alpha \neq \alpha \cap S .$$

Let C^* be the set of limit pts. of C. Then

$$\alpha \in C^* \rightarrow \bigwedge \gamma \in C_\alpha \quad S_\gamma \neq \gamma \cap S_\alpha .$$

~~Hence:~~

$$\cancel{\alpha, \beta \in C^*}$$

By the subtlety of κ, however, there are $\alpha, \beta \in C^*$ s.t. $\alpha < \beta$, $C_\alpha = \alpha \cap C_\beta$, $S_\alpha = \alpha \cap S_\beta$. Hence

$$\alpha \in C_\beta \wedge S_\alpha = \alpha \cap S_\beta .$$

Contradiction! QED